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CONDENSED MATTER

Spatial Equation for Water Waves[1]

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A compact spatial Hamiltonian equation for gravity waves on deep water has been derived. The equation is dynamical and can describe extreme waves. The equation for the envelope of a wave train has also been obtained.

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1. INTRODUCTION

Surface gravity waves generated in the laboratory tank (or flume) is one of the most studied examples of nonlinear wave evolution. Numerical simulation of such wave evolution has to be its integral part. These waves are usually described by the classical Hamiltonian system of equations for potential flows with the truncated Hamiltonian [1]:

$$H = \frac{1}{2}\int g\eta^2 + \psi\hat{k}\psi dx - \frac{1}{2}\int \{(\hat{k}\psi)^2 - (\psi_x)^2\}\eta dx + \frac{1}{2}\int \{\psi_{xx}\eta^2\hat{k}\psi + \psi\hat{k}[\eta\hat{k}(\eta\hat{k}\psi)]\}dx \tag{1}$$

with the Hamiltonian variables, where $\eta(x, t)$ is the surface profile and $\psi(x, t)$ is the potential at the surface. Equations of motions are the following:

$$\frac{\partial\psi}{\partial t} = -\frac{\delta H}{\delta\eta}, \quad \frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta\psi}. \tag{2}$$

These equations describe Cauchy problem in time, one has to set up initial conditions $\eta(x, 0)$ and $\psi(x, 0)$ at $t = 0$ at all x. However, in the flume situation is different. Typically, at the one end of the flume there is a wavemaker (piston or paddle) which generates (in the ideal case) $\eta(0, t)$ and $\psi(0, t)$. Thus, we have to solve Cauchy problem in space. If we restrict ourselves to an envelope of the wave train, than the equations for spatial Cauchy problem were derived in [2, 3] directly from Zakharov equation. They derived special analogies of both the nonlinear Schrödinger and Dysthe equations. Their Hamiltonian structures and new invariants were studied in [4]. However, to study waves with extreme amplitudes, freak-waves, the envelope approximation is not enough. In other words, to simulate real nonlinear waves, we *need spatial dynamical* equation for water waves. Below, this equation is derived for the case of one horizontal direction (narrow flume).

2. SUPERCOMPACT EQUATION

Let us recall very briefly the Zakharov equation for water waves. It can be derived by two steps.

First, instead of η and ψ, normal canonical variable a_k is introduced:

$$\eta_k = \sqrt{\frac{\omega_k}{2g}}(a_k + a^*_{-k}), \quad \psi_k = -i\sqrt{\frac{g}{2\omega_k}}(a_k - a^*_{-k}),$$
$$\omega_k = \sqrt{gk}.$$

Second, canonical transformation from a_k to b_k is chosen to cancel all nonresonant terms in the Hamiltonian, both the third and fourth orders.

As a result, the Hamiltonian acquires the form:

$$H = \int \omega_k b_k b_k^* dk + \frac{1}{2}\int T_{kk_1}^{k_2k_3} b_k^* b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk dk_1 dk_2 dk_3. \tag{3}$$

The explicit (and cumbersome) expression for $T_{kk_1}^{k_2k_3}$ be found in [1, 5]. The motion equation is the following:

$$\frac{\partial b_k}{\partial t} + i\frac{\delta H}{\delta b_k^*} = 0. \tag{4}$$

For one-dimensional waves, $T_{kk_1}^{k_2k_3}$ has very important for further—it is equal to zero on the four resonant

[1] The article is published in the original.

manifold [6]. This property allows applying another canonical transformation from b_k to c_k, namely,

$$b_k = c_k - i\int \tilde{B}_{kk_1}^{k_2k_3} c_{k_1}^* c_{k_2} c_{k_3} \delta_{k+k_1-k_2-k_3} dk_1 dk_2 dk_3 + \ldots$$

with

$$\tilde{B}_{kk_1}^{k_2k_3} = i\frac{\tilde{T}_{kk_1}^{k_2k_3} - T_{kk_1}^{k_2k_3}}{\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}}. \quad (5)$$

This transformation replaces $T_{kk_1}^{k_2k_3}$ by $\tilde{T}_{kk_1}^{k_2k_3}$ in (3). Coefficient $\tilde{T}_{kk_1}^{k_2k_3}$ can be any function having the same values on the four-wave resonant manifold:

$$\begin{aligned} k + k_1 &= k_2 + k_3, \\ \omega_k + \omega_{k_1} &= \omega_{k_2} + \omega_{k_3}. \end{aligned} \quad (6)$$

In [7, 8], the choice of $\tilde{T}_{kk_1}^{k_2k_3}$ allowed obtaining the Hamiltonian in a compact way. However, it was shown in [9, 10] that the best choice for $\tilde{T}_{kk_1}^{k_2k_3}$ is the following:

$$\begin{aligned} \tilde{T}_{k_2k_3}^{kk_1} = \frac{(kk_1k_2k_3)^{1/2}}{2\pi} \min(k, k_1, k_2, k_3) \\ \times \theta_k \theta_{k_1} \theta_{k_2} \theta_{k_3}, \end{aligned} \quad (7)$$

where $\theta_k = \theta(k)$ is the Heaviside step function.

