



Solutions to the Kaup–Broer system and its (2+1) dimensional integrable generalization via the dressing method

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ABSTRACT

In this paper we formulate the nonlocal dbar problem dressing method of Manakov and Zakharov (Zakharov and Manakov, 1984, 1985; Zakharov, 1989) for the 4 scaling classes of the (1+1) dimensional Kaup–Broer system (Broer, 1975; Kaup, 1975). The applications of the method for the (1+1) dimensional Kaup–Broer systems are reductions of a method for a complex valued (2+1) dimensional completely integrable partial differential equation first introduced in Rogers and Pashaev (2011). This method allows computation of solutions to all scaling classes of the Kaup–Broer system. We then consider the case of non-capillary waves with gravitational forcing, and use the dressing method to compute N-soliton solutions and more general solutions in the closure of the N-soliton solutions in the topology of uniform convergence in compact sets called primitive solutions. These more general solutions are analogous to the solutions derived in (Dyachenko and Zakharov, 2016; Zakharov and Dyachenko, 2016; Zakharov et al., 2016) for the KdV equation. We derive dressing functions for finite gap solutions, and compute counter propagating dispersive shockwave type solutions numerically.

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1. Introduction

1.1. General introduction

It was recently demonstrated experimentally by Redor et al. that a soliton gas consisting of many solitons propagating in both directions can be formed in a flume [1]. Moreover, they demonstrated that the counter propagating Kaup–Broer (Kaup–Boussinesq) 2-soliton collision solution compares favorably to laboratory data provided that we ignore non-integrable effects, which are of small amplitude relative to the solitons [1]. Because of the finite size of the flume, experimental observations were necessarily of counter propagating solitons to achieve the length of propagation needed to produce the soliton gas [1]. In this paper we consider the problem of computing solitons to the Kaup–Broer system consisting of many interacting solitons. This is done by considering the primitive solutions, which were first introduced for the KdV equation [2–5]. We also discuss how the method appears as a reduction of the dressing method for the complete complexification of the (2+1) dimensional (two spatial dimensional and one temporal dimension) completely integrable generalization of the Kaup–Broer system originally considered by Rogers and Pashaev [6].

It should be noted that the Kaup–Broer system has been shown to be ill-posed in Sobolev space [7]. This does not mean there can be no practical application of particular solutions to the Kaup–Broer system in the small amplitude and long wave regime. However, this does mean trying to apply conventional numerical or perturbation theory methods to the Kaup–Broer system is not a good idea. Given that the collision of two counter propagating solitons modeled by the Kaup–Broer system compares favorably to experiment [1], a soliton gas described by the Kaup–Broer system could be interesting from a practical point of view. Moreover, as mentioned in [1], the observations of Chen and Yeh [8] show that the head on-collision of solitons produces an amplification over a simple linear superposition of KdV solitons, which means we can expect a counter propagating soliton gas to have larger amplitude statistics than a sum of counter propagating KdV soliton gases. The splitting of the scattering theory into left and right moving components makes the Kaup–Broer system an intuitive model for describing a soliton gas consisting of counter propagating solitons.

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1.2. Background

In 1975 Kaup [9] and Broer [10] independently derived the following system of nonlinear partial differential equations

$$\eta_t + h_0 \varphi_{xx} + (\eta \varphi_x)_x + \left(\frac{h_0^3}{3} - \frac{h_0 \tau}{\rho g} \right) \varphi_{xxxx} = 0, \quad (1)$$

$$\varphi_t + \frac{1}{2}(\varphi_x)^2 + g\eta = 0, \quad (2)$$

which we have expressed in dimensional form. We will call this system of equations the Kaup–Broer system. This system describes weakly nonlinear long shallow water waves in a channel of constant depth. In system (1), (2) the constant h_0 is the quiescent water depth, g is the gravitational acceleration, τ is the surface tension of the fluid, and ρ is the density of the fluid. The variable η is the free surface displacement from quiescent water depth, and φ is the velocity potential evaluated on the free surface.

Broer arrived on the system (1), (2) by considering Hamiltonian approximations to the Hamiltonian equations for water–waves in a narrow channel. Kaup showed that system (1), (2) when $g > 0$ and $h_0^2/3 > \tau/(g\rho)$ can be solved by the Inverse Scattering Method (ISM); he studied the direct and inverse scattering problem, and found one-soliton solutions. Periodic and quasiperiodic finite gap solutions and their N-soliton limit were found by Matveev and Yavor [11]. The N-soliton solutions have also been computed using the Hirota bilinear form [12]. Complete integrability of the Kaup–Broer system was proved by Kupershmidt [13].

Consider the general form of the Kaup–Broer system

$$\eta_t + \mu_1 \varphi_{xx} + \mu_2 (\eta \varphi_x)_x + \mu_3 \varphi_{xxxx} = 0, \quad (3)$$

$$\varphi_t + \varepsilon_1 \frac{1}{2}(\varphi_x)^2 + \varepsilon_2 \eta = 0. \quad (4)$$

The system (3), (4) has 4 distinct scaling classes that can be represented by systems of the form

$$\eta_t + \varphi_{xx} + (\eta \varphi_x)_x + \mu \varphi_{xxxx} = 0, \quad (5)$$

$$\varphi_t + \frac{1}{2}(\varphi_x)^2 + \varepsilon \eta = 0, \quad (6)$$

determined by the 4 possible choices defined by $\varepsilon = \pm 1$ and $\mu = \pm 1/4$. The 4 scaling class of the Kaup–Broer system are analogous to the 4 scaling classes of (1+1) dimensional Boussinesq equations discussed by Bogdanov and Zakharov in [14]. The 4 scaling classes of the Kaup–Broer system have the following physical interpretations:

1. $\varepsilon = 1, \mu = 1/4$ corresponds to gravitational non-capillary waves.
2. $\varepsilon = 1, \mu = -1/4$ corresponds to gravitational capillary waves.
3. $\varepsilon = -1, \mu = 1/4$ corresponds to non-capillary waves in reversed gravity.
4. $\varepsilon = -1, \mu = -1/4$ corresponds to capillary waves in reversed gravity.

It is known (see, for instance [15]) that any integrable system in (1+1) dimensions admits a generalization to an equation in (2+1) dimensions, preserving integrability. For instance, the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \quad (7)$$

admits generalizations to the KP equation

$$\frac{\partial}{\partial x}(u_t + 6uu_x + u_{xxx}) = \pm 3u_{yy}, \quad (8)$$

and the Veselov–Novikov equation

$$u_t = u_{zzz} + u_{\bar{z}\bar{z}\bar{z}} + (uv)_z + (u\bar{v})_{\bar{z}} \quad (9)$$

$$v_{\bar{z}} = -3u_z, u = \bar{u}. \quad (10)$$

A completely integrable (2+1) dimensional generalization of the Kaup–Broer system was discovered by Rogers and Pashaev using the Hirota bilinear form [6].

All four scaling classes of the Kaup–Broer system are dimensional reductions of the completely complexified version of the system

$$s_t + \alpha s_u^2 - \beta s_v^2 + \Pi = 0 \quad (11)$$

$$c_t + 2\alpha(s_u c)_u - 2\beta(s_v c)_v - 2aPs + \frac{1}{2}Ps_{uv} = 0 \quad (12)$$

with α, β arbitrary constants,

$$P = \alpha \frac{\partial^2}{\partial u^2} - \beta \frac{\partial^2}{\partial v^2}, \quad (13)$$

and Π defined in terms of c by solving

$$\Pi_{uv} = 2Pc. \quad (14)$$

By complete complexification, we mean all coordinates, fields, and coefficients are taken to possibly be complex numbers. The Π dependence in (11) can be removed by differentiating (11) by u and v and applying (14). This completely integrable generalization of the Kaup–Broer system was first introduced by Rogers and Pashev [6] (for a particular choice of parameters). The system (11), (12) is Hamiltonian in the sense that it is the canonical system

$$s_t = -\frac{\delta H}{\delta c}, \quad c_t = \frac{\delta H}{\delta s} \quad (15)$$

for the (complex) Hamiltonian functional

$$H = \iiint_{\mathbb{C}^2} \left[\alpha c s_u^2 - \beta c s_v^2 + a s P s - \frac{1}{4} s P s_{uv} + \frac{1}{2} c \Pi \right] d^2 u d^2 v. \quad (16)$$

We can see that all 4 scaling classes of the Kaup–Broer system are achievable as the dimensional reduction of the complete complexification of integrable (2+1) dimensional generalization of the Kaup–Broer system. Consider the choice of parameters $\alpha = -\beta = 1/4$ and $a = \pm 1$, and suppose there exists a solution s, c to (11), (12) such that

$$\sigma s(u, v, \sigma t) = \varphi(u + v, t), \quad -ac(u, v, \sigma t) = \eta(u + v, t). \quad (17)$$

If we write $x = u + v$, then P acts on the (1+1) dimensional fields as $(1/2)\partial^2/\partial x^2$, and the equation for Π becomes $\Pi_{xx} = -a\eta_{xx}$. It is thus easy to see that $\varphi(x, t)$ and $\eta(x, t)$ solve

$$\varphi_t + \frac{1}{2}\varphi_x^2 - a\sigma^2\eta = 0 \quad (18)$$

$$\eta_t + (\varphi_x\eta)_x + \varphi_{xx} - \frac{a}{4}\varphi_{xxxx} = 0. \quad (19)$$

We see that all 4 scaling classes are attainable from the complete complexification based on the two choices $a = \pm 1$ and $\sigma = 1, i$. The complex (2+1) dimensional Hamiltonian then reduces to the real (1+1) dimensional Hamiltonian

$$H = \int \left[\frac{1}{2}\eta\varphi_x^2 - \frac{1}{2}\varphi\varphi_{xx} - \frac{a}{4}\varphi\varphi_{xxxx} - \frac{a\sigma^2}{2}\eta^2 \right] dx. \quad (20)$$

Remark 1. The fact that we can solve the complete complexification allows us to produce solutions to other interesting reductions of the completely complexified equations. A particularly interesting reduction from the complete complexification corresponds to taking $u = z = x + iy$ and $v = \bar{z} = x - iy$ with $\alpha = \beta = 1$ so that

$$\frac{\partial^2}{\partial u \partial v} = \frac{1}{4} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) = \frac{1}{4} \Delta, \quad (21)$$

$$P = \frac{1}{2} \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) = \frac{1}{2} \square, \quad (22)$$

and the (2+1) dimensional equation becomes

$$s_t - \frac{i}{2}s_x s_y + \Pi = 0, \quad (23)$$

$$c_t - i(s_x c)_y - i(s_y c)_x - a\square s + \frac{1}{16}\Delta\square s = 0, \quad (24)$$

$$\Delta\Pi = 4\square c. \quad (25)$$

The linearization of this system can be expressed in terms of c and Π as

$$c_{tt} + a\square\Pi - \frac{1}{16}\Delta\square\Pi = 0, \quad (26)$$

$$\Delta\Pi = 4\square c. \quad (27)$$

If we take the Fourier transform

$$c_{pq} = \frac{1}{2\pi} \iint c(x, y) e^{-i(px+qy)} dx dy, \quad (28)$$

then equation (26) becomes

$$(c_{pq})_{tt} + a(p^2 - q^2)\Pi_{pq} - \frac{1}{16}(p^2 + q^2)(p^2 - q^2)\Pi_{pq} = 0. \quad (29)$$

Moreover, (27) implies

$$\Pi_{pq} = 4 \frac{p^2 - q^2}{p^2 + q^2} c_{pq}. \quad (30)$$

so we can remove Π_{pq} from the equation for c_{pq} to get

$$(c_{pq})_{tt} + 4a \frac{(p^2 - q^2)^2}{p^2 + q^2} c_{pq} - \frac{1}{4}(p^2 - q^2)^2 c_{pq} = 0. \quad (31)$$

Therefore, this version of the (2+1) dimensional system is a nonlinear wave equation with cubic nonlinearity, and dispersion relation

$$\omega^2 = \frac{1}{4}(p^2 - q^2)^2 - 4a \frac{(p^2 - q^2)^2}{p^2 + q^2}. \quad (32)$$

When $a < 0$ this is a stable wave equation in the long wave limit, while for $a > 0$ this is stable only in the long wave limit with $p \gg q$ or $q \ll p$. These could potentially be physically relevant wave equations.

In this paper we will construct solutions to system (5), (6) and its generalization (11), (12) by the dressing method developed in papers of Zakharov and his collaborators [15–18]. The dressing method makes it possible to find much broader classes of solutions of integrable systems than the ISM, and is based on the use of the nonlocal $\bar{\partial}$ -problem first used by Ablowitz, Fokas, and Bar-Yaacov [19] for the solution of the KP-2 equation (the KP-2 is equation (8) with the positive choice of sign). Moreover, since u, v, t and α, β, a will appear as parameters in the Kaup–Broer system, this method leads immediately to solutions of the completely complexified system as well.

