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INTEGRATION OF A DEEP FLUID EQUATION WITH A FREE SURFACE

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We show that the Euler equations describing the unsteady potential flow of a two-dimensional deep fluid with a free surface in the absence of gravity and surface tension can be integrated exactly under a special choice of boundary conditions at infinity. We assume that the fluid surface at infinity is unperturbed, while the velocity increase is proportional to distance and inversely proportional to time. This means that the fluid is compressed according to a self-similar law. We consider perturbations of a self-similarly compressible fluid and show that their evolution can be accurately described analytically after a conformal map of the fluid surface to the lower half-plane and the introduction of two arbitrary functions analytic in this half-plane. If one of these functions is equal to zero, then the solution can be written explicitly. In the general case, the solution appears to be a rapidly converging series whose terms can be calculated using recurrence relations.

Keywords: integrability, conformal transformation, drop, bubble, singularity

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1. Problem statement

The results obtained in this paper are not new. They were repeatedly reported at scientific conferences, the last time at the Russian–French symposium on mathematical hydrodynamics held in August 2016 in Novosibirsk [1]. But these results are published for the first time. The obtained solution coincides with the solution recently obtained by Karabut and Zhuravleva [2] in a somewhat different way.

It is well known that the "shallow water" equations generate many integrable systems, in particular, the Korteweg–de Vries equation, the Boussinesq equation, the Kaup–Broer system of equations, and the nonlinear Schrödinger equation. The (2+1)-dimensional generalizations of these systems—the Kadomtsev–Petviashvili equation and the Davey–Stewartson equation—are also integrable. These results can be found in standard monographs devoted to the method of the inverse scattering problem (see, e.g., [3]). It was recently shown that a (2+1)-dimensional generalization of the Kaup–Broer model is also integrable [4]. In this connection, the question naturally arises about the integrability of the equations of a deep fluid.

The hypothesis of the integrability of the deep-fluid equations, moreover in the presence of gravity, was first formulated in [5], where small-amplitude waves were studied and it was established that in the leading order of the perturbation theory, waves traveling in one direction cannot generate waves propagating in the opposite direction. This result was recently strengthened [6]: it turns out that the total wave actions

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of waves propagating in different directions are integrals of motion. It is still unknown whether this result can be extended to waves of finite amplitude, but it turned out that there are completely different integrals of motion [7], [8] not yet physically interpreted. Their number is not fixed but depends on the initial conditions. Nevertheless, their existence is confirmed by detailed numerical experiments. There are also exact solutions describing capillary waves of arbitrary amplitude [9].

Here, we consider a deep fluid without gravity and surface tension. We show that if several artificial boundary conditions are imposed at infinity, then the Euler equation describing this fluid can be integrated exactly. Unfortunately, it is not yet possible to replace the "artificial" boundary conditions with "natural" ones. But the obtained results are undoubtedly interesting from the standpoint of mathematics. In addition, we can hope that they can be used to describe the behavior of small-scale perturbations against the background of large-scale ones, and this will already have significant practical benefits.

2. Basic equations

We consider the potential flow of a deep fluid in (1+1)-geometry. The fluid occupies the area

$$-\infty < x < \infty, \qquad -\infty < y < \eta(x,t), \qquad \operatorname{div}\mathbf{v} = 0. \tag{1}$$

The flow is potential, $\mathbf{v} = \nabla\Phi$, and the potential satisfies the Laplace equation $\Delta\Phi = 0$ and the Bernoulli equation

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2 + P = 0. \tag{2}$$

On the surface, the kinematic conditions

$$\frac{\partial\eta}{\partial t} + \frac{\partial\Phi}{\partial x}\frac{\partial\eta}{\partial x} = \frac{\partial\Phi}{\partial y}\bigg|_{y=\eta} \tag{3}$$

and dynamical conditions

$$P = 0, \qquad \frac{\partial\Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2 = 0\big|_{y=\eta} \tag{4}$$

hold. If the fluid surface is flat and stationary (such that $\eta = 0$), then the conditions give

$$\frac{\partial\Phi}{\partial y}\bigg|_{y=0} = 0, \qquad \frac{\partial\Phi}{\partial t} + \frac{1}{2}\left(\frac{\partial\Phi}{\partial x}\right)^2\bigg|_{y=0} = 0. \tag{5}$$

The only nontrivial solution of Bernoulli equation (2) satisfying conditions (5) has the form

$$\Phi = \frac{1}{2}\frac{(x-x_0)^2 - y^2}{t-t_0}, \qquad P = -\frac{y^2}{(t-t_0)^2}, \tag{6}$$

where x_0 and t_0 are arbitrary constants.