The Hamiltonian can be written in the x-space:

$$\begin{aligned} H &= \int c^* \hat{V} c dx \\ &+ \frac{1}{2}\int\left[\frac{i}{4}\left(c^2 \frac{\partial}{\partial x} c^{*2} - c^{*2} \frac{\partial}{\partial x} c^2\right) - |c|^2 \hat{K}(|c|^2)\right] dx. \end{aligned} \quad (8)$$

Here, the operator $\hat{V}$ in k-space is so that $V_k = \omega_k/k$. When introducing along with this the Gardner–Zakharov–Faddeev bracket

$$\partial_x^+ \Leftrightarrow ik\theta_k \quad (9)$$

the equation of motion becomes the following:

$$\frac{\partial c}{\partial t} + \partial_x^+ \frac{\delta H}{\delta c^*} = 0. \quad (10)$$

Introducing advection velocity

$$\mathcal{U} = \hat{K}|c|^2 \quad (11)$$

and taking variational derivative, one can write Eq. (10) in the form

$$\frac{\partial c}{\partial t} + i\hat{\omega} c - i\partial_x^+\left(|c|^2 \frac{\partial c}{\partial x}\right) = \partial_x^+(\mathcal{U} c). \quad (12)$$

3. DERIVE SPATIAL EQUATION

Equation (12) for water waves can be written in k-space:

$$\begin{aligned} i\dot{c}_k &= \omega_k c_k \\ &+ \frac{k\theta_k}{2\pi}\int T_{k_2k_3}^{kk_1} c_{k_1}^* c_{k_2} c_{k_3} \delta_{k+k_1-k_2-k_3} dk_1 dk_2 dk_3, \\ T_{k_2k_3}^{kk_1} &= \min(k, k_1, k_2, k_3). \end{aligned} \quad (13)$$

Performing Fourier transformation over time and multiplying the result by $\omega + \omega_k$ one can easily get:

$$\begin{aligned} (\omega^2 - gk)c_{k\omega} &= \frac{(\omega + \omega_k)k\theta_k}{(2\pi)^2} \\ &\times \int T_{k_2k_3}^{kk_1} c_{k_1\omega_1}^* c_{k_2\omega_2} c_{k_3\omega_3} \\ &\times \delta_{k+k_1-k_2-k_3} \delta_{\omega+\omega_1-\omega_2-\omega_3} dk_1 dk_2 dk_3 d\omega_1 d\omega_2 d\omega_3. \end{aligned} \quad (14)$$

For the waves with small amplitudes, all harmonics $c_{k\omega}$ are focused near the dispersion curve:

$$\omega = \sqrt{gk} + \tilde{\omega}_{nl}. \quad (15)$$

Here, $\tilde{\omega}_{nl}$ is the nonlinear frequency shift. Obviously,

$$\tilde{\omega}_{nl} \sim c^2.$$

Thus, gk on the right-hand side of Eq. (14) can be replaced by ω^2. The inclusion of $\tilde{\omega}_{nl}$ would give terms of higher order in Eq. (14). Such terms should be omitted. Therefore,

$$\begin{aligned} (\omega^2 - gk)c_{k\omega} &= \frac{2\omega^3}{g^2}\frac{1}{(2\pi)^2} \\ &\times \int T_{\omega_2^2\omega_3^2}^{\omega^2\omega_1^2} c_{k_1\omega_1}^* c_{k_2\omega_2} c_{k_3\omega_3} \\ &\times \delta_{k+k_1-k_2-k_3} \delta_{\omega+\omega_1-\omega_2-\omega_3} dk_1 dk_2 dk_3 d\omega_1 d\omega_2 d\omega_3. \end{aligned} \quad (16)$$

Now we can perform backward Fourier transformation of the Eq. (16) over space and get spatial equation for water waves:

$$\begin{aligned} \frac{\partial}{\partial x} c_\omega - i\frac{\omega^2}{g} c_\omega &= -\frac{2\omega^3}{g^3}\frac{i}{2\pi} \\ &\times \int T_{\omega_2^2\omega_3^2}^{\omega^2\omega_1^2} c_{\omega_1}^* c_{\omega_2} c_{\omega_3} \delta_{\omega+\omega_1-\omega_2-\omega_3} d\omega_1 d\omega_2 d\omega_3. \end{aligned}$$

This equation can be written in the Hamiltonian form

$$\frac{\partial}{\partial x} c_\omega = i\omega^3 \frac{\delta H}{\delta c_\omega^*}$$

with the third-order bracket

$$i\omega^3 \leftrightarrow \frac{\partial^3}{\partial t^3}$$

and the Hamiltonian

$$H = \frac{1}{g}\int \frac{1}{\omega}|c_\omega|^2 d\omega - \frac{1}{2\pi}\frac{1}{g^3}$$
$$\times \int T^{\omega^2\omega_1^2}_{\omega_2^2\omega_3^2} c^*_\omega c^*_{\omega_1} c_{\omega_2} c_{\omega_3} \delta_{\omega+\omega_1-\omega_2-\omega_3} d\omega d\omega_1 d\omega_2 d\omega_3.$$