One important property of the (2+1) dimensional generalization of the Kaup–Broer system is that the nonlocal $\bar{\partial}$ (dbar) problem for the 4 cases of the Kaup–Broer system are dimensional reductions of the nonlocal $\bar{\partial}$ problem for the (2+1) dimensional generalization. The operator for the scattering problem for our (2+1) dimensional generalization of the Kaup–Broer system is the non-relativistic quantum Hamiltonian for a charged particle under the influence of an electromagnetic field.

In the case of $\varepsilon = 1$ and $\mu = 1/4$, we will explicitly construct the N-soliton solutions, and also construct solutions that can be interpreted as a limit of the N-soliton solutions as the number of solitons diverges to ∞ . We will provide numerical evidence that the second type of solution can describe counter propagating dispersive shockwave type solutions.

The problem of computing solutions to completely integrable partial differential equations that are limits of sequences N-soliton solutions as the number of solitons diverges to infinity is an intriguing area of research with many interesting unsolved problems. Recent progress on this problem was made by Dyachenko, Zakharov and Zakharov who demonstrated in [2–4] that periodic, dispersive shockwave, and turbulent solutions to the KdV equation can be computed as limits of N-soliton solutions. From the point of view of the inverse spectral theory of one dimensional Schrödinger operators, a continuum limit was taken in which discrete eigenvalue coalesce into spectral bands. These potentials were dubbed primitive potentials. This work was continued in [5] where the case of symmetric primitive potentials was considered in detail. An analytic procedure for computing all the Taylor coefficients of symmetric primitive potentials was presented, as well as the special case of an elliptic potential. The asymptotic behavior of genus one dispersive shockwave behavior of a soliton was analyzed rigorously by Girotti, Grave and McLaughlin [20] using the Riemann–Hilbert problem formulation of the infinite soliton limit presented by Dyachenko, Zakharov and Zakharov [2–4]. Nabelek [21] showed how to produce all finite gap solutions using primitive solutions.

One drawback of using the KdV equation as a model of (1+1) dimensional (one spatial and one temporal dimension) shallow water long waves is that the KdV equation is derived by assuming wave motion in a single direction. A common completely integrable alternative to the KdV equation is the (1+1) dimensional Boussinesq equation. However, one issue is that the scattering problem used to solve the Boussinesq equation by the inverse scattering transform (IST) is of degree three making application of nonlinear steepest descent more difficult [14]. In contrast, the Kaup–Broer system has a degree 2 scattering problem that separates into left and right moving KdV scattering problems.

A more in depth comparison of the dressing method for the Kaup–Broer system to the dressing method for the Korteweg–de Vries and the Kadomtsev–Petviashvili equations can be found in chapter 2 of the first author's PhD dissertation [22] done under the supervision of the second author. However, in the PhD dissertation only N-soliton solutions were computed by the dressing method. In this paper we extend the results first presented in the dissertation to compute solutions to the Kaup–Broer system in the closure of the N-soliton solutions with respect to the topology of uniform convergence on compact sets.

1.3. Outline of paper

This paper is structured as follows:

- In Section 2 we provide the Manakov triple formulation of the (2+1) dimensional generalization of the Kaup–Broer system.
- In Section 3 we give the nonlocal $\bar{\partial}$ formulation of the dressing method for the (2+1) dimensional generalization of Kaup–Broer system and use it to formally compute a new class of solutions to this generalization in Section 3.1. We then discuss how the dimensional reduction of the (2+1) dimensional system to the (1+1) dimensional Kaup–Broer system is achieved from the point of view of the nonlocal $\bar{\partial}$ problem in Section 3.2.
- In Section 4 we discuss the most studied case of the (1+1) dimensional Kaup–Broer system ($\varepsilon = 1, \mu = 1/4$), and show how the dressing method can be used to produce a new class of solutions to the (1+1) dimensional Kaup–Broer system called primitive solutions that can be interpreted as the infinite soliton limit of the N-soliton solutions.
- In Section 5 we discuss how finite gap solutions to the Kaup–Broer system can be computed.
- In Section 6 we compute numerical approximations (corresponding to exact N-soliton solutions with large N) to some of these new solutions to the (1+1) dimensional Kaup–Broer system.
- In Section 7 we provide some concluding remarks.

2. The Manakov triple for the (2+1) dimensional generalization of the Kaup–Broer system

We will construct the Manakov triple and the corresponding linear system for the (2+1) dimensional generalization of the Kaup–Broer system. The existence of the Manakov triple is what justifies the use of the dressing method discussed in the next section. The system (11), (12) is equivalent to a solution of the Manakov triple operator equation

$$L_t = [M, L] + QL, \quad (33)$$

where

$$L = \frac{\partial^2}{\partial u \partial v} + A \frac{\partial}{\partial v} + B - a, \quad (34)$$

$$M = \alpha \frac{\partial^2}{\partial u^2} + \beta \frac{\partial^2}{\partial v^2} + F \frac{\partial}{\partial v} + G, \quad (35)$$

and $Q = F_v - 2\alpha A_u$ are defined in terms of constants a, α, β and coefficients A, B, F, G depending on the coordinates u, v, t . The equation for the Manakov triple is equivalent to the simultaneous solvability of the two linear PDEs making up the linear system

$$L\psi = 0, \quad \psi_t = M\psi. \quad (36)$$

In the next section we will discuss how the nonlocal $\bar{\partial}$ problem can be used to compute a solution ψ to (36), corresponding potentials, and a solution to the (2+1) dimensional generalization of the Kaup–Broer system.

In terms of A, B, F and G , the Manakov triple equation is

$$\begin{aligned} A_t \frac{\partial}{\partial v} + B_t = & (2\beta A_v - F_u) \frac{\partial^2}{\partial u^2} + (2\alpha B_u - G_v) \frac{\partial}{\partial u} \\ & + (\alpha A_{uu} + \beta A_{vv} + 2\beta B_v - F_{uv} + F A_v - F_v A - G_u) \frac{\partial}{\partial v} \\ & + \alpha B_{uu} + \beta B_{vv} + F B_v - G_{uv} - A G_v - a F_v + 2\alpha a A_u. \end{aligned} \quad (37)$$

The coefficients for the constant, $\partial/\partial v$, $\partial/\partial u$, $\partial^2/\partial u^2$ terms solve the nonlinear system

$$A_t = \alpha A_{uu} + \beta A_{vv} + 2\alpha B_u + 2\beta B_v - F_{uv} + F A_v - F_v A - G_u \quad (38)$$

$$B_t = \alpha B_{uu} + \beta B_{vv} + F B_v - G_{uv} - A G_v - a F_v + 2\alpha a A_u \quad (39)$$

$$F_u = 2\beta A_v, \quad (40)$$

$$G_v = 2\alpha B_u. \quad (41)$$

We will now show that this system is equivalent to the complete complexification of the (2+1) dimensional completely integrable generalization of the Kaup–Broer system. If we suppose that A has the potential form $A = s_u$ for some potential s then $F = 2\beta s_v + C$ for some constant C (we will show that this potential exists when we discuss the dressing method in the next section). Since we will remove the F dependence we can take $C = 0$ and produce a solution to the nonlinear system (38)–(41). There also exists some antiderivative ∂_v^{-1} such that $G = 2\alpha \partial_v^{-1} B_u$.

Eliminating the F and G dependence in (38), (39), replacing A with the potential s and anti-differentiating equation (38) with respect to u and v implies

$$s_t + \alpha(s_u)^2 - \beta(s_v)^2 - P s + \rho = 0, \quad (42)$$

$$B_t + \alpha 2(s_u B)_u - \beta 2(s_v B)_v + P B - a P s = 0, \quad (43)$$

where ρ is defined in terms of B as a solution to

$$\rho_{uv} = 2P B, \quad (44)$$

and we recall that P is the differential operator

$$P = \alpha \frac{\partial^2}{\partial u^2} - \beta \frac{\partial^2}{\partial v^2}.$$

The ρ dependence in the system can be removed by differentiating (42) by u and v and applying (44). We can change variables to fields c and s related by

$$c = B - \frac{1}{2} s_{uv} \quad (45)$$

so that we end up with the system

$$s_t + \alpha s_u^2 - \beta s_v^2 + \Pi = 0$$

$$c_t + 2\alpha(s_u c)_u - 2\beta(s_v c)_v - 2a P s + \frac{1}{2} P s_{uv} = 0$$

with Π defined in terms of c as the solution to

$$\Pi_{uv} = 2P c.$$

This is the (2+1) dimensional generalization of the Kaup–Broer system. The function Π so defined is related to s and ρ by $\Pi = \rho - P s$. This is the system discussed in the introduction. Moreover, this change of variables is a canonical transformation.

Remark 2. If we transfer to complex coordinates $u = x + iy$ and $v = x - iy$, then operator L is

$$L = \frac{1}{4} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{A}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) + B - a, \quad (46)$$

which has the physical interpretation as being proportional to the quantum mechanical Hamiltonian for a 2 dimensional non-relativistic charged quantum particle under the influence of an electromagnetic field. The solutions to the nonlocal $\bar{\partial}$ problem will allow us to compute solutions to $L\psi = 0$. The inverse spectral theory of periodic and finite gap operators of this form was studied extensively by Novikov and his collaborators. See [23,24] for a review of this theory.

3. The dressing method for the (1+1) dimensional Kaup–Broer system and its (2+1) dimensional generalization

The nonlocal $\bar{\partial}$ problem can be used to find solutions ψ to the system (36) with coefficients in the linear operators depending on the solution. Since the coefficients depend on ψ , this method computes a solution to the (2+1) dimensional generalization of the Kaup–Broer system using ψ . We can produce a meromorphic family of solutions $\psi(\lambda; u, v, t)$ to (36) by taking $\psi(\lambda; u, v, t) = e^{\phi(\lambda; u, v, t)} \chi(\lambda; u, v, t)$ where

$$\phi(\lambda; u, v, t) = \lambda u + a\lambda^{-1}v + (\alpha\lambda^2 + \beta\lambda^{-2})t \quad (47)$$

and χ solves the nonlocal $\bar{\partial}$ problem

$$\frac{\partial \chi}{\partial \bar{\lambda}}(\lambda; u, v, t) = \iint_{\mathbb{C}} R(\lambda, w) e^{\phi(w; u, v, t) - \phi(\lambda; u, v, t)} \chi(w; u, v, t) dA(w) \quad (48)$$

and is normalized so $\chi \rightarrow 1$ as $\lambda \rightarrow \infty$. Equivalently, χ solves the equivalent integral equation

$$\chi(\lambda; u, v, t) = 1 + \frac{1}{\pi} \iint_{\mathbb{C}} \iint_{\mathbb{C}} \frac{e^{\phi(w; u, v, t) - \phi(\zeta; u, v, t)}}{\lambda - \zeta} R(\zeta, w) \chi(w; u, v, t) dA(w) dA(\zeta) \quad (49)$$

where we can interpret the integral appearing in (49) as a principal value integral because

$$\frac{1}{\lambda - \zeta} = \lim_{\epsilon \rightarrow 0} \frac{\bar{\lambda} - \bar{\zeta}}{|\lambda - \zeta|^2 + \epsilon^2}. \quad (50)$$

Define the operators

$$L_1 \chi = D_u D_v \chi + A D_v \chi + (B - a) \chi \quad (51)$$

$$= \chi_{uv} + \lambda \chi_v + a\lambda^{-1} \chi_u + A \chi_v + A a \lambda^{-1} \chi + B \chi, \quad (52)$$

and

$$L_2 \chi = D_t \chi - \alpha D_u^2 \chi - \beta D_v^2 \chi - F D_v \chi - G \chi \quad (53)$$

$$= \chi_t - \alpha(\chi_{uu} + 2\lambda \chi_u) - \beta(\chi_{vv} + 2a\lambda^{-1} \chi_v) - F(\chi_v + a\lambda^{-1} \chi) - G \chi, \quad (54)$$

where

$$D_u = \frac{\partial}{\partial u} + \lambda, \quad D_v = \frac{\partial}{\partial v} + a\lambda^{-1}, \quad D_t = \frac{\partial}{\partial t} + (\alpha\lambda^2 + \beta\lambda^{-2})t. \quad (55)$$

Applying L_j to the nonlocal $\bar{\partial}$ problem (48) we find that $L_j \chi$ solves the nonlocal $\bar{\partial}$ problem

$$\frac{\partial L_j \chi}{\partial \bar{\lambda}}(\lambda) = \gamma_j \delta(\lambda) + \iint_{\mathbb{C}} R(\lambda, w) e^{\phi(w; u, v, t) - \phi(\lambda; u, v, t)} L_j \chi(w; u, v, t) dA(w), \quad (56)$$

for $L_j \chi$, where

$$\gamma_1 = \pi a(\chi_u(0) + A \chi(0)), \quad \gamma_2 = -\pi a(2\beta \chi_v(0) + F \chi(0)) \quad (57)$$

are such that

$$\left[\frac{\partial}{\partial \bar{\lambda}}, L_j \right] = \gamma_j \delta(\lambda). \quad (58)$$

The condition that $\gamma_j = 0$ is equivalent to the assumption that $L_j \chi(\lambda)$ is holomorphic at $\lambda = 0$.