Formulas (6) give a unique nontrivial solution of the unsteady Bernoulli equation that does not disturb the free surface of the fluid. We set $x_0 = 0$ and $t_0 = 0$. Then

$$\Phi = \frac{1}{2}\frac{x^2 - y^2}{t}. \tag{7}$$

This solution is a special case of a more general class of self-similar solutions

$$\eta = t^\alpha F\left(\frac{x}{t^\alpha}\right), \qquad \Phi = t^{2\alpha-1}\Phi\left(\frac{x}{t^\alpha}, \frac{y}{t^\alpha}\right), \tag{8}$$

where α is an arbitrary real constant. In the considered case, $\alpha = 0$ and $F \equiv 0$. The choice $\alpha = 3$ leads to a solution of the Dirichlet class with a parabolic surface profile. If $\alpha = 2$, then self-similar substitution (8) is compatible with the equation in the presence of gravity.

We note that the equations in this section are invariant under the change $t \to -t$, $\Phi \to -\Phi$, $y \to -y$, and $\eta \to -\eta$. Therefore, it suffices to consider the case $t > 0$.

3. Conformal map

We conformally map the region occupied by the fluid on the physical plane $z = x + iy$ to the complex half-plane $w = u + iv$, $v < 0$, using a function $Z = Z(w, t)$ that is analytic in the lower half-plane. We assume that such functions have a given power-law asymptotic behavior in the lower half-plane as $v \to -\infty$. They are then completely determined by their values on the real axis, from which they can be continued to the entire complex plane.

Kinematic boundary condition (3) generates the equation

$$Z_t \overline{Z}_u - \overline{Z}_t Z_u + \Phi_u - \overline{\Phi}_u = 0. \tag{9}$$

It is convenient to write dynamical boundary condition (4) in the form

$$\Psi_t Z_u - \Psi_u Z_t + \frac{1}{2}\frac{\overline{\Phi}_u^2}{\overline{Z}_u} = 0, \tag{10}$$

where

$$\Psi = \frac{1}{2}(\Phi + \overline{\Phi}\,). \tag{11}$$

A detailed derivation of these equations can be found in [10], [8]. We note that the complex velocity V_z has the form

$$V = \Phi_z = \frac{\Phi_u}{Z_u}. \tag{12}$$

Equations (9) and (10) have the exact solution

$$Z = Z_0 = (t - t_0)u, \qquad \Phi = \Phi_0 = \frac{1}{2}(t - t_0)u^2. \tag{13}$$

In this case, $Z = \overline{Z}$ and $\Phi = \overline{\Phi} = \Psi$.

Now

$$V = \frac{x}{t - t_0}. \tag{14}$$

Therefore, this solution is a conformal image of the self-similar solution constructed above. We emphasize that this solution describes a contracting fluid for $t < t_0$ and an expanding fluid for $t > t_0$. In what follows, we consider only the case of an expandable fluid $t > t_0$. Solutions in the case of a compressible fluid can be obtained by replacing the variables described above.

At $t = t_0$, the solution has a singularity, and the pressure becomes infinite in the entire volume of the fluid. Therefore, the solution has a physical meaning only for $t \neq t_0$. Nevertheless, solution (13) can be regarded as the first terms of the Taylor expansion of a more general locally analytic solution of Eqs. (9) and (10).

We now consider the perturbation $Z = Z_0 + \widetilde{Z}(u)$, $\Phi = \Phi_0 + \widetilde{\Phi}(u)$ of solution (13). The perturbations $\widetilde{Z}(u)$ and $\widetilde{\Phi}(u)$ are analytic in the lower half-plane and decrease sufficiently rapidly as $u \to \infty$. Further, we simplify the notation and set $t - t_0 = \tau$ and $\partial/\partial t = \partial/\partial\tau$, $\tau > 0$.

