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# Numerical Study of Isotropic Ocean Swell

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**Abstract**—A new algorithm is used for detailed numerical study of the evolution of isotropic swell in a homogeneous ocean. It is shown that the Zakharov-Filonenko spectrum occurs in an explosive manner in a short time. The Kolmogorov constant of the solution is estimated numerically.

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## 1. INTRODUCTION

This article is a continuation of the previous article of two authors [1], devoted to the same issue. In both articles, a numerical solution of the kinetic equation for water waves (the Hasselmann equation) is performed in the absence of wind forcing, i.e. the evolution of a free ocean swell is reproduced. The difference from the article [1] is the use of a new numerical algorithm. In the previous article, we used the Resio-Tracy code [2], improved and used by our group for a long time. In this paper, we use a completely new code developed by one of the authors (V. V. Geogjaev). Details of the algorithm have not yet been published, but the new code is significantly more accurate and faster than the one we already used [3].

The new code allows us to convincingly confirm the conclusions of the weakly turbulent theory of wind waves in their full and in the important details. Note that a comparison with the experiment of the main predictions of this theory has already been done in [1]. Thus, this article is primarily of theoretical value. The main results obtained in this article are as follows:

1. For moderate values of initial steepness ($\mu < 0.1$), the power-law asymptotic spectrum $\varepsilon \simeq \omega^{-4}$ arises over several hundred periods of the initial waves, i.e. almost an order of magnitude faster than dimensional estimates predict. The establishment of the asymptotic occurs in an explosive way, in a finite time.

2. The Kolmogorov constant found in the numerical experiments agrees perfectly well with the analytical estimate of the stationary theory presented in [3]. In addition, the work shows that the new numerical method for the Hasselmann equation has good prospects.

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## 2. WEAK TURBULENCE THEORY OF OCEAN WAVES (WTT)

The history of the weak turbulence theory (WTT) has began with the work of Phillips [4], who suggested that the four-wave interaction is the main physical process for the sea surface waves. For four waves (a quadruplet) to interact their wave vectors should obey the resonance conditions

$$\begin{aligned}\mathbf{k}_1 + \mathbf{k}_2 &= \mathbf{k}_3 + \mathbf{k}_4,\\ \omega_1 + \omega_2 &= \omega_3 + \omega_4\end{aligned} \tag{1}$$

($\omega_{\mathbf{k}} = \sqrt{g|\mathbf{k}|}$ is the dispersion law of deep water waves).

In 1962, Klauss Hasselmann derived the kinetic equation for the wave action spectrum $N_{\mathbf{k}}$ [5–7]. Taking into account the wave generation by wind and dissipation due to wave breaking, the kinetic equation takes the form

$$\frac{\partial N_{\mathbf{k}}}{\partial t} + \frac{\partial \omega_{\mathbf{k}}}{\partial \mathbf{k}}\frac{\partial N_{\mathbf{k}}}{\partial \mathbf{r}} = S_{\mathrm{nl}} + S_{\mathrm{in}} + S_{\mathrm{diss}}. \tag{2}$$

Here $S_{\mathrm{in}}$ describes the wind wave generation, $S_{\mathrm{diss}}$ is the term of dissipation by wave breaking, $S_{\mathrm{nl}}$ represents the effect of nonlinear interactions of waves obeying the Phillips resonance conditions (1). The $S_{\mathrm{nl}}$ term naturally splits into "pumping" and "damping" terms of [8]:

$$S_{\mathrm{nl}} = F_{\mathbf{k}} - \Gamma_{\mathbf{k}} N_{\mathbf{k}}. \tag{3}$$

Here

$$\begin{aligned}F_{\mathbf{k}} = \pi g^2 \int\limits_{\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} |T(\mathbf{k},\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)|^2\, N_2 N_3 N_4 \delta(\omega + \omega_2\\ - \omega_3 - \omega_4)\delta(\mathbf{k} + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4)d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4,\end{aligned} \tag{4}$$

$$\Gamma_{\mathbf{k}} = \pi g^2 \int\limits_{\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} |T(\mathbf{k},\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)|^2 \times (N_3N_4 - N_2N_3 - N_2N_4)\delta(\omega + \omega_2 - \omega_3 - \omega_4) \times \delta(\mathbf{k} + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4)d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4. \quad (5)$$

The kernel $T_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}$ is a homogeneous function of degree 3 that obeys the symmetry conditions. Its simplest expression can be found in [3].