The crux is that if A, B, F and G are set according to

$$s = -\log(\chi_0^0), \quad A = s_u, \quad B = -(\chi_1^\infty)_v, \quad F = 2\beta s_v, \quad G = -2\alpha(\chi_1^\infty)_u, \quad (59)$$

then $L_j \chi \rightarrow 0$ as $\lambda \rightarrow \infty$, and $\gamma_j = 0$ making $L_j \chi$ holomorphic off of $\lambda \in \mathbb{C} \setminus \{\text{poles}\}$. If it can be proved that 0 is the unique solution to the homogeneous Fredholm integral equation

$$\xi(\lambda; u, v, t) = \frac{1}{\pi} \iint_{\mathbb{C}} \iint_{\mathbb{C}} \frac{e^{\phi(w; u, v, t) - \phi(\zeta; u, v, t)}}{\lambda - \zeta} R(\zeta, w) \xi(w; u, v, t) dA(w) dA(\zeta), \quad (60)$$

then

$$L_1 \chi(\lambda; u, v, t) = 0, \quad L_2 \chi(\lambda; u, v, t) = 0. \quad (61)$$

It is easy to see that χ solves (61) if and only if $e^{\phi(\lambda; u, v, t)} \chi(\lambda; u, v, t)$ solves (36). Since we have produced a meromorphic family of solutions to (36), the functions s and $c = B - \frac{1}{2}s_{uv}$ solve the (2+1) dimensional Kaup–Broer system. In the soliton case, the integral equation is clearly Fredholm because it is equivalent to a finite dimensional linear equation.

Theorem 3. Suppose that $R : \mathbb{C}^2 \rightarrow \mathbb{C}$ is a generalized function with compact support on $\{(z, w) \in \mathbb{C}^2 : z \neq 0, w \neq 0\}$ that has been chosen so that:

1. The operator $\mathcal{F} : D \rightarrow D$ determined by

$$\mathcal{F}\chi(\lambda; u, v, t) = \frac{1}{\pi} \iint_{\mathbb{C}} \iint_{\mathbb{C}} \frac{e^{\phi(w; u, v, t) - \phi(\zeta; u, v, t)}}{\lambda - \zeta} R(\zeta, w) \chi(w; u, v, t) dA(w) dA(\zeta) \quad (62)$$

is a well defined operator on some function space D (the appropriate choice of function space D may depend on the choice of dressing function R , see Remark 4).

2. If $(\mathcal{I} - \mathcal{F})\xi = 0$, then $\xi = 0$ ($\mathcal{I} : D \rightarrow D$ is the identity operator on D).
3. There is a solution χ to the integral equation

$$(\mathcal{I} - \mathcal{F})\chi(\lambda; u, v, t) = 1. \quad (63)$$

Then the function χ solves the nonlocal $\bar{\partial}$ problem

$$\frac{\partial \chi}{\partial \bar{\lambda}}(\lambda; u, v, t) = \iint_{\mathbb{C}} R(\lambda, w) e^{\phi(w; u, v, t) - \phi(\lambda; u, v, t)} \chi(w; u, v, t) dA(w), \quad (64)$$

and is the unique solution normalized by $\chi(\lambda) \rightarrow 1$ as $\lambda \rightarrow \infty$.

The function χ has expansions

$$\chi(\lambda; u, v, t) = \sum_{n=0}^{\infty} \chi_n^0(u, v, t) \lambda^n, \quad \chi(\lambda; x, t) = \sum_{n=0}^{\infty} \chi_n^\infty(u, v, t) \lambda^{-n}. \quad (65)$$

The functions c and s computed from

$$s(u, v, t) = -\log(\chi_0^0(u, v, t)), \quad c(u, v, t) = -(\chi_1^\infty)_v(u, v, t) - \frac{1}{2} s_{uv}(u, v, t), \quad (66)$$

and Π determined from c by solving the linear differential equation

$$\Pi_{uv} = 2Pc \quad (67)$$

solve the complete complexification of the integrable (2+1) dimensional generalization of the Kaup–Broer system (11), (12).

Remark 4. We are free to choose any dressing function $R(\lambda, w)$ provided conditions 1–3. are satisfied. Conditions 2. and 3. on the dressing function amount to stating that $\mathcal{I} - \mathcal{F}$ satisfies the Fredholm alternative. We verify that \mathcal{F} is well defined and satisfies the Fredholm alternative directly for all special solutions we discuss in this paper. A sufficient condition for the conditions to be satisfied is that the kernel of \mathcal{F} satisfies the Hilbert–Schmidt condition.

Remark 5. The appropriate choice of function space D could depend on the choice dressing function R . For example:

1. When considering N -soliton solutions, the appropriate function space D is the space of meromorphic functions that are bounded at ∞ .
2. For the extension of the N -soliton solutions described in Section 3.1, the appropriate function space D is the space of continuous functions that restrict of holomorphic function in $\mathbb{C} \setminus K$ for some compact subset $K \subset \mathbb{C} \setminus \{0\}$. The reduction of these solutions to the N -soliton solutions is a degeneration in which the continuous function develops poles.
3. For the solutions discussed in Section 4, the appropriate function space D will be the space of sectionally holomorphic functions that are bounded at ∞ .

Remark 6. The operators $D_{u,v,t}$ and $L_{1,2}$ with $a = 1$ discussed in this section are closely related to those used to solve the Veselov–Novikov equation with the dressing method [25]. The differences are:

- We replace the “light cone” coordinates u, v with complex conjugate coordinates $u \rightarrow \bar{z}$, and $v \rightarrow z$.
- We replace $\alpha\lambda^2 + \beta\lambda^{-2}$ in the definitions of D_t and ϕ with $\lambda^3 + \lambda^{-3}$.

The Riemann–Hilbert problem used in [25] is really just a nonlocal $\bar{\partial}$ problem in the case where the dressing function R is supported on an arc. By setting $A = 0$ and replacing u and v with the complex conjugated coordinates \bar{z} and z , we end up with the operator L evolving in the Lax equation for the Veselov–Novikov equation [25–27].

3.1. A class of exact solutions to the (2+1) dimensional Kaup–Broer system generalizing the N -soliton solutions

We can consider a completely solvable case of the nonlocal $\bar{\partial}$ problem complex (2+1) dimensional system corresponding to a dressing function of the form

$$R(\lambda, w) = \sum_{n=1}^N f_n(\lambda) g_n(w). \quad (68)$$

Then the solution to the nonlocal $\bar{\partial}$ problem is of the form

$$\chi(\lambda; u, v, t) = 1 + \frac{1}{\pi} \sum_{n=1}^N \Phi_n(u, v, t) \hat{f}_n(u, v, t), \quad (69)$$

where \hat{f}_n denotes the integral transform

$$\hat{f}_n(\lambda; u, v, t) = \iint_{\mathbb{C}} \frac{f_n(\zeta) e^{-\phi(\zeta; u, v, t)}}{\lambda - \zeta} dA(\zeta) \quad (70)$$

of f_n , and

$$\Phi_n(u, v, t) = \iint_{\mathbb{C}} g_n(w) e^{\phi(w; u, v, t)} \chi(w; u, v, t) dA(w). \quad (71)$$

Substitution of (69) into the definition of Φ_n gives the equation

$$\Phi_n(u, v, t) = G_n(u, v, t) + \sum_{m=1}^N M_{nm}(u, v, t) \Phi_m(u, v, t), \quad (72)$$

where

$$G_n(u, v, t) = \iint_{\mathbb{C}} g_n(w) e^{\phi(w; u, v, t)} dA(w), \quad (73)$$

and

$$M_{nm}(u, v, t) = \frac{1}{\pi} \iint_{\mathbb{C}} \iint_{\mathbb{C}} \frac{f_m(\zeta) g_n(w) e^{\phi(w; u, v, t) - \phi(\zeta; u, v, t)}}{w - \zeta} dA(\zeta) dA(w). \quad (74)$$

If $\det(I - M) \neq 0$ then the functions $\chi_0^0(u, v, t)$ and $\chi_1^\infty(u, v, t)$ given by

$$\chi_1^\infty(u, v, t) = \frac{1}{\pi} \sum_{n=1}^N \Phi_n(u, v, t) F_n(u, v, t) \quad (75)$$

where

$$F_n(u, v, t) = \iint_{\mathbb{C}} f_n(\zeta) e^{-\phi(\zeta; u, v, t)} dA(\zeta), \quad (76)$$

and

$$\chi_0^0(u, v, t) = 1 + \hat{f}_n(0; u, v, t), \quad (77)$$

can then be used to compute a solution s, c to the (2+1) dimensional Kaup–Broer system by Theorem 3. When f_n and g_n are taken to be constant multiples of delta functions supported on some points z_n, w_n , this solution reduces to an N -soliton solution.

3.2. Reduction of the (2+1) dimensional dressing method to (1+1) dimensions

We now present the dressing method for the Kaup–Broer system. The dressing method allows solutions to all 4 scaling classes of the Kaup–Broer system to be computed by solving a nonlocal $\bar{\partial}$ problem.

From (17) we see that dimensional reduction to the 4 scaling classes of the Kaup–Broer system can be achieved provided the solution depends only on $x = u + v$, and $\alpha = -\beta = 1/4$. The u, v , and t dependence enters the $\bar{\partial}$ problem only in terms of the combination

$$\phi(w; u, v, t) - \phi(\lambda; u, v, t) = -(\lambda - w)u - a(\lambda^{-1} - w^{-1})v - (\alpha\lambda^2 + \beta\lambda^{-2} - \alpha w^2 - \beta w^{-2})t. \quad (78)$$

This will depend only on $x = u + v$ if and only if

$$\lambda - w = a(\lambda^{-1} - w^{-1}), \quad (79)$$

which is solved if w is related to a by $w = -a\lambda^{-1}$. Therefore suppose that

$$R(\lambda, w) = \delta(w + a\lambda^{-1}) R_0(\lambda) \quad (80)$$

where R_0 has compact support on $\mathbb{C} \setminus \{-1, 0, 1\}$. Then (48) reduces to the nonlocal $\bar{\partial}$ problem

$$\frac{\partial \chi}{\partial \lambda}(\lambda; x, t) = R_0(\lambda) e^{-2\tilde{\phi}(\lambda; x, t)} \chi(-a\lambda^{-1}; x, t), \quad (81)$$

where

$$\tilde{\phi}(\lambda; x, t) = \frac{1}{2} (\lambda + a\lambda^{-1})x + \frac{1}{4} (\lambda^2 - \lambda^{-2})t. \quad (82)$$

This nonlocal $\bar{\partial}$ problem depends only on x , so the solution to the (2+1) dimensional generalization of the Kaup–Broer system does as well. We look for a solution χ to the nonlocal $\bar{\partial}$ problem normalized so that $\chi(\lambda) \rightarrow 1$ as $\lambda \rightarrow \infty$.

Therefore the following is a corollary to Theorem 1:

Corollary 7. Suppose that $R_0 : \mathbb{C} \rightarrow \mathbb{R}$ is a generalized function with compact support on $\mathbb{C} \setminus \{1, 0, -1\}$ that has been chosen so that:

1. The operator $\mathcal{F}_0 : D \rightarrow D$ determined by

$$\mathcal{F}_0 \chi(\lambda; x, t) = \frac{1}{\pi} \iint_{\mathbb{C}} \frac{R_0(\zeta) e^{-2\tilde{\phi}(\zeta; x, t)}}{\lambda - \zeta} \chi(-a\zeta^{-1}; x, t) dA(\zeta) \quad (83)$$

is a well defined operator on some function space D (the appropriate choice of function space D may depend on the choice of dressing function R , see Remark 4).

2. If $(\mathcal{I} - \mathcal{F}_0)\xi = 0$, then $\xi = 0$ ($\mathcal{I} : D \rightarrow D$ is the identity operator on D).

3. There is a solution χ to the integral equation

$$(\mathcal{I} - \mathcal{F}_0)\chi(\lambda; u, v, t) = 1. \quad (84)$$

Then the function χ solves the nonlocal $\bar{\partial}$ problem

$$\frac{\partial \chi}{\partial \bar{\lambda}}(\lambda; x, t) = R_0(\lambda) e^{-2\tilde{\phi}(\lambda; x, t)} \chi(-a\lambda^{-1}; x, t), \quad (85)$$

and is the unique solution normalized by $\chi(\lambda) \rightarrow 1$ as $\lambda \rightarrow \infty$.

The function χ has expansions

$$\chi(\lambda; x, t) = \sum_{n=0}^{\infty} \chi_n^0(x, t) \lambda^n, \quad \chi(\lambda; x, t) = 1 + \sum_{n=1}^{\infty} \chi_n^{\infty}(x, t) \lambda^{-n}. \quad (86)$$

Then the functions φ and η defined by

$$\varphi(x, t) = -\sigma \log(\chi_0^0(x, \sigma t)), \quad \eta(u, v, t) = a(\chi_1^{\infty})_x(x, \sigma t) + \frac{a\sigma}{2} \varphi_{xx}(x, t), \quad (87)$$

solve the (1 + 1) dimensional Kaup–Broer system (5), (6) with $\varepsilon = -a\sigma^2$ and $\mu = -a/4$.

