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NONLINEAR PHENOMENA

Numerical and Analytical Calculations of the Parameters of Power-Law Spectra for Deep Water Gravity Waves[1]

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We determine the asymptotic behavior of the coupling coefficient for four-wave interactions of gravity waves in deep water in the limiting case when two wave vectors of interacting waves are small with respect to the other two ("long–short interactions"). It makes possible to find numerically dimensionless Kolmogorov constants for the power-law Kolmogorov–Zakharov spectra. The results obtained are crucially important for comparison of the weak turbulent theory with the experiments and natural observations.

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INTRODUCTION

Gravity waves spectra in the wind-driven sea are shaped by the four-wave nonlinear interactions described by the Hasselmann kinetic equation [1]. This equation has a broad family of stationary solutions known as the Kolmogorov–Zakharov (KZ) spectra (see, for instance [2]).

In the simplest isotropic case, the spectra are power-law

$$N_1 = c_p \frac{P^{1/3}}{k^4}, \tag{1}$$

$$N_2 = c_q \frac{Q^{1/3}}{k^{23/6}}, \tag{2}$$

where P is the energy flux (directed to the high wavenumbers) and Q is the wave action flux (directed to the low wavenumbers). We assume the water density $\rho = 1$ and the acceleration of gravity $g = 1$ here and below.

The features of these spectra are widely observed both in the ocean measurements and in wave tank experiments (see, e.g., [3]).

In this paper, we calculate c_p and c_q numerically. The accuracy of this numerical calculation is supported by the analytical estimation of the asymptotic behaviors of power-law spectra (see below).

[1] The article is published in the original.

ASYMTOTIC BEHAVIOR OF THE COUPLING COEFFICIENT

The dispersion relation for the gravity waves in deep water is:

$$\omega = \sqrt{|\mathbf{k}|} \tag{3}$$

(here and below, we put the acceleration of gravity $g = 1$).

We will use the Hamiltonian description of gravity waves (see [4]). After the canonical transformation, the complex normal variables $b_{\mathbf{k}}$ satisfy the equation [5, 6]

$$\frac{\partial b_{\mathbf{k}}}{\partial t} + i\frac{\partial H}{\partial b^*_{\mathbf{k}}} = 0, \tag{4}$$

where the Hamiltonian function

$$H = H_0 + H_{\text{int}} \tag{5}$$

is presented as a series in formally small wave amplitudes. The lowest order term corresponds to the linear approximation:

$$H_0 = \int \omega_{\mathbf{k}} b_{\mathbf{k}} b^*_{\mathbf{k}} d\mathbf{k}. \tag{6}$$

The weak nonlinearity is described by the next order term

$$H_{\text{int}} = \int T_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} b^*_{\mathbf{k}_1} b^*_{\mathbf{k}2} b_{\mathbf{k}_3} b_{\mathbf{k}4} \times \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4. \tag{7}$$

The kernel T was first derived in [4–6]. Different forms of the kernel expression may be compared in [7, 8]. We present here the new form of T which seems to be the simplest one

$$
\begin{aligned}
T_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} &= -\frac{1}{4}\frac{1}{(k_1k_2k_3k_4)^{1/4}} \\
&\times\Big\{\frac{1}{2}\Big(k_{1+2}^2-(\omega_1+\omega_2)^4\Big) \\
&\times(\mathbf{k}_1\mathbf{k}_2-k_1k_2+\mathbf{k}_3\mathbf{k}_4-k_3k_4) \\
&-\frac{1}{2}\Big(k_{1-3}^2-(\omega_1-\omega_3)^4\Big) \\
&\times(\mathbf{k}_1\mathbf{k}_3+k_1k_3+\mathbf{k}_2\mathbf{k}_4+k_2k_4) \\
&-\frac{1}{2}\Big(k_{1-4}^2-(\omega_1-\omega_4)^4\Big) \\
&\times(\mathbf{k}_1\mathbf{k}_4+k_1k_4+\mathbf{k}_2\mathbf{k}_3+k_2k_3) \\
&+\left(\frac{4(\omega_1+\omega_2)^2}{k_{1+2}-(\omega_1+\omega_2)^2}-1\right) \\
&\times(\mathbf{k}_1\mathbf{k}_2-k_1k_2)(\mathbf{k}_3\mathbf{k}_4-k_3k_4) \\
&+\left(\frac{4(\omega_1-\omega_3)^2}{k_{1-3}-(\omega_1-\omega_3)^2}-1\right) \\
&\times(\mathbf{k}_1\mathbf{k}_3+k_1k_3)(\mathbf{k}_2\mathbf{k}_4+k_2k_4) \\
&+\left(\frac{4(\omega_1-\omega_4)^2}{k_{1-4}-(\omega_1-\omega_4)^2}-1\right) \\
&\times(\mathbf{k}_1\mathbf{k}_4+k_1k_4)(\mathbf{k}_2\mathbf{k}_3+k_2k_3)\Big\}.
\end{aligned} \tag{8}
$$