The explicit form of the non-conservative terms $S_{\rm in}$ and $S_{\rm diss}$ was a subject of long-lasted discussion (see, for example, [2]). However, it has now been decisively shown [8, 9] that these terms, as a rule, are an order of magnitude smaller than the partial terms of pumping and damping $\Gamma_{\mathbf{k}}N_{\mathbf{k}}$ and $F_{\mathbf{k}}$ in $S_{\rm nl}$. Therefore, the building of a consistent analytical theory of wind waves (such a program was presented in [10], [11]) should begin with a thorough study of the temporal evolution of a spatially uniform swell which is a solution to the equation

$$\frac{\partial N_{\mathbf{k}}}{\partial t} = S_{\rm nl}. \quad (6)$$

We solved this equation in the general anisotropic case in [1]. In this paper, we study the isotropic case in detail. We assume that there is no wave action flux from the domain of very short waves. Then the total wave action

$$N = \int N_{\mathbf{k}} d\mathbf{k} \quad (7)$$

is a true integral of motion, whereas the energy

$$E = \int \omega_{\mathbf{k}} N_{\mathbf{k}} d\mathbf{k} \quad (8)$$

is being lost due to the flux into the region of large wave numbers by Kolmogorov cascade. Moreover, the only solution to the stationary equation $S_{\rm nl} = 0$ is

$$N(\mathbf{k}) = c_p \frac{|dE/dt|^{1/3}}{g^{2/3}|\mathbf{k}|^4}. \quad (9)$$

This is the Zakharov–Filonenko spectrum, first found in [12]. $c_p$ is a dimensionless Kolmogorov constant, according to [3] $c_p = 0.194$.

## 3. NUMERICAL EXPERIMENTS

A numerical study was carried out for the idealized problem of the evolution of a wave field in a homogeneous isotropic ocean in the absence of generation and dissipation. The initial state corresponds to an isotropic distribution localized in the frequency domain

$$0.04\pi < \omega < 4\pi \text{ rad/s}, \quad 0.02 < f < 2 \text{ Hz}. \quad (10)$$

Such a formulation allows one to minimize the number of parameters of the problem and focus oneself on the quality issues of the algorithm for calculating the collision integral $S_{\rm nl}$, without being distracted by questions related to the nontrivial dependence of the solutions on the angle (cf. [1]).

We used a logarithmic grid for frequency with an increment $(\omega_{i+1} - \omega_i)/\omega_i = 1.03344$ (141 node). In total, 49152 resonant quadruplets were chosen in a special way to provide optimal coverage of the domains of the most significant interactions [3]. For the frequency spectrum of energy

$$\varepsilon(\omega)d\omega = \frac{4\pi|\mathbf{k}|^2}{g} N(\mathbf{k})d|\mathbf{k}| \quad (11)$$

the weakly turbulent Kolmogorov spectrum (the Zakharov–Filonenko spectrum) has the form

$$\varepsilon(\omega) = 4\pi c_p g^{4/3}\left|\frac{dE}{dt}\right|^{1/3}\omega^{-4}. \quad (12)$$

The initial condition is step-like: $\varepsilon(\omega) = 2$, $0.1 < \omega/2\pi < 0.2$, $\varepsilon(\omega) = 10^{-6}$ outside this area. The characteristic initial period is about $T = 7$ s. The initial steepness turbulentdefined through the mean-over-spectrum frequency $\omega_m$

$$\mu = \sqrt{E\omega_m^4/g^2} \simeq 0.1. \quad (13)$$

A numerical experiment for a physical time of 200000 s seconds (about 56 h) requires less than 2 h of computer time and can easily be continued up to several million seconds. The code is implemented for parallel computing, which brings its performance closer to the requirements of the today operational models of wind waves.