Remark 8. The function $\psi(\lambda; x, t) = e^{\tilde{\phi}(\lambda; x, t)} \chi(\lambda; x, t)$ solves the linear system

$$\tilde{L}(\lambda, t) \psi(\lambda; x, t) = 0, \quad (88)$$

$$\tilde{M}(\lambda, t) \psi(\lambda; x, t) = \psi_t(\lambda; x, t), \quad (89)$$

where

$$\tilde{L}(\lambda, t) = \frac{\partial^2}{\partial x^2} + A \frac{\partial}{\partial x} + B - \frac{\lambda - a\lambda^{-1}}{2} A - \left(\frac{\lambda + a\lambda^{-1}}{2} \right)^2, \quad (90)$$

$$\tilde{M}(\lambda, t) = \left(\frac{\lambda - a\lambda^{-1}}{2} + F \right) \frac{\partial}{\partial x} + G - \frac{\lambda - a\lambda^{-1}}{2} F. \quad (91)$$

The compatibility condition to be able to find a simultaneous solution to both equations in the system is equivalent to the Kaup–Broer system. The potentials are set according to (59) except we use coefficients from (86) and the u and v derivatives reduce to x derivatives.

Any choice of R_0 leads to a solution to the Kaup–Broer system; however most choices of R_0 will lead to complex solutions with singularities. Reality conditions on R_0 must be imposed to guarantee the solutions thus produced are real and bounded.

4. N-soliton solutions and the limit $N \rightarrow \infty$ for the (1+1) dimensional Kaup–Broer system

In this section we will consider the case of the N -soliton solutions in the (1+1) dimensional case with $a = -1$, $\sigma = 1$ (the most studies case). We then take the infinite soliton limit using the primitive solution method.

To produce an N -soliton solution consider parameters $\{\lambda_n, r_n\}_{n=1}^N$, which we will call the discrete spectral data for the N -soliton solutions. We will assume $\lambda_n \in \mathbb{R} \setminus \{-1, 0, 1\}$, $r_n > 0$ if $\lambda_n \in (-1, 0) \cup (1, \infty)$, and $r_n < 0$ if $\lambda_n \in (-\infty, -1) \cup (0, 1)$. The positive λ_n correspond to left moving solitons, while the negative λ_n correspond to right moving solitons.

We take the dressing function

$$R_0(\lambda) = \sum_{n=1}^N r_n \delta(\lambda - \lambda_n) \quad (92)$$

so that the integral equation becomes the algebraic equation

$$\chi(\lambda; x, t) = 1 + \frac{1}{\pi} \sum_{n=1}^N \frac{r_n e^{-\tilde{\phi}(\lambda_n; x, t)}}{\lambda - \lambda_n} \chi(\lambda_n^{-1}; x, t). \quad (93)$$

In particular, the solution is a rational function of the form

$$\chi(\lambda; x, t) = 1 + \frac{1}{\pi} \sum_{n=1}^n \frac{f_n(x, t)}{\lambda - \lambda_n}. \quad (94)$$

Plugging the rational form of the solution into the algebraic equation (93) gives the linear equation

$$\sum_{m=1}^N \left(\delta_{nm} + \frac{1}{\pi} \frac{r_n e^{-\tilde{\phi}(\lambda_n; x, t)}}{\lambda_m - \lambda_n^{-1}} \right) f_m = r_n e^{-\tilde{\phi}(\lambda_n; x, t)}. \quad (95)$$

This linear equation can be solved numerically for the values f_m , $m = 1, 2, \dots, N$. From these values of f_m an N-solution φ , η to the Kaup–Broer system can be computed by Theorem 1 via

$$\varphi(x, t) = -\log \left(1 - \frac{1}{\pi} \sum_{n=1}^N \frac{f_n(x, t)}{\lambda_n} \right), \quad \eta(x, t) = -\frac{1}{2} \varphi_{xx} - \frac{1}{\pi} \sum_{n=1}^N (f_n)_x(x, t). \quad (96)$$

We describe these solutions as N-solitons, because as $t \rightarrow \pm\infty$ the field η that could represent a free surface water wave has the form of N well separated solitary waves. The corresponding field φ limits to a solution with N kinks as $t \rightarrow \pm\infty$. The constituent solitons of η have well defined velocities $v_n^{\text{sol}} = -(\lambda_n + \lambda_n^{-1})/2$ when the solitons are well separated and noninteracting.

Consider the contour

$$\Gamma = [-\gamma_2, -\gamma_1] \cup [-\gamma_1^{-1}, -\gamma_2^{-1}] \cup [\gamma_2^{-1}, \gamma_1^{-1}] \cup [\gamma_1, \gamma_2] \quad (97)$$

with $1 < \gamma_1 < \gamma_2 < \infty$ oriented from left to right on all components, and let R_1 be a real valued functions on Γ that is nonnegative on $[-\gamma_1^{-1}, -\gamma_2^{-1}] \cup [\gamma_1, \gamma_2]$ and nonpositive on $[-\gamma_2, -\gamma_1] \cup [\gamma_2^{-1}, \gamma_1^{-1}]$. Moreover, we assume that R_1 only vanishes on a finite number of intervals, and that R_1 is Hölder continuous on the support of R_1 . The pair $\{\Gamma, R_1\}$ are the spectral data for a class of solutions determined by the dressing functions

$$R_0(\lambda) = \int_{\Gamma} \delta(\lambda - s) R_1(s) ds. \quad (98)$$

The spectral data give rise to the nonlocal $\bar{\partial}$ problem

$$\frac{\partial \chi}{\partial \bar{\lambda}}(\lambda; x, t) = \int_{\Gamma} \delta(\lambda - s) e^{-2\tilde{\phi}(s; x, t)} R_1(s) \chi(s^{-1}) ds, \quad (99)$$

and the integral equation giving the solution χ to the nonlocal $\bar{\partial}$ problem normalized so that $\chi \rightarrow 1$ as $\lambda \rightarrow \infty$ is

$$\chi(\lambda) - \frac{1}{\pi} \int_{\Gamma} \frac{e^{-2\tilde{\phi}(s; x, t)} R_1(s)}{\lambda - s} \chi(s^{-1}) ds = 1. \quad (100)$$

A subtlety here is that when $R_1(s) \neq 0$ and $R_1(s^{-1}) \neq 0$ simultaneously, then χ can become singular on Γ . This means that we should interpret the nonlocal $\bar{\partial}$ problem (99) as a nonlocal scalar Riemann–Hilbert problem with jump

$$\chi_+(s) - \chi_-(s) = R_1(s) e^{-2\tilde{\phi}(s; x, t)} (\chi_+(s^{-1}) + \chi_-(s^{-1})) \quad (101)$$

where the solution has the form

$$\chi(\lambda; x, t) = 1 + \frac{1}{\pi} \int_{\Gamma} \frac{f(s)}{\lambda - s} ds. \quad (102)$$

The function f solves the 1 dimensional integral equation

$$f(s; x, t) - e^{-2\tilde{\phi}(s; x, t)} R_1(s) \mathcal{H}_{\Gamma} f(s^{-1}; x, t) = e^{-2\tilde{\phi}(s; x, t)} R_1(s), \quad (103)$$

where \mathcal{H}_{Γ} is the Hilbert transform on Γ defined by

$$\mathcal{H}_{\Gamma} g(s) = \frac{1}{\pi} \text{P.V.} \int_{\Gamma} \frac{g(s'; x, t)}{s - s'} ds'. \quad (104)$$

Note that the support of the function f solving (103) is the same as the support of R_1 . Once the solution $f(s)$ to integral equation (103) has been computed, a solution φ , η to the Kaup–Broer system can be computed by Theorem 1 as

$$\begin{cases} \varphi(x, t) = -\log \left(1 - \frac{1}{\pi} \int_{\Gamma} \frac{f(s; x, t)}{s} ds \right) \\ \eta(x, t) = -\frac{1}{2} \varphi_{xx}(x, t) - \frac{1}{\pi} \int_{\Gamma} f_x(s; x, t) ds. \end{cases} \quad (105)$$

We call these solutions primitive solutions; these are analogous to the primitive solutions to the KdV equation [2–5].

Theorem 9. Let R_1 be a real valued functions on Γ that is nonnegative on $[-\gamma_1^{-1}, -\gamma_2^{-1}] \cup [\gamma_1, \gamma_2]$ and nonpositive on $[-\gamma_2, -\gamma_1] \cup [\gamma_2^{-1}, \gamma_1^{-1}]$. Suppose that $f(s, x, t)$ solves (103), then η and φ defined from f according to (105) solve the (1+1) dimensional Kaup–Broer system with $a = -1$ and $\sigma = 1$.

Consider the following property (I) on R_1 : If s is in the support of R_1 , then s^{-1} is not in the support of R_1 . If R_1 satisfies this property, then the principle value integral appearing in Eq. (103) reduces to a regular integral and the equation is a regular Fredholm integral equation of the second kind on the support of R_1 . Otherwise, Eq. (103) is a nonlocal singular integral equation.

Suppose that property (I) is satisfied, then the integral equation (103) is approximated via an N point quadrature rule on the support of R_1 , and the integrals appearing in (105) are approximated by the same quadrature rule. Then the approximations of (103) and (105)

by quadrature have the form of (95) and (96), and thus the approximation to the solution determined by $\{\Gamma, R_1\}$ is an exact N -soliton solution. In this manner, the solution determined by $\{\Gamma, R_1\}$ can be interpreted as an element of the closure of the N -soliton solutions to the Kaup–Broer system in the topology of uniform convergence in compact sets. If property (I) is not satisfied, then care needs to be taken in dealing with the singularity in defining such a quadrature rule. However, even when property (I) is not satisfied, the interpretation of the solution as an element of the closure of the N -solitons solutions is still valid.

The primitive solutions to the Kaup–Broer system determined by $\{\Gamma, R_1\}$ can also be computed via the solution to a Riemann–Hilbert problem. To see this, consider the function $\chi(\lambda) = [\chi(\lambda), \chi(\lambda^{-1})]$ and the contour

$$\tilde{\Gamma} = \{s \in \Gamma : R_1(s) \neq 0 \text{ or } R_1(s^{-1}) \neq 0\}. \quad (106)$$

The contour $\tilde{\Gamma}$ is a subset of \mathbb{R} consisting of Γ with a finite number of intervals removed by the assumption that R_1 vanishes only on a finite number of intervals in Γ . Then χ solves the following Riemann–Hilbert problem:

Riemann–Hilbert Problem 10. For all x, t find a 1×2 vector valued function $\chi(\lambda; x, t)$ such that

1. χ is a holomorphic function of $\lambda \in \mathbb{C} \setminus \tilde{\Gamma}$.
2. The boundary values

$$\chi_+(s; x, t) = \lim_{\epsilon \rightarrow 0^+} \chi(s + i\epsilon; x, t), \quad \chi_-(s; x, t) = \lim_{\epsilon \rightarrow 0^+} \chi(s - i\epsilon; x, t) \quad (107)$$

of χ for $s \in \tilde{\Gamma} \setminus \{\text{endpoints of } \tilde{\Gamma}\}$ from above or below are continuous.

3. The function χ has singularities that are less severe than poles on the endpoints of $\tilde{\Gamma}$.
4. The boundary values $\chi_{\pm}(\lambda; x, t)$ of $\chi(\lambda; x, t)$ from above and below for $\lambda \in \Gamma$ are related by

$$\chi_+(s; x, t) = \chi_-(s; x, t)V(s; x, y) \quad (108)$$

where

$$V(s; x, t) = \begin{pmatrix} \frac{1+R_1(s)R_1(s^{-1})}{1-R_1(s)R_1(s^{-1})} & \frac{2iR_1(s^{-1})}{1-R_1(s)R_1(s^{-1})}e^{2\tilde{\phi}(s;x,t)} \\ -\frac{2iR_1(s)}{1-R_1(s)R_1(s^{-1})}e^{-2\tilde{\phi}(s;x,t)} & \frac{1+R_1(s)R_1(s^{-1})}{1-R_1(s)R_1(s^{-1})} \end{pmatrix}. \quad (109)$$

5. The function χ has the limiting behaviors $\chi_1(\lambda) \rightarrow 1$ as $\lambda \rightarrow \infty$ and $\chi_2(\lambda) \rightarrow 1$ as $\lambda \rightarrow 0$.
6. χ satisfies the symmetry

$$\chi(s^{-1}; x, t) = \chi(s; x, t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (110)$$

Proposition 11. A solution to the Riemann–Hilbert problem has Taylor expansions of the form

$$\chi(\lambda; x, t) = (\chi_0^\infty(x, t), \chi_0^0(x, t)) + (\chi_1^\infty(x, t), \chi_1^0(x, t))\lambda^{-1} + O(\lambda^{-2}), \quad \lambda \rightarrow \infty, \quad (111)$$

$$\chi(\lambda; x, t) = (\chi_0^0(x, t), \chi_0^\infty(x, t)) + (\chi_1^0(x, t), \chi_1^\infty(x, t))\lambda + O(\lambda^2), \quad \lambda \rightarrow 0, \quad (112)$$

where $\chi_0^\infty = 1$. We can construct primitive solutions to the Kaup–Broer system using the Taylor coefficients via

$$\varphi(x, t) = -\log(\chi_0^0(x, t)), \quad \eta(x, t) = -\frac{1}{2}\varphi_{xx}(x, t) - (\chi_0^0)_x(x, t) \quad (113)$$

by Corollary 7.