Further, we assume that

$$Z \to u\tau + Z, \qquad \Phi \to \frac{1}{2}\,u\tau^2 + \Phi. \tag{15}$$

The corrections Z and Ψ satisfy the equations

$$\Phi_u - uZ_u + \tau Z_\tau + Z_\tau \overline{Z}_u - \overline{Z}_\tau Z_u = \overline{\Phi}_u + \tau\overline{Z}_\tau - u\overline{Z}_u, \tag{16}$$

$$\frac{1}{2}\tau\Phi_t - u\tau Z_t + \frac{1}{2}u^2 Z_u - \frac{1}{2}u\Phi_u + \Psi_t Z_u + Z_t\Psi_u = -\frac{\tau}{2}\,\overline{\Phi}_\tau + \frac{1}{2}u\overline{\Phi}_u + \frac{1}{2}\frac{(u\tau + \overline{\Phi}_u)^2}{\tau + \overline{Z}_u} - \frac{1}{2}u^2\tau. \tag{17}$$

On the lower half-plane, we introduce the projection operator $P^{-} = (1 + i\widehat{H})/2$, where $\widehat{H}$ is the Hilbert transform. We require that applying the operator P^{-} to Eqs. (16) and (17) lead to the right-hand sides vanishing. Equations (16) and (17) are then simplified:

$$\Phi_u - uZ_u + \tau Z_\tau = A, \tag{18}$$

$$\tau \frac{\partial}{\partial \tau}(\Phi - uZ) = uA - B, \tag{19}$$

where

$$A = P^{-}(\,\overline{Z}_\tau Z_u - Z_\tau \overline{Z}_u), \tag{20}$$

$$B = \Phi_\tau Z_u - \Phi_u Z_\tau + P^{-}(\,\overline{\Phi}_\tau Z_u - \overline{\Phi}_u Z_\tau). \tag{21}$$

The right-hand sides of Eqs. (16) and (17) vanish if we require that

$$P^{-}(u^2 Z_u) = 0, \tag{22}$$

$$P^{-}(u\Phi_u) = 0. \tag{23}$$

Equations (18) and (19) have a simple solution that we say is *elementary.* Let Z and Φ be independent of τ. Then $A = 0$, $B = 0$, and Eq. (19) is satisfied automatically. Equation (18) is satisfied if we set

$$Z = \alpha(u), \qquad \Phi = \partial^{-1} u\alpha_u(u), \tag{24}$$

where $\alpha(u)$ is an analytic function in the lower half-plane and satisfies condition (22). Condition (23) is then automatically satisfied.

We study the elementary solutions in the next section. In the meantime, we consider more general solutions. We assume that the general solution is represented by a series in inverse powers of τ:

$$Z = \sum_{n=0}^{\infty} \frac{Z_n}{\tau^n}, \qquad \Phi = \sum_{n=0}^{\infty} \frac{\Phi_n}{\tau^n}. \tag{25}$$

The nonlinear terms A and B can also be expanded in inverse powers of τ, but their expansion contains only lower-order terms,

$$Z_n = \frac{1}{n-1} A_n + \frac{1}{n(n-1)} \frac{\partial}{\partial u}(B_n - uA_n). \tag{26}$$

The term Φ_n is determined from the equation

$$\Phi_n = uZ_n - \frac{1}{n}(uA_n - B_n). \tag{27}$$

The nonlinear terms A_n and B_n include the coefficients Z_k and Φ_k for $k \le n-1$, and the expressions for Z_n and Φ_n are therefore recurrence relations. As $n \to \infty$, the expressions for Z_n and Φ_n arise as a result of division by large numbers. They therefore decrease exponentially as n increases, and series (25) hence converge at any finite value of τ.

For $n = 2$, we have

$$A_2 = -P^{-}(\,\overline{Z}_1\alpha - \bar{\alpha} Z_1), \qquad B_2 = -P^{-}(\bar{u}\overline{Z}_1\alpha - u\bar{\alpha} Z_1), \tag{28}$$

and the remaining terms in the expression for B_2 cancel.

The rapid decay of Z_n and Φ_n means that the general solution is constructed explicitly from two arbitrary analytic functions $\alpha(u)$ and $Z_1(u)$.