As expected, the evolution of this initial condition leads to the appearance of the Kolmogorov tail spectrum described by (12). The Kolmogorov asymptotic behavior is established "in an explosive manner" in a finite time. This experimental fact still needs a theoretical explanation. Based on the dimensional arguments, one has

$$\frac{T}{\mu^4} \simeq 70\,000 \text{ s}, \quad (14)$$

which is an order of magnitude less than the experimentally observed 3000 s. In Fig. 1 the initial spectrum and the result of its evolution are combined at $t = 3000$ s. It can be seen that the Kolmogorov asymptotic behavior is already fully developed for a time of about 400 initial periods.

Figure 2 represents the compensated spectrum of $\varepsilon(\omega)\omega^4/g^2$ at time $t \approx 15$ h. It can be seen that the asymptotics (12) dominates at frequencies $\omega > 4$ rad/s (period of about 1.5 s, wavelength 3.5 m).

It was shown [1] that the evolution of a wide class of initial conditions leads to the establishment of a self-similar solution:

$$\varepsilon(\omega, t) = \varepsilon(\omega t^{1/11}). \quad (15)$$

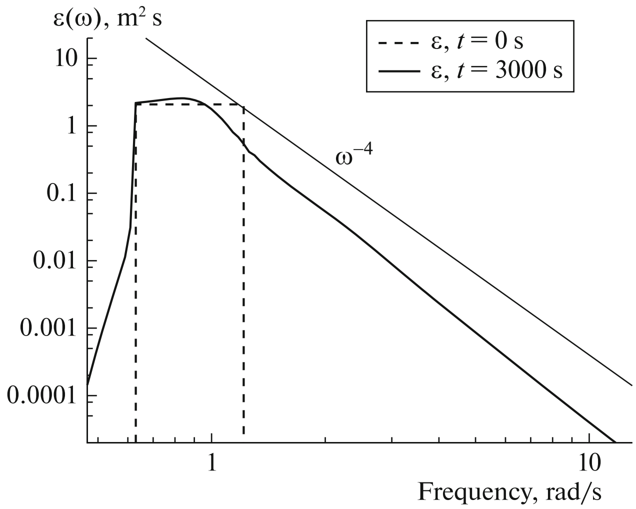


**Fig. 1.** Frequency spectra of energy at initial $t = 0$ and at $t = 3000$ s. Establishing a power-law distribution $\omega^{-4}$ is clearly seen.

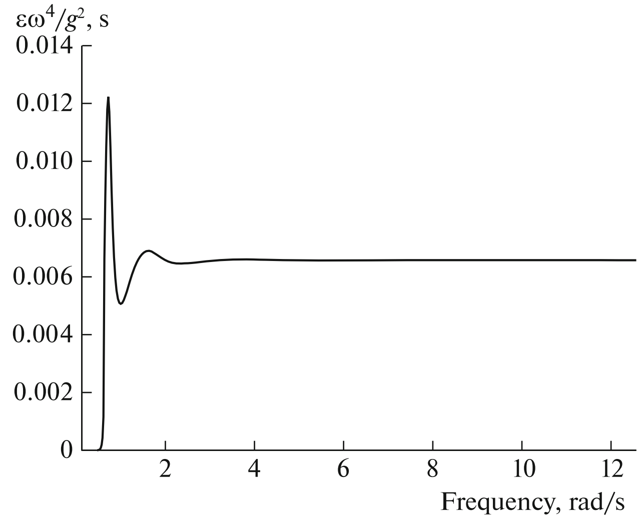


**Fig. 2.** Compensated spectrum $\varepsilon(\omega)\omega^4/g^2$ at $t = 50\,400$ s.

Our numerical experiment shows that the establishment of this regime requires a rather long time of 50000 s or more, i.e. more than 7000 initial wave periods. The establishment of a self-similar regime is shown in Fig. 3 as the evolution of the spectra normalized to the maximum value vs the dimensionless frequency $\omega/\omega_p$ ($\omega_p$ is the frequency of the spectral peak). Thus, the tendency to the self-similar spectral shapes is markedly weaker than the establishment of the Kolmogorov "tail" and the behavior of the integral swell parameters. This feature was discussed in detail in [1].