5. Finite gap primitive solutions

In this section we construct a family of algebro-geometric finite gap solutions to the Kaup–Broer system that can be constructed as primitive solutions. We thus end up with an effective way to compute sequences of N -soliton solutions converging to these algebro-geometric finite gap solutions. These are the algebro-geometric finite gap solutions that can be thought of as nonlinear superpositions of counter propagating solutions to the KdV equation. We work out these solutions in detail for the case $a = -1$ and $\sigma = 1$. This construction is analogous to the construction for the KdV equation appearing in [21].

We can consider the hyperelliptic projective curve Σ' defined by

$$w^2 = P_{4g}(\lambda), \quad P_{4g}(\lambda) = \prod_{n=1}^{2g_\ell} (\lambda - \eta_n^{-1})(\lambda - \eta_n) \prod_{m=1}^{2g_r} (\lambda + \xi_m)(\lambda + \xi_m^{-1}). \quad (114)$$

This curve has genus $g' = 2(g_\ell + g_r) - 1$. Consider the involution

$$\iota(\lambda, w) = (\lambda^{-1}, -w) \quad (115)$$

and define the curve $\Sigma = \Sigma' / \langle \iota \rangle$ where $\langle \iota \rangle$ is the group $\{id., \iota\}$ which acts on Σ . We put the branch cuts of $\sqrt{P_{4g}(\lambda)}$ on $[\eta_{2n-1}, \eta_{2n}]$, $[\eta_{2n}^{-1}, \eta_{2n-1}^{-1}]$ for $n = 1, \dots, g_\ell$ and $[-\xi_{2n}, -\xi_{2n-1}]$, $[-\xi_{2n-1}^{-1}, -\xi_{2n}^{-1}]$ for $n = 1, \dots, g_r$. One sheet of Σ' gives a coordinate $\lambda \in \mathbb{C} \setminus \tilde{\Gamma}$ where

$$\tilde{\Gamma} = \bigcup_{n=1}^{g_\ell} [\eta_{2n-1}, \eta_{2n}] \cup [\eta_{2n}^{-1}, \eta_{2n-1}^{-1}] \cup \bigcup_{m=1}^{g_r} [-\xi_{2m}, -\xi_{2m-1}] \cup [-\xi_{2m-1}^{-1}, -\xi_{2m}^{-1}], \quad (116)$$

which covers all of Σ except g circles. We will not need to make reference to the points on the interiors of the branch cuts; however the branch points will be important. If λ_0 is a branch point of $\sqrt{P_{4g}(\lambda)}$, then λ_0^{-1} is also a branch point of $\sqrt{P_{4g}(\lambda)}$. Moreover, λ_0 and λ_0^{-1} correspond to the same point on Σ , and we use the notation $\langle \lambda_0 \rangle$ to refer to this point. Since Σ' is a double covering of Σ it follows from the Euler characteristic that Σ has genus $g = g_\ell + g_r$.

We will now compute a basis of abelian differentials of the first kind on Σ as follows: Consider the basis

$$\alpha'_n = \frac{\lambda^{n-1} d\lambda}{\sqrt{P_{4g}(\lambda)}} \quad (117)$$

for $j = 1, 2, \dots, 2g - 1$ of Abelian differential of the second kind on Σ' . The involution ι acts on the above basis of abelian differential of the first kind by ι^* as

$$\iota^* \alpha'_n = \frac{\lambda^{2g-n-1} d\lambda}{\sqrt{\lambda^{4g} P_{4g}(\lambda^{-1})}} = \frac{\lambda^{2g-n-1} d\lambda}{\sqrt{P_{4g}(\lambda)}} = \alpha'_{2g-n} \quad (118)$$

where it is easy to verify that $\lambda^{4g} P_{4g}(\lambda^{-1})$ because each root of $P_{4g}(\lambda)$ is the multiplicative inverse of another root of $P_{4g}(\lambda)$.

The Abelian differentials of the first kind α_n on Σ that can be represented as

$$\alpha_n = \alpha'_n + \alpha'_{2g-n} = \frac{\lambda^{n-1} + \lambda^{2g-n-1}}{\sqrt{P_{4g}(\lambda)}} d\lambda \quad (119)$$

for $n = 1, 2, \dots, g$ form a basis. Let a_j and b_j for $j = 1, 2, \dots, g$ be a canonical homology basis for the first homology group $H_1(\Sigma)$ satisfying $a_i \circ b_j = \delta_{ij}$, $a_i \circ a_j = 0$ and $b_i \circ b_j = 0$. We can then form a basis of normalized abelian differentials of the first kind ω_n satisfying

$$\int_{a_j} \omega_i = 2\pi i \delta_{ij} \quad (120)$$

as linear combinations of α_n .

The normalized basis of Abelian differentials of the first kind allows us to define the Able map **A** with entries

$$A_n(\lambda) = \int_{\infty}^{\lambda} \omega_n \quad (121)$$

mapping Σ into the Jacobi variety. The Riemann matrix for Σ is

$$B_{ij} = \int_{b_j} \omega_i. \quad (122)$$

The vector of Riemann constants **K** has entries

$$K_j = \frac{2\pi i + B_{jj}}{2} - \frac{1}{2\pi i} \sum_{\ell \neq j} \int_{a_\ell} A_j(\lambda) \omega_\ell. \quad (123)$$

We can also define the Abelian differentials of the second kind $\omega^{(n)}$ with asymptotic behavior $\omega^{(n)} = (n\lambda^{n-1} + O(1))d\lambda$ as $\lambda \rightarrow \infty$, asymptotic behavior $\omega^{(n)} = (n\lambda^{-n-1} + O(1))d\lambda$ as $\lambda \rightarrow 0$, and

$$\int_{a_j} \omega^{(n)} = 0. \quad (124)$$

These differentials are given by

$$\omega^{(n)} = \frac{n(\lambda^{2g+n-1} + \lambda^{-n-1})}{\sqrt{P_{4g}(\lambda)}} d\lambda + \sum_{j=1}^g c_j^{(n)} \omega_j \quad (125)$$

where

$$c_j^{(n)} = \frac{i}{2\pi} \int_{a_j} \frac{n(\lambda^{2g+n-1} + \lambda^{-n-1})}{\sqrt{P_{4g}(\lambda)}} d\lambda. \quad (126)$$

The integrals of these differentials satisfy

$$\int_{\infty}^{\lambda} \omega^{(n)} = \lambda^n + O(1), \text{ as } \lambda \rightarrow \infty; \quad \int_{\infty}^{\lambda} \omega^{(n)} = -\lambda^{-n} + O(1), \text{ as } \lambda \rightarrow 0. \quad (127)$$

An important aspect of $\omega^{(n)}$ is the vector $\Omega^{(n)}$ with entries

$$\Omega_j^{(n)} = \int_{b_j} \omega^{(n)}, \text{ for } j = 1, 2, \dots, g. \quad (128)$$

Let us pick a degree g divisor \mathfrak{P} as a direct sum of points

$$P_1 \in [\eta_1^{-1}, \eta_1], \quad (129)$$

$$P_j \in [\eta_{2j+1}^{-1}, \eta_{2j}^{-1}] \cup [\eta_{2j}, \eta_{2j+1}] \text{ for } j = 2, 3, \dots, g_\ell, \quad (130)$$

$$P_{g_\ell+1} \in [-\xi_1, -\xi_1^{-1}], \quad (131)$$

$$P_{g_\ell+j} \in [-\xi_{2j+1}, -\xi_{2j}] \cup [-\xi_{2j}^{-1}, -\xi_{2j+1}^{-1}] \text{ for } j = 2, 3, \dots, g_r \quad (132)$$

This divisor is nonspecial because all points are distinct [28,29].

The Baker–Akhiezer 2-point function for the Kaup–Broer system is the unique meromorphic function ψ on Σ such that ψ has simple poles on the points of \mathfrak{P} and asymptotic behavior

$$\psi(\lambda; x, t) = e^{\tilde{\phi}(\lambda; x, t)}(1 + O(\lambda^{-1})), \lambda \rightarrow \infty \text{ and } \psi(\lambda; x, t) = e^{\tilde{\phi}(\lambda; x, t)}O(1), \lambda \rightarrow 0. \quad (133)$$

The function $e^{-\tilde{\phi}(\lambda; x, t)}\psi(\lambda; x, t)$ is holomorphic at ∞ and 0 respectively, and therefore $\psi(\lambda; x, t)$ has the following uniformly convergent expansions:

$$\psi(\lambda; x, t) = e^{\tilde{\phi}(\lambda; x, t)} \left(1 + \sum_{n=1}^{\infty} \chi_n^\infty(x, t) \lambda^{-n} \right) \quad (134)$$

as $\lambda \rightarrow \infty$ and

$$\psi(\lambda; x, t) = e^{\tilde{\phi}(\lambda; x, t)} \sum_{n=0}^{\infty} \chi_n^0(x, t) \lambda^n \quad (135)$$

as $\lambda \rightarrow 0$.

Remark 12. A standard application of the Riemann–Roch theorem tells us that the space of functions satisfying all properties of the Baker–Akhiezer function (allowing the more general asymptotic behavior $\psi(\lambda; x, t) = e^{\tilde{\phi}(\lambda; x, t)}O(1)$, $\lambda \rightarrow \infty$) is one dimensional [23].

Therefore the Baker–Akhiezer 2-point function is unique. In fact, the Baker–Akhiezer function has the explicit formula

$$\psi(\lambda; x, t) = \exp \left(\frac{1}{2} \int_{\infty}^{\lambda} \omega^{(1)} x + \frac{1}{4} \int_{\infty}^{\lambda} \omega^{(2)} t \right) \frac{\theta(\mathbf{A}(\lambda) + \Omega^{(1)} x + \Omega^{(2)} t - \mathbf{A}(\mathfrak{P}) - \mathbf{K}; B) \theta(-\mathbf{A}(\mathfrak{P}) - \mathbf{K}; B)}{\theta(\mathbf{A}(\lambda) - \mathbf{A}(\mathfrak{P}) - \mathbf{K}; B) \theta(\Omega^{(1)} x + \Omega^{(2)} t - \mathbf{A}(\mathfrak{P}) - \mathbf{K}; B)}, \quad (136)$$

where $\theta(\mathbf{z}; B)$ is the Riemann theta function

$$\theta(\mathbf{z}, B) = \sum_{\mathbf{n} \in \mathbb{Z}^g} \exp \left(\frac{1}{2} \mathbf{n} \cdot B \mathbf{n} + \mathbf{n} \cdot \mathbf{z} \right). \quad (137)$$

Remark 13. This representation of the Kaup–Broer system spectral curve is different than that of Matveev–Yavor [11]. The spectral curve Σ is isomorphic to a spectral curve of the form considered by Matveev–Yavor. Let us consider the projective curve Σ_{MY} defined by

$$r^2 = P_{2g+2}(k), \quad P_{2g+2}(k) = (k^2 - 1) \prod_{n=1}^{g_\ell} (k - \eta_n) \prod_{n=1}^{g_r} (k + \xi_n) = P_{2g+2}(k). \quad (138)$$

The mapping

$$h(\lambda) = (k(\lambda), r(\lambda)), \quad k(\lambda) = \frac{\lambda + \lambda^{-1}}{2}, \quad r = \text{sgn} \circ \log(|\lambda|) \sqrt{P_{2g+2}(k(\lambda))} \quad (139)$$

maps Σ bijectively onto Σ_{MY} . Therefore, Σ_{MY} and Σ are isomorphic. If π is the projection $\pi(k, r) = k$ then $\pi(h(\mu_n)) \in [\eta_{2n-1}, \eta_{2n}]$ and $\pi(h(\mu_{g_\ell+n})) \in [-\xi_{2n}, -\xi_{2n-1}]$. Therefore, our spectral data satisfied the reality and regularity conditions determined by Smirnov [30].