4. Elementary solutions

We consider the elementary solution determined by the function $\alpha(u)$. Because this function continues analytically to the lower half-plane, we use the notation $\alpha(w)$, $w = u + iv$. In the general case,

$$Z = w(t - t_0) + \alpha(w), \qquad Z_w = t - t_0 + \alpha_w, \tag{29}$$

$$\Phi = \frac{1}{2}\, w^2(t - t_0) + \partial^{-1} w\alpha_w, \qquad \Phi_w = w(t - t_0 + \alpha_w), \tag{30}$$

$$\Phi_Z = V = \frac{\Phi_w}{Z_w} = w, \tag{31}$$

where $V = u_x - iv_y$ is the complex velocity, which turns out to coincide with the conformal variable w in the case of elementary solutions.

Equation (29) defines $w = w(z, t)$. This is a function defined on the physical plane and analytic in the area occupied by the fluid.

We differentiate Eq. (29) with respect to z and t sequentially:

$$\begin{aligned} &\frac{\partial w}{\partial z}(t - t_0 + \alpha'(w)) = 1, \\ &w + (t - t_0 + \alpha'(u)w_t) = 0. \end{aligned} \tag{32}$$

The equation

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial z} = 0 \tag{33}$$

follows from relations (32), i.e.,

$$\frac{\partial V}{\partial t} + V\,\frac{\partial V}{\partial z} = 0. \tag{34}$$

Identities (33) and (34) extend to the entire area of the physical plane occupied by the fluid. In addition, the boundary condition

$$V \to \frac{z}{t - t_0}, \quad |z| \to \infty, \tag{35}$$

is satisfied.

In Eq. (34). we separate the real and imaginary parts using the relations

$$\frac{\partial V_x}{\partial x} = -\frac{\partial V_y}{\partial y}, \qquad \frac{\partial V_x}{\partial y} = \frac{\partial V_y}{\partial x}. \tag{36}$$

We obtain

$$\begin{aligned} &\frac{\partial V_x}{\partial t} + V_x\frac{\partial V_x}{\partial x} - V_y\frac{\partial V_x}{\partial y} = 0, \\ &\frac{\partial V_y}{\partial t} + V_x\frac{\partial V_y}{\partial x} - V_y\frac{\partial V_y}{\partial y} = 0. \end{aligned} \tag{37}$$

Comparing with the standard Euler equations, we obtain the results

$$\frac{\partial}{\partial x}(P + Vy^2) = 0, \qquad \frac{\partial}{\partial y}(P + V_y^2) = 0. \tag{38}$$

Integrating relations (38) gives $P = -V_y$.

Equation (34) was independently obtained by Karabut and Zhuravleva [1]. We do not use this equation further and return to Eq. (29) considering it on the real axis, i.e., setting $w = u$.

We now have

$$Z = u(t - t_0) + \alpha(u). \tag{39}$$

Let

$$\alpha(u) = \sum_{n=0}^{N} \frac{A_n}{u - iv} \tag{40}$$

be a rational function (we do not consider the case of multiple poles). If $\alpha(u)$ is a rational function, then the real and imaginary parts of Z are related by an algebraic curve of genus zero for which there are inevitably singular points. On the real plane (x, y), they are manifested as points of self-intersection. Therefore, Eq. (39) in this case naturally describes the appearance of drops and bubbles.

To satisfy condition (22), we must require the condition

$$\sum A_n = 0, \tag{41}$$

but this condition can be relaxed if at least one of the poles tends to infinity. We consider the simplest case of a single pole located on the imaginary axis. We assume that the residue at this pole is real. Then Eq. (39) becomes

$$Z = ut + \frac{A}{u - ia}, \quad a > 0. \tag{42}$$

Separating the real and imaginary parts, we obtain

$$x = ut + \frac{Au}{u^2 + a^2}, \tag{43}$$

$$y = \frac{aA}{u^2 + a^2}. \tag{44}$$

The coordinates x and y satisfy the equation of an algebraic curve $F(x, y) = 0$,

$$F(x, y) = ay[x^2 + (y + at)^2] - A(y + at)^2 = 0. \tag{45}$$

The function $x(u)$ is odd, while the function $y(u)$ is even. As a result, we find that $y(x)$ is also an even function, $y(-x) = y(x)$.