Here, k_{1+2}, k_{1-3}, and k_{1-4} are the lengths of the vectors $\mathbf{k}_1+\mathbf{k}_2$, $\mathbf{k}_1-\mathbf{k}_3$, and $\mathbf{k}_1-\mathbf{k}_4$, respectively. It should be stressed that Eq. (8) holds only at the resonant manifold:

$$
\begin{aligned}
\mathbf{k}_1+\mathbf{k}_2 &= \mathbf{k}_3+\mathbf{k}_4, \\
\omega_1+\omega_2 &= \omega_3+\omega_4.
\end{aligned} \tag{9}
$$

The coupling coefficient satisfies the symmetry conditions

$$T_{1234} = T_{2134} = T_{1243} = T_{3412}. \tag{10}$$

Now suppose that the two wave vectors, e.g., $\mathbf{k}_1$ and $\mathbf{k}_3$ are much shorter than the other two ($\mathbf{k}_2$ and $\mathbf{k}_4$). Taking into account Eq. (9), we see that $\mathbf{k}_1$ and $\mathbf{k}_3$ have nearly equal length. The vectors $\mathbf{k}_2$ and $\mathbf{k}_4$ are nearly equal, both in length and direction. An example of such configuration is shown in Fig. 1.

Thereafter, we define $k_1 = |\mathbf{k}_1|$, $k_2 = |\mathbf{k}_2|$, etc. We have $k_1 \approx k_3 \ll k_2 \approx k_4$.

After the tedious algebra, one may find the following asymptotic behavior for the coupling coefficient:

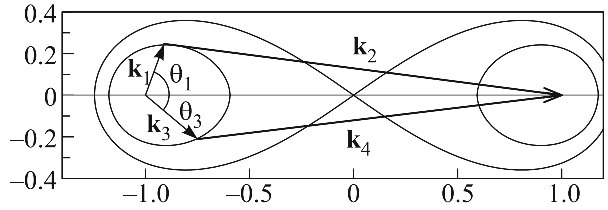

Fig. 1. Wave vector quadruplet of the long–short interaction. A curve $\omega_1+\omega_2 = \mathrm{const}$ is drawn; any two points of the curve constitute a resonant quadruplet. The angles θ_1 and θ_3 are given with respect to the vector $\mathbf{k}_1+\mathbf{k}_2 = \mathbf{k}_3+\mathbf{k}_4$. The eight-shape figure is the Phillips curve.

$$
\begin{aligned}
T_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} &= \frac{1}{2}k_1^2k_2T_{\theta_1\theta_3}+o(k_1^2), \\
T_{\theta_1\theta_3} &= (\cos\theta_1+\cos\theta_3)(1+\cos(\theta_1-\theta_3)),
\end{aligned} \tag{11}
$$

θ_1 is the angle between the small vectors $\mathbf{k}_1$ and $\mathbf{k}_1+\mathbf{k}_2$ (see Fig. 1). Same stands for θ_3.

In the diagonal case $\theta_1 = \theta_3$, $\mathbf{k}_1 = \mathbf{k}_3$, $\mathbf{k}_2 = \mathbf{k}_4$,

$$T(\mathbf{k}_1,\mathbf{k}_2) = 2k_1^2k_2\cos(\theta_1). \tag{12}$$

POWER-LAW SOLUTION OF THE KINETIC EQUATION

A random field of gravity waves is statistically described by the wave action spectrum $N_{\mathbf{k}}$,

$$\langle b_{\mathbf{k}}b_{\mathbf{k}'}^*\rangle = N_{\mathbf{k}}(t)\delta_{\mathbf{k}-\mathbf{k}'}. \tag{13}$$