Recall that \tilde{L} and \tilde{M} were defined in Remark 8. We set the potentials in \tilde{L} and \tilde{M} according to (59) except we use coefficients from (134), (135) and the u and v derivatives reduce to x derivatives. The functions $e^{-\tilde{\phi}(\lambda; x, t)} \tilde{L}\psi(\lambda)$ and $e^{-\tilde{\phi}(\lambda; x, t)} \tilde{M}\psi(\lambda)$ are regular at 0 and satisfy the asymptotic conditions $e^{-\tilde{\phi}(\lambda; x, t)} \tilde{L}\psi(\lambda) \rightarrow 0$ and $e^{-\tilde{\phi}(\lambda; x, t)} \tilde{M}\psi(\lambda) \rightarrow 0$, $\lambda \rightarrow \infty$. Moreover, it can be verified that $\tilde{L}\psi(\lambda)$ and $\tilde{M}\psi(\lambda)$ satisfy all the properties of the Baker–Akhiezer function except that we allow the more general asymptotic behavior $\psi(\lambda; x, t) = e^{\tilde{\phi}(\lambda; x, t)}O(1)$, $\lambda \rightarrow \infty$. This means $\tilde{L}\psi = 0$ and $\tilde{M}\psi = 0$ by Remark 12. Remark 8 then leads to the following proposition:

Proposition 14. We can construct solutions to the Kaup–Broer system using asymptotic expansion (134), (135) by taking

$$\varphi(x, t) = -\log(\chi_0^0(x, t)), \quad \eta(x, t) = -\frac{1}{2} \varphi_{xx}(x, t) - (\chi_0^0)_x(x, t). \quad (140)$$

This argument is analogous to the corresponding argument justifying the $\bar{\partial}$ dressing method.

Remark 15. The g values η_n, ξ_n determining Σ and the degree g divisor \mathfrak{P} on Σ constitute the spectral data for the finite gap solutions. The positive spectral data corresponds to a left moving finite gap KdV scattering problem, and the negative spectral data corresponds to a right moving finite gap KdV scattering problem.

Let us introduce an auxiliary Baker–Akhiezer function $\psi^{(aux)}(\lambda)$ set by the zero divisor \mathfrak{P} and pole divisor \mathfrak{P}_0 consisting of a direct sum of the points $\langle \eta_{2j-1} \rangle$ and $\langle -\xi_{2j-1} \rangle$. This function is given by

$$\psi(\lambda; x, t) = \exp \left(\sum_{n=1}^g \int_{\infty}^{\lambda} \omega^{(n)} t_n \right) \frac{\theta(\mathbf{A}(\lambda) - \mathbf{A}(\mathfrak{P}_0) - \mathbf{K}; B) \theta(-\mathbf{A}(\mathfrak{P}) - \mathbf{K}; B)}{\theta(\mathbf{A}(\lambda) - \mathbf{A}(\mathfrak{P}) - \mathbf{K}; B) \theta(-\mathbf{A}(\mathfrak{P}_0) - \mathbf{K}; B)} \quad (141)$$

where the coefficient t_j are determined by solving

$$\Omega \mathbf{t} \equiv \mathbf{A}(\mathfrak{P}) - \mathbf{A}(\mathfrak{P}_0) \quad (142)$$

where Ω is the matrix with entries $\Omega_{jn} = \Omega_j^{(n)}$, and \equiv is equivalence within the Jacobi variety.

This construction is analogous to the construction used by Trogdon and Deconinck in studying the KdV equation. The construction of Trogdon and Deconinck was used in [29] to compute finite gap primitive solutions to the KdV equation.

Proposition 16. *The matrix Ω is invertible and the solution \mathbf{t} has real entries t_n .*

Proof.

The proof of this theorem is similar to the analogous proof for the KdV equation given in [29]. The proof of invertibility is based off arguments presented in [31].

Let $\epsilon < \min\{\eta_1, \xi_1, \eta_{2g_\ell}^{-1}, \xi_{2g_\ell}^{-1}\}$. Let C_0 be the circle of radius ϵ centered at 0 oriented counterclockwise, and let C_∞ be the circle of radius ϵ^{-1} centered at 0 oriented clockwise. Let D_0 be the disc with boundary C_0 containing 0, and let D_∞ be the disc with boundary C_∞ containing ∞ . Inversion maps C_0 into C_∞ and vice versa. Let us choose the a_j, b_j so that they avoid D_0 and D_∞ , and let λ_0 be a point on Σ that is in neither D_0 nor D_∞ and avoid a_j, b_j . Using Stokes theorem and the fact primitives of abelian differentials are single valued on $\Sigma \setminus \bigcup_j a_j \cup b_j$, we can compute [31]

$$0 = \iint_{\Sigma} \alpha_m \wedge \omega^{(n)} = \sum_{j=1}^g \int_{a_j} \alpha_m \int_{b_j} \omega^{(n)} - \int_{C_0} \left(\int_{\lambda_0}^{\lambda} \alpha_m \right) \omega^{(n)} - \int_{C_\infty} \left(\int_{\lambda_0}^{\lambda} \alpha_m \right) \omega^{(n)}, \quad (143)$$

or equivalently

$$\Omega_{jn} = \sum_{j=1}^g A_{jm}^{-1} D_{mn} \quad (144)$$

where

$$A_{mj} = \int_{a_j} \alpha_m, \quad D_{mn} = \int_{C_0} \left(\int_{\lambda_0}^{\lambda} \alpha_m \right) \omega^{(n)} + \int_{C_\infty} \left(\int_{\lambda_0}^{\lambda} \alpha_m \right) \omega^{(n)}. \quad (145)$$

It is well known that a matrix of a-periods of any basis of abelian differentials of the first kind on a Riemann surface is invertible; therefore if D_{mn} is invertible then Ω_{jn} is invertible.

The first integral appearing in D_{mn} can be computed using the calculus of residues as

$$\int_{\tilde{C}_0} \left(\int_{\lambda_0}^{\lambda} \alpha_m \right) \omega^{(n)} = \int_{C_0} \int_{\lambda_0}^{\lambda} \frac{\lambda_1^{m-1} + \lambda_1^{2g-m-1}}{\sqrt{P_{4g}(\lambda_1)}} d\lambda_1 \frac{n\lambda^{-n-1}}{\sqrt{P_{4g}(\lambda)}} d\lambda \quad (146)$$

$$= \frac{2\pi i}{(n-1)!} \frac{d^{n-1}}{d\lambda^{n-1}} \frac{\lambda^{m-1} + \lambda^{2g-m-1}}{\sqrt{P_{4g}(\lambda)}} \Big|_{\lambda=0} \quad (147)$$

where we only need to keep the singular part of $\omega^{(n)}$ because the integral involving the regular part is 0. We have made use of the fact that $P_{4g}(0) = 1$. The second integral appearing in D_{mn} is seen to be equal to the first because it is represented in the z -plane as

$$\int_{C_\infty} \left(\int_{\lambda_0}^{\lambda} \alpha_m \right) \omega^{(n)} = - \int_{C_0} \int_{P_0}^{z^{-1}} \frac{\lambda_1^{n-1} + \lambda_1^{2g-n-1}}{\sqrt{P_{4g}(\lambda_1)}} d\lambda_1 \frac{nz^{-n-1}}{\sqrt{P_{4g}(z)}} dz \quad (148)$$

$$= \int_{C_0} \int_{P_0}^z \frac{z_1^{n-1} + z_1^{2g-n-1}}{\sqrt{P_{4g}(z_1)}} dz_1 \frac{nz^{-n-1}}{\sqrt{P_{4g}(z)}} dz. \quad (149)$$

Therefore,

$$D_{mn} = \frac{4\pi i}{(n-1)!} \frac{d^{n-1}}{d\lambda^{n-1}} \frac{\lambda^{m-1} + \lambda^{2g-m-1}}{\sqrt{P_{4g}(\lambda)}} \Big|_{\lambda=0}. \quad (150)$$

We see that $D_{mn} = 0$ for $n < m$, $D_{nn} = 4\pi i/(n-1)!$ for $n = 1, 2, \dots, g-1$ and $D_{gg} = 8\pi i/(g-1)$. The matrix D_{mn} is invertible because it is triangular with nonzero entries on the diagonal.

We will prove that t_n are real by choosing a canonical homology basis a_j and b_j so that Ω is real, and choosing contours for the Abel map so that $\mathbf{A}(\mathfrak{P}) - \mathbf{A}(\mathfrak{P}_0)$ is real.

Let b_1 be the contour traversing $[\eta_1^{-1}, \eta_1]$ oriented from right to left. Let b_n for $n = 2, 3, \dots, g_\ell$ be the contour traversing $[\eta_{2n-2}, \eta_{2n}]$ and then traversing $[\eta_{2n}^{-1}, \eta_{2n-2}^{-1}]$ oriented from right to left (while this is disjoint in the λ plane, it is a single connected loop on the Riemann surface Σ). Let $b_{g_\ell+1}$ be the contours traversing $[-\xi_1, -\xi_1^{-1}]$ oriented from right to left. Let $b_{g_\ell+n}$ for $n = 2, 3, \dots, g_r$ be the contour traversing $[-\xi_{2n-1}, -\xi_{2n-2}]$ and then traversing $[-\xi_{2n-2}^{-1}, -\xi_{2n-1}^{-1}]$ oriented from right to left. Let a_n for $n = 1, 2, \dots, g_\ell$ be the contour looping once $[\eta_{2g_\ell}^{-1}, \eta_{2n-1}^{-1}]$ counterclockwise that can be deformed down to this interval. Let $a_{g_\ell+n}$ for $n = 1, 2, \dots, g_r$ be the contour looping once around $[-\xi_{2g_r}, -\xi_{2n-1}]$ counter clockwise that can be deformed down to this interval.

The integral of α_j around a_i can be written as integrals on the bottom and top of branch cuts of $\sqrt{P_{4g}(\lambda)}$ on which $\sqrt{P_{4g}(\lambda)}$ is purely imaginary. This means that $\int_{a_i} \alpha_j$ are purely imaginary because α_j are purely imaginary on the top and bottom of the branch cuts.

Therefore, the coefficients for expanding the normalized differentials ω_j in terms of α_n are purely real. The function $\sqrt{P_{4g}(\lambda)}$ is purely real on b_i so $\omega^{(n)}$ is purely real on b_i . Therefore the entries $\int_{b_i} \omega^{(n)}$ of Ω are purely real.

We can write

$$A_j(\mathfrak{P}) - A_j(\mathfrak{P}_0) = \sum_{n=1}^{g_\ell} \int_{\langle \eta_{2n-1} \rangle}^{P_n} \omega_j + \sum_{m=1}^{g_r} \int_{\langle -\xi_{2m-1} \rangle}^{P_{g_\ell+m}} \omega_j. \quad (151)$$

We can choose the contour from $\langle \eta_{2n-1} \rangle$ to P_n so that it stays in $[\eta_1^{-1}, \eta_1]$ for $n = 1$ or $[\eta_{2n-1}^{-1}, \eta_{2n-2}^{-1}] \cup [\eta_{2n-2}, \eta_{2n-1}]$ for $n = 2, 3, \dots, g_\ell$. We can choose the contour from $\langle -\xi_{2m-1} \rangle$ to $P_{g_\ell+m}$ so that it stays in $[-\xi_1, -\xi_1^{-1}]$ for $m = 1$ or $[-\xi_{2m-1}, -\xi_{2m-2}^{-1}] \cup [-\xi_{2m-2}, -\xi_{2m-1}^{-1}]$ for $m = 2, 3, \dots, g_r$. The differentials α_j are real on these intervals which means ω_j are real on these intervals too. Therefore, $\mathbf{A}(\mathfrak{P}) - \mathbf{A}(\mathfrak{P}_0)$ is real for this choice of contours. \square

We define the function

$$\chi(\lambda; x, t) = \xi(\lambda) e^{-\tilde{\phi}(\lambda; x, t) - \phi^{(aux)}(\lambda)} \psi^{(aux)}(\lambda; x, t) \psi(\lambda; x, t) \quad (152)$$

where

$$\xi(\lambda) = \prod_{n=1}^{g_\ell} \frac{\sqrt[4]{\lambda - \eta_{2n-1}} \sqrt[4]{\lambda - \eta_{2n-1}^{-1}}}{\sqrt[4]{\lambda - \eta_{2n}} \sqrt[4]{\lambda - \eta_{2n}^{-1}}} \prod_{m=1}^{g_r} \frac{\sqrt[4]{\lambda + \xi_{2m-1}} \sqrt[4]{\lambda + \xi_{2m-1}^{-1}}}{\sqrt[4]{\lambda + \xi_{2m}} \sqrt[4]{\lambda + \xi_{2m}^{-1}}} \quad (153)$$

where we choose ξ to have branch cuts on \tilde{r} and asymptotic behaviors $\xi = 1 + O(\lambda^{-1})$ as $\lambda \rightarrow \infty$.

If I is an interval then $\mathbb{1}_I(s)$ is the indicator function of the interval I .