Formula (44) shows that $y(x)$ has the same sign as A. If $A > 0$, then

$$0 < y(x) < y_{\max}, \qquad y_{\max} = \frac{A}{a}, \tag{46}$$

and as $|x| \to \infty$, we have the asymptotic behavior

$$y \to \frac{aAt^2}{x^2 + a^2t^2}. \tag{47}$$

We consider the equation $x(u) = 0$ and recall that $t > 0$. If $A > 0$, then this equation has a unique solution $u = 0$. But if A is negative, $A = -|A|$, then in the time interval

$$0 < t < \frac{|A|}{a^2}, \tag{48}$$

there are two more solutions

$$u = \pm\sqrt{\frac{A}{a^2} - t}. \tag{49}$$

In this time interval, curve point (45) is located on the negative segment of the imaginary axis at the point

$$y_0 = -at > -y_{\max}. \tag{50}$$

At this point, the self-intersection of the surface occurs, and conformal transformation (42) ceases to be reversible. Until now in [10], [8], we avoided ambiguous reversible conformal transformations, but this does not mean that they have no physical meaning. Physically, the ambiguity of a conformal transformation means the existence of a bubble. At $t = 0$, the bubble has the shape of a circle touching the horizontal surface from below (we recall that we consider the case $A < 0$). As time increases, the bubble decreases and completely disappears at

$$t_0 = -\frac{A}{a^2}. \tag{51}$$

At this instant, a singularity that looks like a Whitney fold forms on the surface,

$$y = -y_{\max} + \left(\frac{A}{a}\,x^2\right)^{1/3}. \tag{52}$$

For $A > 0$, the conformal transformation is nonsingular for all $t > 0$, but there is a singularity at the time $t = 0$. The surface is a round drop of radius $A/(2a)$. Curve (45) takes the simple form

$$F(x, y) = y\left[x^2 + \left(y - \frac{1}{2}\frac{A}{a}\right)^2 - \frac{1}{4}\frac{A^2}{a^2}\right] = 0. \tag{53}$$

For $t > 0$, $x(u)$ is a single-valued function, and the drop turns out to be connected with the main surface by a “stem.” To describe the formation of a drop at negative times, we can use the reflection of time and the vertical coordinate described in Sec. 2. The end result is that we can set $A = a = 1$ and at all times use the equations

$$x = u\left(t + \frac{1}{u^2 + a^2}\right), \qquad y = \frac{1}{u^2 + a^2}. \tag{54}$$

The numerical solution of these equations is shown in Figs. 1 and 2.

For $A < 0$, solution (42) describes the formation of a bubble. The corresponding plots can be obtained by replacing $y \to -y$ and $t \to -t$.