Figure 4 shows the evolution of the total energy and frequency of the spectral peak in the conservative kinetic equation (1). For 200000 s, no more than 30% of energy is lost, however, this loss rate is enough to form a weakly turbulent Kolmogorov spectrum. The decrement $\beta \simeq (\partial E/\partial t)^{1/3} \sim t^{-4/11}$ decreases with time. The estimate of the dimensionless constant $c_p$ gives $c_p = 0.203$, which is consistent with the theoretical value found in [1].

## 4. RESULTS AND CONCLUSIONS

As noted above, the main result is a demonstration of weakly turbulent asymptotics for the numerical solutions of the kinetic equation for water waves (the Hasselmann equation). It is important, that the results were obtained using a completely new numerical algorithm developed by V. V. Geogjaev. This algorithm allows one to get closer to the operational model requirements at incomparably higher accuracy and resolution of the scale of waves (in frequencies and directions). Earlier the numerical studies of the properties of solutions of the kinetic equation [13, 14] being consistent with the main results of the theory of weak turbulence, found fundamental constraints on Webb-Resio-Tracy algorithm [2, 15] both in terms of accuracy and speed.

We emphasize that the developed algorithm uses its own parametrization of resonant surface (1), which makes it possible to more accurately describe spectral fluxes and achieve higher performance with a relatively small number of resonant quadruplets. This distinguishes it from other approaches, in particular,

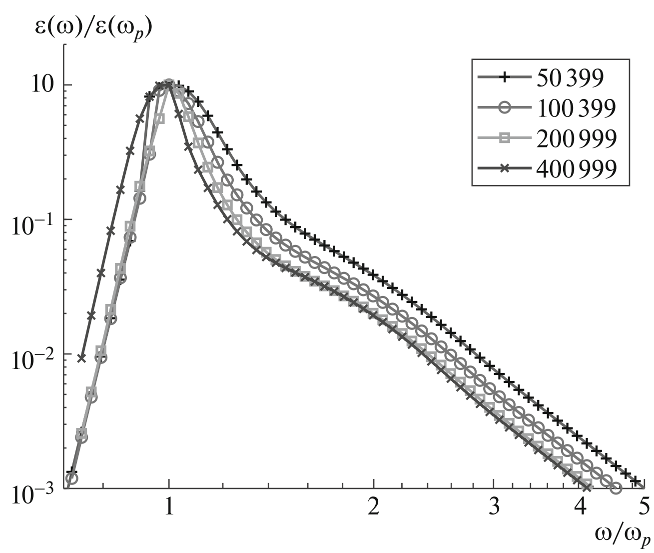


**Fig. 3.** Normalized spectra of energy as functions of the dimensionless frequency at different times (in legend, in seconds).

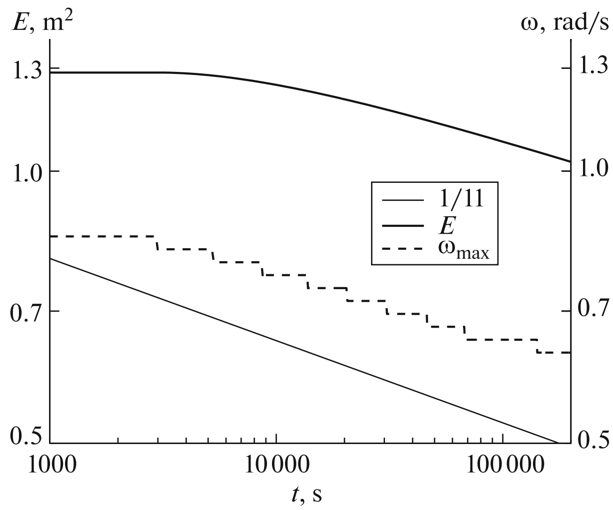


**Fig. 4.** Evolution of total energy and spectral peak frequency in the conservative kinetic equation (1).

from the versions of Discrete Interaction Approximation used in operational models.

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