Theorem 17. The vector valued function $\chi(\lambda; x, t) = [\chi(\lambda; x, t), \chi(\lambda^{-1}; x, t)]$ solves [Riemann–Hilbert Problem 10](#) with $R_1(s) = e_1(s)f_1(s)$ where

$$e_1(s) = \exp \left(-2 \sum_{n=1}^g t_n (s^n - s^{-n}) \right) \quad (154)$$

$$f_1(s) = \sum_{n=1}^{g_\ell} \left(\mathbb{1}_{[\eta_{2n-1}, \eta_{2n}]}(s) - \mathbb{1}_{[\eta_{2n-1}^{-1}, \eta_{2n-1}^{-1}]}(s) \right) + \sum_{m=1}^{g_r} \left(\mathbb{1}_{[-\xi_{2m-1}^{-1}, -\xi_{2m-1}^{-1}]}(s) - \mathbb{1}_{[-\xi_{2m}, -\xi_{2m-1}]}(s) \right). \quad (155)$$

These finite gap solutions can alternatively be computed from $R_1(s)$ by solving (103) for $f(s, x, t)$ and then computing a solution to the Kaup–Broer system from $f(s, x, t)$ using (105).

Proof. The proof of this theorem is a straight-forward verification of the properties of the Riemann–Hilbert problem. The steps differ only minimally from the proof of the analogous result for the KdV equation appearing in [21]. We will verify conditions (3) and (4). The proofs of the other conditions are straight forward, and we leave their verification to the reader.

We define a singularity that locally behaves like z^{-p} to have order p . The poles of $\tilde{\psi}(\lambda)$ on Σ at $\langle \eta_{2n-1} \rangle$ and $\langle -\xi_{2m-1} \rangle$ manifest themselves in the λ plane as order 1/2 singularities at $\eta_{2n-1}, \eta_{2n-1}^{-1}$ and $-\xi_{2m-1}, -\xi_{2m-1}^{-1}$. This means that χ has order 1/4 singularities on the endpoints of \tilde{r} . These are less severe than poles verifying condition (3).

Let us write $\tilde{\psi}(s) = \psi^{(aux)}(s) \psi(s; x, t)$, which is a meromorphic function on $\Sigma \setminus \{0, \infty\}$. Its representation in the λ plane therefore satisfy the nonlocal jump relations

$$\tilde{\psi}_+(s) = \tilde{\psi}_+(s^{-1}), \quad \tilde{\psi}_-(s) = \tilde{\psi}_-(s^{-1}) \quad (156)$$

for $s \in \tilde{r}$. The function ξ satisfies the nonlocal jump relations

$$\xi_+(s) = -i\xi_+(s^{-1}), \quad \xi_-(s) = -i\xi_-(s^{-1}) \quad (157)$$

for $s \in [-\xi_{2m-1}^{-1}, -\xi_{2m-1}^{-1}]$ or $s \in [\eta_{2n-1}, \eta_{2n}]$. Let $\tilde{e}(s) = e^{-\tilde{\phi}(s) - \phi^{(aux)}(s)}$, then $\tilde{e}(s)$ satisfies $\tilde{e}(s) = e^{-2\tilde{\phi}(s) - 2\phi^{(aux)}(s)} \tilde{e}(s^{-1})$ for all s , and in particular for $s \in \tilde{r}$. Putting these nonlocal jump relations together gives the nonlocal jump conditions

$$\chi_+(s; x, t) = -ie^{-2\tilde{\phi}(s; x, t)} R_1(s) \chi_+(s^{-1}; x, t), \quad \chi_-(s^{-1}; x, t) = ie^{2\tilde{\phi}(s; x, t)} R_1(s^{-1}) \chi_-(s; x, t) \quad (158)$$

for $s \in \tilde{r}$, where we use $R_1(s) = e_1(s)f_1(s)$. Using $\chi_+(s; x, t) = [\chi_+(s; x, t), \chi_-(s^{-1}; x, t)]$ and $\chi_-(s) = [\chi_-(s; x, t), \chi_+(s^{-1}; x, t)]$ we see that $\chi_+(s; x, t) = \chi_-(s; x, t)V(s; x, t)$ where

$$V(s; x, t) = \begin{pmatrix} 0 & iR_1(s)e^{2\tilde{\phi}(s; x, t)} \\ -iR_1(s)e^{-2\tilde{\phi}(s; x, t)} & 0 \end{pmatrix}. \quad (159)$$

Using the fact that $R_1(s^{-1}) = -R_1(s)^{-1}$ for the choice of dressing function $R_1(s) = e_1(s)f_1(s)$, we see that the jump matrix $V(s; x, t)$ is equal to the jump matrix given by (109). \square

6. Numerical primitive solutions

In this section we compute numerical primitive solutions to the Kaup–Broer system determined by the continuum spectral data with $\gamma_1 = 9/8, \gamma_2 = 2$ using various functions $R_0(s)$ satisfying property (I). These particular solutions appear to describe counter propagating

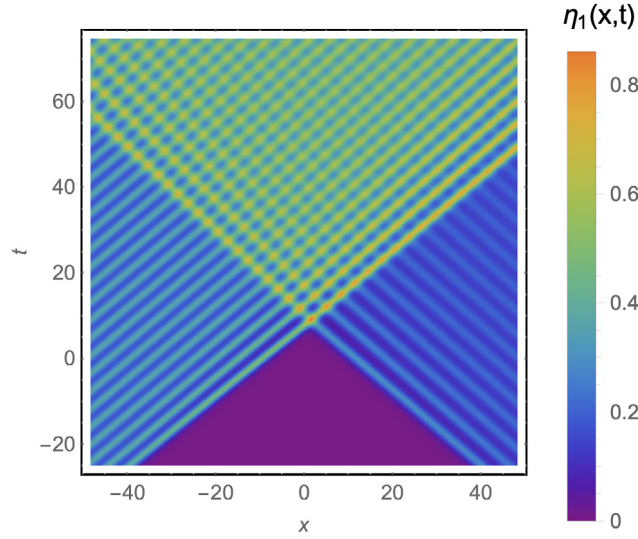


Fig. 1. A spacetime plot for the field η_1 for the 100-soliton case 1 solution to the Kaup–Broer system.

dispersive shockwaves with spacetime regions that appear to be described by finite gap solutions with various numbers of periods. We believe that these solutions are locally described by the finite-gap solutions discussed in the previous section. In producing solutions to the Kaup–Broer system in this section we will make use of the invariance of the Kaup–Broer system with respect to the time reversal symmetry $\eta(x, t) \rightarrow \eta(x, -t)$, $\varphi(x, t) \rightarrow -\varphi(x, -t)$.

6.1. Case 1:

The case of

$$R_1(s) = \pi e^{-10} \left(-\mathbb{1}_{\left[-\frac{7}{4}, -\frac{5}{4}\right]}(s) + \mathbb{1}_{\left[\frac{3}{2}, 2\right]}(s) \right). \quad (160)$$

When numerically solving (103) and numerically computing the integrals in (105) we use Gauss–Legendre quadrature with 50 points on each interval $[-7/4, -5/4]$ and $[3/2, 2]$. The solutions were computed on a uniform spacetime grid, and the x derivatives appearing in (105) were computed spectrally with fast Fourier transforms. In particular, we considered a continuous periodic extension of the functions being differentiated with a period twice the spatial interval, and we applied an order 3 Butterworth filter to remove artifacts. The matrix approximation of the integral equation (103) has a large condition number in this case, and was solved in Mathematica with 90 digits of precision. From the solutions η, φ produced by the numerical approximation to (105), we define the time reversed solution $\eta_1(x, t) = \eta(x, -t)$, $\varphi_1(x, t) = -\varphi(x, -t)$ to the Kaup–Broer system. While this is a numerical approximation to (105), it can also be interpreted as a 100-soliton solution (96).

Remark 18. Using Gaussian quadrature to discretize the integral equation is likely not the optimal way to solve the integral equation (103) because of the large condition numbers described in the previous paragraph. We choose to use Gaussian quadrature because it leads to an intuitive picture in which the solutions computed are N -soliton solutions. That said, it may be possible to use a g function and contour deformations to regularize the corresponding Riemann–Hilbert problem. We have so far been unable to successfully formulate this procedure explicitly, however doing so would lead to a more practical numerical method for computing primitive solutions than the method used in the present paper.

Spacetime plots for the solutions η_1 and φ_1 are provided in Figs. 1 and 2, and spatial plots of these solutions are provided in Figs. 3 and 4. At early times we see that the η_1 component of this solution approximates counter propagating dispersive shockwaves with 1-period trailing waves. These dispersive shockwaves collide, and when the 1-period trailing waves begin to interact we see what we believe to be described by a 2-period solution to the Kaup–Broer system.

6.2. Case 2:

We now consider the case of

$$R_1(s) = \pi e^{-10} \left(-\mathbb{1}_{\left[-\frac{15}{8}, -\frac{13}{8}\right]}(s) - \mathbb{1}_{\left[-\frac{6}{4}, -\frac{5}{4}\right]}(s) + \mathbb{1}_{\left[\frac{11}{8}, \frac{13}{8}\right]}(s) + \mathbb{1}_{\left[\frac{7}{4}, 2\right]}(s) \right). \quad (161)$$

When numerically solving (103) and numerically computing the integrals in (105) we use Gauss–Legendre quadrature with 25 points on each of the intervals $[-15/8, -13/8]$, $[-6/4, -5/4]$, $[11/8, 13/8]$ and $[7/4, 2]$. The solutions were computed on a uniform spacetime grid, and the x derivatives appearing in (105) were computed spectrally with fast Fourier transforms. In particular, we extended the functions and an order 3 Butterworth filter was applied. The matrix approximation of the integral equation (103) has a large condition number in this case, and was solved in Mathematica with 90 digits of precision. From the solutions η, φ produced by the numerical

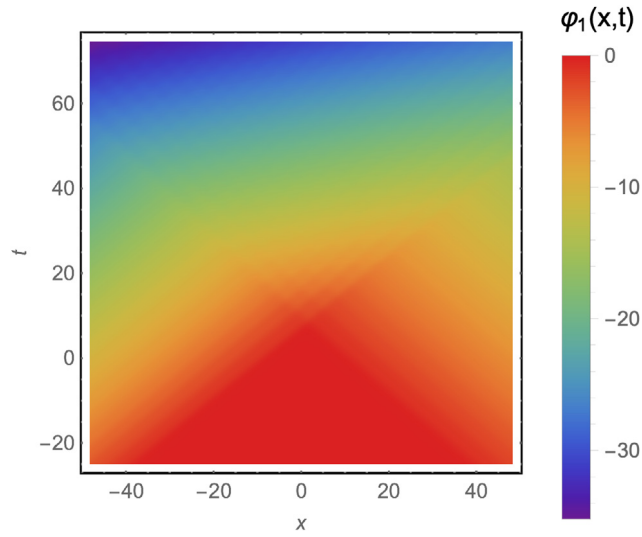


Fig. 2. A spacetime plot for the field φ_1 for the case 1 solution to the Kaup–Broer system.

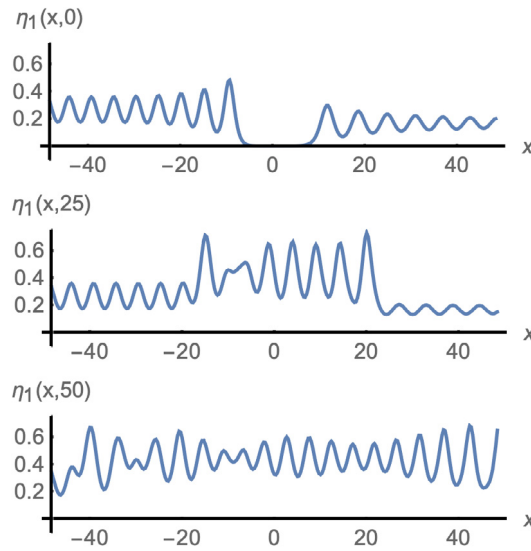


Fig. 3. Spatial plots for the field η_1 for the case 1 solution to the Kaup–Broer systems plotted at times $t = 0$, $t = 25$ and $t = 50$.

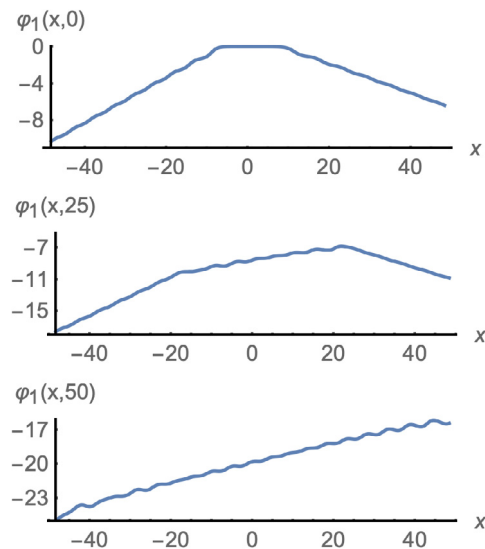


Fig. 4. Spatial plots for the field φ_1 for the case 1 solution to the Kaup–Broer systems plotted at times $t = 0$, $t = 25$ and $t = 50$.