We note that in the region of times satisfying condition (48), the conformal map

$$Z = wt + \frac{A}{w - ia} \tag{55}$$

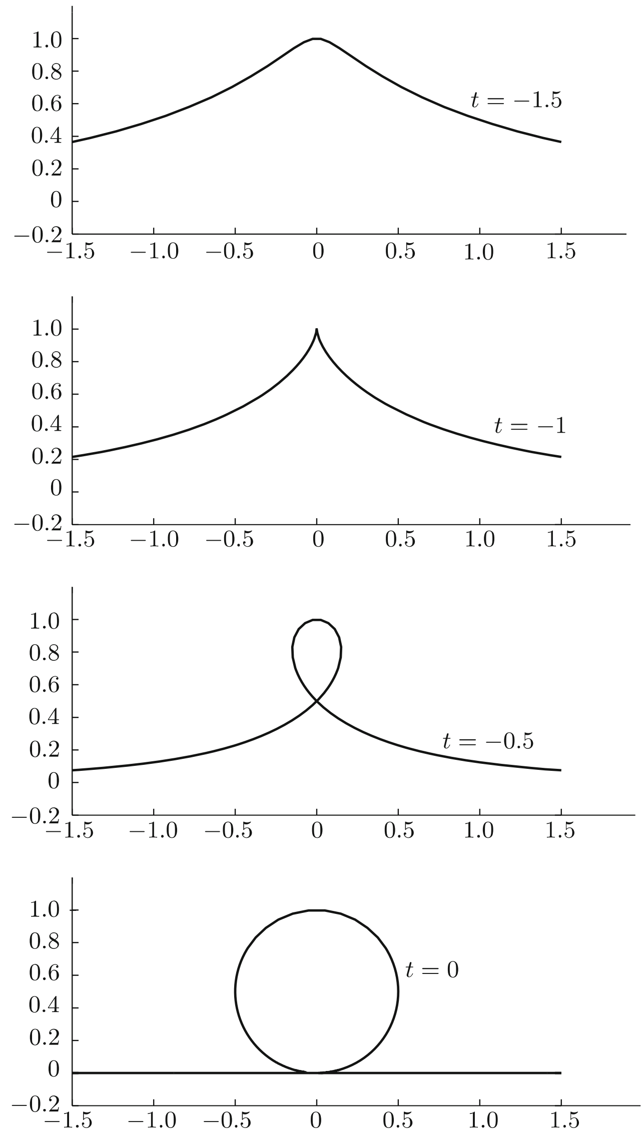

Fig. 1. The appearance of a drop.

stops mapping the negative half-plane $\operatorname{Im} w < 0$ onto a simply connected region because the expression

$$\frac{\partial Z}{\partial w} = t - \frac{A}{(w - ia)^2} \tag{56}$$

acquires a zero in the negative half-plane.

The question of the physical realizability of the obtained solutions in the domain of ambiguous reversibility of conformal transformation (55) needs additional research. But in the domain of its uniqueness outside interval (48), it is beyond doubt. In this domain, the inverse transformation, which determines the complex velocity, is given by solving quadratic equation (55),

$$w = \frac{Z + iat + \sqrt{(Z - iat)^2 - 4At}}{2t}, \tag{57}$$

as $Z \to \infty$ and $w \to Z/t$ in accordance with the above theory. In this paper, we do not discuss the question of whether the second solution of this equation has a physical meaning.

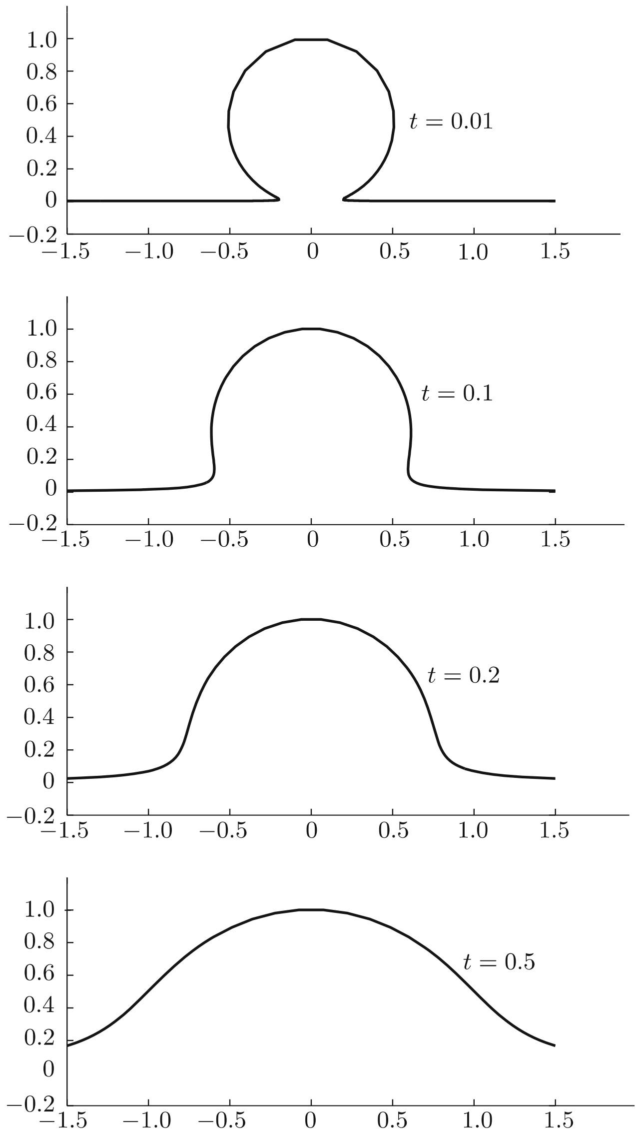

Fig. 2. The disappearance of a drop.