The wave action $N_{\mathbf{k}(t)}$ obeys the kinetic (Hasselmann) equation

$$\frac{dN}{dt} = S_{\mathrm{nl}}, \tag{14}$$

$$
\begin{aligned}
S_{\mathrm{nl}} &= \pi g^2\int\limits_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}(T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3})^2 \\
&\times\big(N_{\mathbf{k}}N_{\mathbf{k}_2}N_{\mathbf{k}_3}+N_{\mathbf{k}_1}N_{\mathbf{k}_2}N_{\mathbf{k}_3} \\
&-N_{\mathbf{k}}N_{\mathbf{k}_1}N_{\mathbf{k}_2}-N_{\mathbf{k}}N_{\mathbf{k}_1}N_{\mathbf{k}_3}\big) \\
&\times\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\delta(\omega+\omega_1-\omega_2-\omega_3) \\
&\times d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3.
\end{aligned} \tag{15}
$$

We are looking for the solution of the stationary equation

$$S_{\mathrm{nl}} = 0. \tag{16}$$

We assume that the solution of (16) is a power-law function

$$N = ak^{-x}. \tag{17}$$

Then

$$S_{\mathrm{nl}} = a^3 g^{\frac{3}{2}} k^{-3x+\frac{19}{2}} F(x), \qquad (18)$$

where F is a dimensionless function depending of x only.

INTEGRAL FORM FOR THE F FUNCTION

The dimensionless F function may be presented in integral form.

We introduce the base vector

$$\mathbf{k}_{\mathrm{b}} = \frac{1}{2}(\mathbf{k}_1 + \mathbf{k}_2) = \frac{1}{2}(\mathbf{k}_3 + \mathbf{k}_4) \qquad (19)$$

(the resonant condition (9) is used here). Then, we norm the wave vectors $\tilde{\mathbf{k}}_1$, $\tilde{\mathbf{k}}_2$, $\tilde{\mathbf{k}}_3$, $\tilde{\mathbf{k}}_4$ as follows

$$\mathbf{k}_i = k_{\mathrm{b}} \tilde{\mathbf{k}}_i. \qquad (20)$$

The normalized coupling coefficient is $\tilde{T}$

$$T_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} = k_{\mathrm{b}}^3 \tilde{T}_{\tilde{\mathbf{k}}_1\tilde{\mathbf{k}}_2\tilde{\mathbf{k}}_3\tilde{\mathbf{k}}_4}. \qquad (21)$$

Instead of normalized ω_i, we introduce the variables s and s'

$$s = \frac{\omega_1 + \omega_2}{2\sqrt{g k_{\mathrm{b}}}}, \qquad s' = \frac{\omega_3 + \omega_4}{2\sqrt{g k_{\mathrm{b}}}}. \qquad (22)$$

(The resonance condition becomes just $s = s'$.)

Now, after choosing the normalized vectors $\tilde{\mathbf{k}}_1$ and $\tilde{\mathbf{k}}_3$ as independent variables, we get the following integral form for the F function:

$$F(x) = 2\pi \int\limits_{\tilde{k}_1<\tilde{k}_2, \tilde{k}_3<\tilde{k}_4} \left(\tilde{T}_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}\right)^2 (\tilde{k}_1\tilde{k}_2\tilde{k}_3\tilde{k}_4)^{-x}$$
$$\times \left(\tilde{k}_1^{-\frac{23}{2}+3x} + \tilde{k}_2^{-\frac{23}{2}+3x} - \tilde{k}_3^{-\frac{23}{2}+3x} - \tilde{k}_4^{-\frac{23}{2}+3x}\right) \qquad (23)$$
$$\times \left(\tilde{k}_1^x + \tilde{k}_2^x - \tilde{k}_3^x - \tilde{k}_4^x\right) \delta(s - s') d\tilde{\mathbf{k}}_1 d\tilde{\mathbf{k}}_3.$$

Here, the variables s and $\tilde{\mathbf{k}}_2$ are defined by $\tilde{\mathbf{k}}_1$, whereas the variables s' and $\tilde{\mathbf{k}}_4$ are defined by $\tilde{\mathbf{k}}_3$.