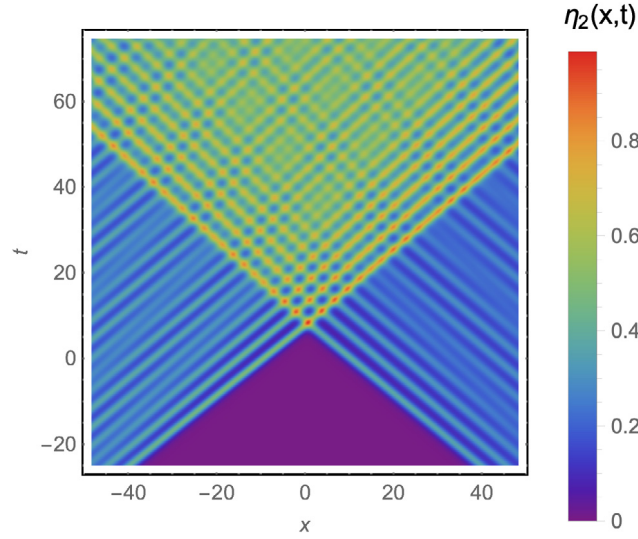


Fig. 5. A spacetime plot for the field η_2 for the case 2 solution to the Kaup–Broer system.

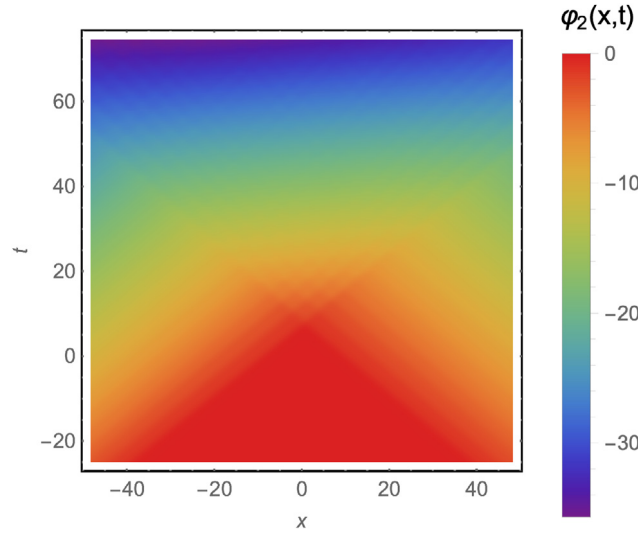


Fig. 6. A spacetime plot for the field φ_2 for the case 2 solution to the Kaup–Broer system.

approximation to (105), we define the time reversed solution $\eta_2(x, t) = \eta(x, -t)$, $\varphi_2(x, t) = -\varphi(x, -t)$. The solution η_2, φ_2 can be interpreted as a 100-soliton solution to the Kaup–Broer system.

Spacetime plots for the solutions η_2 and φ_2 are provided in Figs. 5 and 6, and spatial plots of these solutions are provided in Figs. 7 and 8. At early times we see that the η_2 component of this solution approximates counter propagating dispersive shockwaves with trailing waves that we believe to be described by 2-period solutions to the Kaup–Broer system. When these solutions collide, we initially see what appears to be a region where counter propagating modulated 1-period solutions are interacting. We then see regions with modulated 1-period solutions interacting with the 2-period trailing waves in a region where we believe the solutions can be described by modulated 3-period solutions to the Kaup–Broer system. There also appear to be spacetime regions where the 2-period trailing waves are interacting in what we believe to be described by a 4-period solution to the Kaup–Broer system.

6.3. Case 3:

The case of

$$R_1(s) = \pi e^{-10} \left(-\mathbb{1}_{[-2, -15/8]}(s) - \mathbb{1}_{[-7/4, -13/8]}(s) - \mathbb{1}_{[-11/8, -5/4]}(s) + \mathbb{1}_{[11/8, 13/8]}(s) + \mathbb{1}_{[7/4, 2]}(s) \right). \quad (162)$$

When numerically solving (103) and numerically computing the integrals in (105) we use Gauss–Legendre quadrature with 30 points on each of the intervals $[-2, -15/8]$, $[-7/4, -13/8]$, $[-11/8, -5/4]$, $[11/8, 13/8]$ and $[7/4, 2]$. The solutions were computed on a uniform spacetime grid, and the x derivatives appearing in (105) were computed spectrally with fast Fourier transforms. The matrix approximation of the integral equation (103) has a large condition number in this case, and was solved in Mathematica with 100

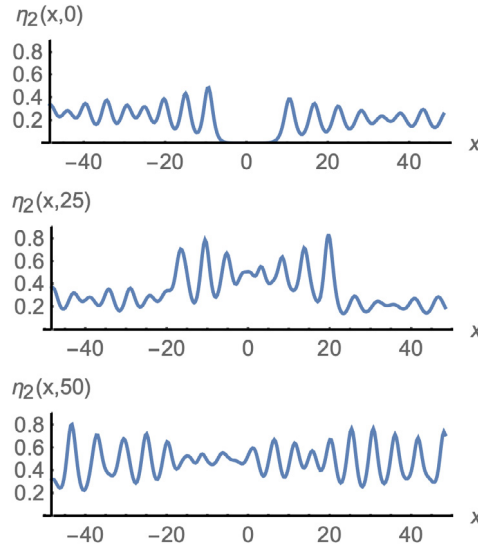


Fig. 7. Spatial plots for the field η_2 for the case 2 solution to the Kaup–Broer systems plotted at times $t = 0$, $t = 25$ and $t = 50$.

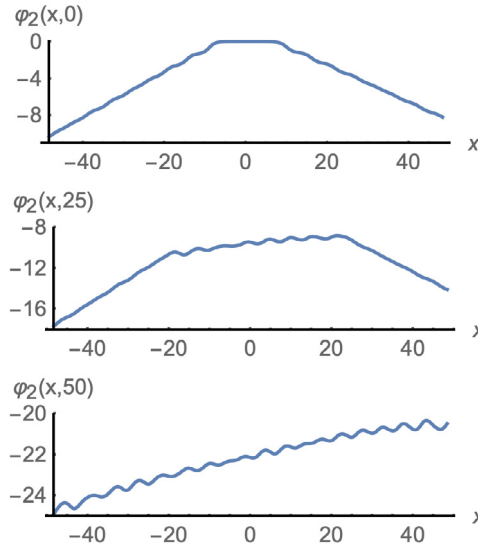


Fig. 8. Spatial plots for the field φ_2 for the case 2 solution to the Kaup–Broer systems plotted at times $t = 0$, $t = 25$ and $t = 50$.

digits of precision. From the solutions η, φ produced by the numerical approximation to (105), we define the time reversed solution $\eta_3(x, t) = \eta(x, -t)$, $\varphi_3(x, t) = -\varphi(x, -t)$. The solution η_3, φ_3 can be interpreted as a 150-soliton solution to the Kaup–Broer system.

Spacetime plots for the solutions η_3 and φ_3 are provided in Figs. 9 and 10, and spatial plots of these solutions are provided in Figs. 11 and 12. This solution has a fairly complicated structure. We believe this solution has regions described by modulated g -period solutions to the Kaup–Broer with g ranging from 0 through 5 ($g = 0$ corresponding to constant solutions).

7. Conclusion

In this paper we derived the $\bar{\partial}$ dressing method for the 4 scaling classes of the Kaup–Broer system as dimension reductions of the complete complexification of a completely integrable (2+1) dimensional generalization of the Kaup–Broer system. We then applied the dressing method to compute examples of interesting solutions to the most common scaling of the Kaup–Broer system describing non-capillary waves with normal gravitation forcing. This method also provides a starting place for the computation of many other solutions. A logical next step is the computation of exact real solutions to other scaling classes of the Kaup–Broer system. We have also barely touched on exact solutions to the (2+1) dimensional generalization of the Kaup–Broer system. In particular, the solutions briefly discussed in Section 3.1 should be analyzed with more detail. In Section 6 we made some conjectures on the structure of some numerical solutions related to the finite gap g -period solutions of Matveev and Yavor [11]. We believe that an approach similar to that Girotti, Grava and McLaughlin [20] can be applied to the Riemann–Hilbert problem discussed in Section 4 of this paper. The isomorphism of the Kaup–Broer system with a complexified NLS equation [11] means that the construction of NLS primitive solutions should be similar. Finally, as mentioned in the introduction, this method should be well suited to analysis of soliton gasses consisting of many counter propagating solitons. These topics will be discussed in future papers.

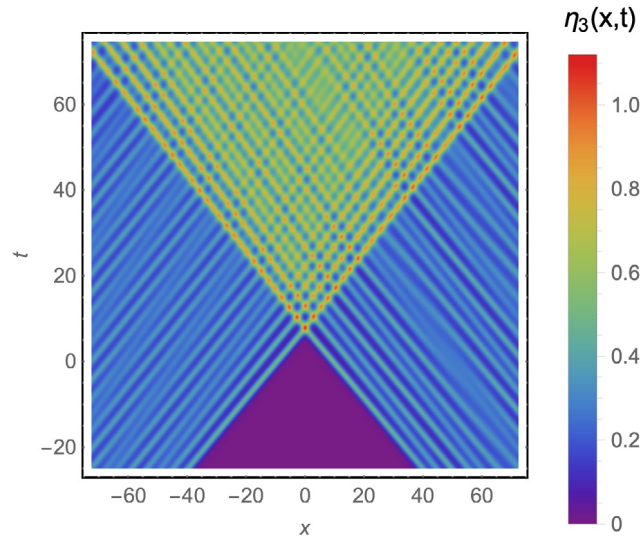


Fig. 9. A spacetime plot for the field η_3 for the case 3 solution to the Kaup–Broer system.

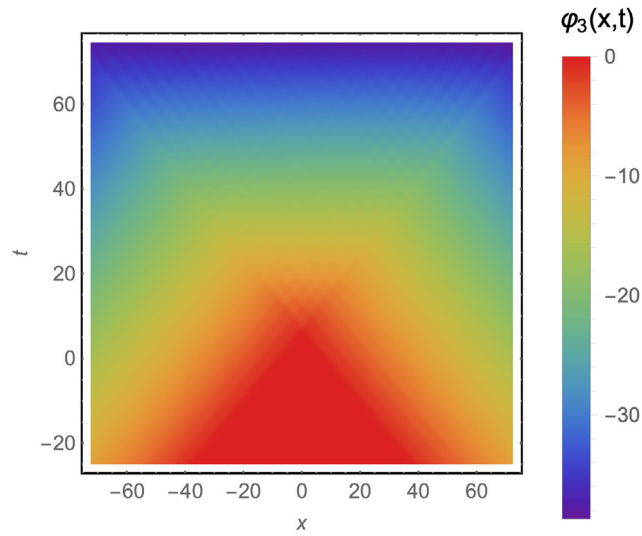


Fig. 10. A spacetime plot for the field φ_3 for the case 3 solution to the Kaup–Broer system.

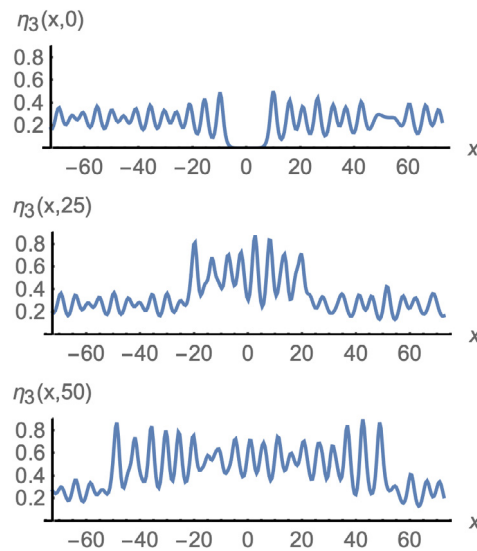


Fig. 11. Spatial plots for the field η_3 for the case 3 solution to the Kaup–Broer systems plotted at times $t=0$, $t=25$ and $t=50$.

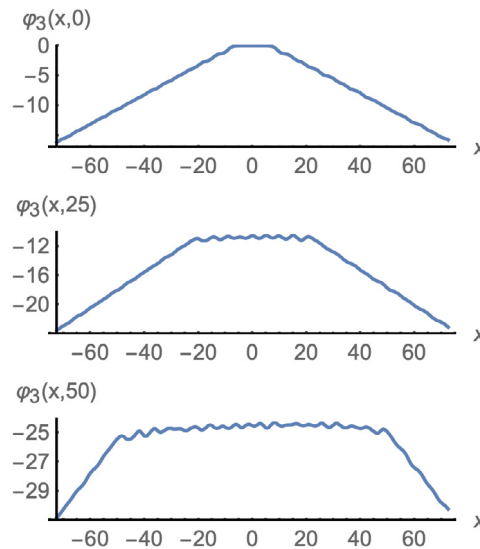


Fig. 12. Spatial plots for the field φ_3 for the case 3 solution to the Kaup–Broer systems plotted at times $t = 0$, $t = 25$ and $t = 50$.

CRediT authorship contribution statement

Patrik V. Nabelek: Conceptualization, Formal analysis, Software, Visualization, Writing - original draft. **Vladimir E. Zakharov:** Conceptualization, Formal analysis, Writing - review & editing, Funding acquisition.

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