5. Conclusion

Although the results are somewhat exotic, they are a strong argument in favor of the hypothesis that deep-fluid equations are integrable, at least in the absence of gravity and surface tension. The solution previously obtained by Karabut and Zhuravleva [11]–[13], which is also somewhat exotic, agrees with our result here. In both cases, exact solutions of the Euler equations are obtained. And as L. D. Landau said, exact solutions adorn life.

We note that the Hopf equation for a deep fluid already existed in [14]. It was not "exotic," because the boundary conditions were standard. But it is approximate.

Some of the results published here were independently obtained by Zubarev and Karabut [15].

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REFERENCES

1. V. E. Zakharov and A. I. Dyachenko, "Are equations of deep water with a free surface integrable?" in: *Russian-French Workshop "Mathematical hydrodynamics." Abstract* (22–27 August 2016, Novosibirsk, Russia), Novosibirsk State University, Novosibirsk (2016).
2. E. A. Karabut and E. N. Zhuravleva, "Reduction of free-boundary problem to the system of differential equation," in: *IX International Scientific Conference "Solitons, Collapses, and Turbulence: Achievements, Developments, and Perspectives"* (SCT-19) *in honor of Vladimir Zakharov's* 80*th birthday: Abstracts* (Yaroslavl, 5–9 August 2019), Demidov Yaroslavl State University, Yaroslavl (2019), pp. 67.
3. V. E. Zakharov, S. Novikov, S. V. Manakov, and L. P. Pitaevskii, *Theory of Solitons: Inverse Problem Method* [in Russian], Nauka, Moscow (1980); English transl.: S. Novikov, S. V. Manakov, L. P. Pitaevskii, and V. E. Zakharov, *Theory of Solitons: The Inverse Scattering Method*, Springer, New York (1984).
4. P. Nabelek and V. E. Zakharov, "Dressing method in application to the Kaup–Broer system," (in preparation).
5. A. I. Dyachenko and V. E. Zakharov, "Is free-surface hydrodynamics an integrable system?" *Phys. Lett. A*, **190**, 144–148 (1994).
6. A. I. Dyachenko, "New integrals of motions for water waves," in: *IX International Scientific Conference "Solitons, Collapses, and Turbulence: Achievements, Developments, and Perspectives"* (SCT-19) *in honor of Vladimir Zakharov's* 80*th birthday: Abstracts* (Yaroslavl, 5–9 August 2019), Demidov Yaroslavl State University, Yaroslavl (2019), pp. 39.
7. V. E. Zakharov and A. I. Dyachenko, "Free-surface hydrodynamics in the conformal variables," arXiv: 1206.2046v1 [physics.flu-dyn] (2012).
8. A. I. Dyachenko, S. A. Dyachenko, P. M. Lushnikov, and V. E. Zakharov, "Dynamics of poles in two-dimensional hydrodynamics with free surface: New constant of motion," *J. Fluid Mech.*, **871**, 891–925 (2019); arXiv:1809.09584v2 [nlin.SI] (2018).
9. L. N. Sretenskii, *Theory of Wave Motions of a Liquid* [in Russian], Nauka, Moscow (1977).
10. A. I. Dyachenko, P. M. Lushnikov, and V. E. Zakharov, "Non-canonical Hamiltonian structure and Poisson brackets for two-dimensional hydrodynamics with free surface," *J. Fluid Mech.*, **869**, 526–552 (2019).
11. E. A. Karabut and E. N. Zhuravleva, "Unsteady flows with a zero acceleration of the free boundary," *J. Fluid Mech.*, **754**, 308–331 (2014).
12. E. A. Karabut and E. N. Zhuravleva, "Unsteady flows with a zero acceleration on the free boundary," *Dokl. Phys.*, **59**, 480–483 (2014).
13. E. A. Karabut and E. N. Zhuravleva, "Reproduction of solutions in the plane problem on motion of a free-boundary fluid," *Dokl. Phys.*, **61**, 347–350 (2016).
14. A. I. Dyachenko, E. A. Kuznetsov, M. D. Spector, and V. E. Zakharov, "Analytical description of the free surface dynamics of ideal fluid (canonical formalism and conformal mapping)," *Phys. Lett. A*, **221**, 73–79 (1999).
15. N. M. Zubarev and E. A. Karabut, "Exact local solutions for the formation of singularities on the free surface of an ideal fluid," *JETP Lett.*, **107**, 412–417 (2018).