PROPERTIES OF THE FUNCTION F

It is easy to show that integrals in (23) converge if

$$\frac{5}{2} < x < \frac{19}{4}. \qquad (24)$$

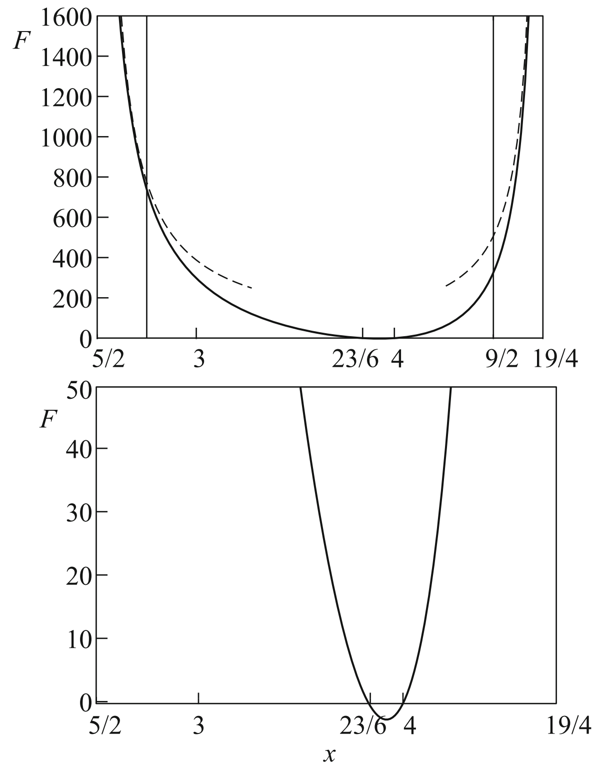

Fig. 2. F function graph and its asymptotes. The second picture is the closeup of the function zeros.

This is the "window of opportunity" for power-law solutions. At the end of this interval, $F \to \infty$. Using the asymptotic expansion (11), we find

$$F \to \frac{25\pi^3}{4} \frac{1}{x - \frac{5}{2}}, \qquad x \to \frac{5}{2},$$
$$F \to \frac{1045\pi^3}{256} \frac{1}{\frac{19}{4} - x}, \qquad x \to \frac{19}{4}. \qquad (25)$$

Notice that (25) is the result of rigorous analytic calculations.

The function F was calculated numerically. The excellent coincidence of the asymptotic behavior of F with analytically predicted asymptotes confirm the accuracy of the numerical code.

The function F is shown in Fig. 2.

According to the general theory [6], the function F has exactly two zeros $x = 4$ and $x = 23/6$. Corresponding Kolmogorov–Zakharov spectra are

$$N_k^{(1)} = c_p P_0^{1/3} \frac{1}{k^4}, \qquad (26)$$

$$N_k^{(2)} = c_q Q_0^{1/3} \frac{1}{k^{23/6}}. \qquad (27)$$

Here, P_0 is the energy flux and Q_0 is the wave action flux. The dimensionless constants c_p and c_q are defined from the first derivatives of F:

$$c_p = \left(\frac{3}{2\pi F'(4)}\right)^{1/3}, \tag{28}$$

$$c_q = \left(-\frac{3}{2\pi F'(23/6)}\right)^{1/3}. \tag{29}$$

Different estimates of c_p and c_q are summarized in [9].

Our numerical calculation of the derivatives F at $x = 4$ and $x = 23/6$ gives

$$c_p = 0.203, \quad c_q = 0.194. \tag{30}$$

It is important to stress that the famous Phillips spectrum [10]

$$N_{\mathbf{k}} = \frac{\alpha}{\omega k^4} \sim k^{-9/2}$$

though not the Kolmogorov spectrum, also belongs to the “window of opportunity” for power-like spectra.

We have calculated that at $x = 9/2$

$$F = 327.$$

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REFERENCES

1. K. Hasselmann, J. Fluid Mech. **12**, 481 (1962).
2. V. E. Zakharov, V. S. Lvov, and G. Falkovich, *Kolmogorov Spectra of Turbulence I. Wave Turbulence* (Springer, Berlin, 1992).
3. V. E. Zakharov, S. I. Badulin, P. A. Hwang, and G. Caulliez, J. Fluid Mech. **780**, 503 (2015).
4. V. E. Zakharov, J. Appl. Mech. Tech. Phys. **9**, 190 (1968–1972).
5. V. P. Krasitskii, J. Fluid Mech. **272**, 1 (1994).
6. V. E. Zakharov, Eur. J. Mech. B: Fluids **18**, 327 (1999).
7. A. I. Dyachenko and Y. V. Lvov, J. Phys. Oceanogr. **25**, 3237 (1995).
8. S. I. Badulin, A. N. Pushkarev, D. Resio, and V. E. Zakharov, Nonlin. Processes Geophys. **12**, 891 (2005).
9. S. I. Badulin and V. E. Zakharov, Nonlin. Processes Geophys. **24**, 237 (2017).
10. O. M. Phillips, J. Fluid Mech. **4**, 426 (1958).