Explicit form of $T^{\omega^2\omega_1^2}_{\omega_2^2\omega_3^2}$ in (13) is the following:

$$T^{\omega_k^2\omega_{k_1}^2}_{\omega_{k_2}^2\omega_{k_3}^2} = \frac{1}{4}(\omega_k^2 + \omega_{k_1}^2 + \omega_{k_2}^2 + \omega_{k_3}^2$$
$$- |\omega_k^2 - \omega_{k_2}^2| - |\omega_k^2 - \omega_{k_3}^2| - |\omega_{k_1}^2 - \omega_{k_2}^2| - |\omega_{k_1}^2 - \omega_{k_3}^2|)$$

and it allows the compact form of the quartic part H_{int} of the Hamiltonian:

$$H_{\text{int}} = \frac{1}{2g^3}\int |c|^2(\ddot{c}^*c + \ddot{c}c^*)dt$$
$$+ \frac{i}{g^2}\int |c|^2 \hat{H}(\dot{c}c^* + \dot{c}c^*)dt.$$

Using the relation

$$\hat{\omega} = \hat{H}\frac{\partial}{\partial t}\ (\hat{H} \text{ is the Hilbert transformation}),$$

the fourth-order part of the Hamiltonian can be written as

$$H_{\text{int}} = \frac{1}{2g^3}\int |c|^2(\ddot{c}^*c + \ddot{c}c^*)dt$$
$$+ \frac{i}{g^2}\int |c|^2 \hat{H}(\ddot{c}c^* + c\ddot{c}^*)dt.$$

The equation of motion has the form

$$\frac{\partial}{\partial x}c = \frac{\partial^3}{\partial t^3}\frac{\delta H}{\delta c^*} \tag{17}$$

or in t-space:

$$\frac{\partial}{\partial x}c + \frac{i}{g}\frac{\partial^2}{\partial t^2}c$$
$$= \frac{1}{2g^3}\frac{\partial^3}{\partial t^3}\left[\frac{\partial^2}{\partial t^2}(|c|^2 c) + 2|c|^2\ddot{c} + \ddot{c}^* c^2\right] \tag{18}$$
$$+ \frac{i}{g^3}\frac{\partial^3}{\partial t^3}\left[\frac{\partial}{\partial t}(c\hat{\omega}|c|^2) + \dot{c}\hat{\omega}|c|^2 + c\hat{\omega}(\dot{c}c^* - c\dot{c}^*)\right].$$

Consequently, Eq. (18) with the Hamiltonian

$$H = \frac{1}{g}\int \frac{1}{\omega}|c_\omega|^2 d\omega$$
$$+ \frac{1}{2g^3}\int |c|^2(\ddot{c}^*c + \ddot{c}c^*)dt + \frac{i}{g^2}\int |c|^2\hat{\omega}(\dot{c}c^* + \dot{c}c^*)dt \tag{19}$$

solves the spatial Cauchy problem for surface gravity waves on deep water.

4. BACK TO η AND ψ

According to canonical transformation η_k and ψ_k are power series of c_k up to the third order:

$$\eta_k = \eta_k^{(1)} + \eta_k^{(2)} + \eta_k^{(3)}, \quad \psi_k = \psi_k^{(1)} + \psi_k^{(2)} + \psi_k^{(3)}. \tag{20}$$

Details of the recovering physical quantities $\eta(x, t)$ and $\psi(x, t)$ are given in [9, 11]. Here, we focus on the η only. Obviously,

$$\eta_k^{(1)} = \frac{1}{2\omega_k}[c_k + c^*_{-k}],$$

or

$$\eta^{(1)}(x,t) = \frac{1}{\sqrt{2}g^{1/4}}[\hat{k}^{1/4}c(x,t) + \hat{k}^{-1/4}c(x,t)^*].$$

Operators $\hat{k}^\alpha$ act in Fourier space as multiplication by $|k|^\alpha$. Following [9, 11], let us consider transformation for η taking into account only first and second order terms.

Then,

$$\eta^{(2)}(x,t) = \frac{\hat{k}}{4\sqrt{g}}[\hat{k}^{-1/4}c(x,t) - \hat{k}^{-1/4}c^*(x,t)]^2. \tag{21}$$

Using approximate relation (15), one can get the following compact formula for the physical observed value η:

$$\eta(x,t) = \frac{\hat{\omega}^{-1/2}}{\sqrt{2}}[c(x,t) + c^*(x,t)]$$
$$- \frac{\partial^2}{\partial t^2}\frac{1}{4g}[\hat{\omega}^{-1/2}[c(x,t) - c^*(x,t)]]^2 + \ldots.$$

5. FREQUENCY NARROW BAND APPROXIMATION

From Eq. (18), one can easily derive equation for envelope of modulated wave train. Obviously, such a wave train propagates with the group velocity and it is convenient to introduce reference system moving with this velocity. So, let $c(x, t)$ be almost monochromatic wave with the frequency ω_0:

$$c(x,t) = C\left(x, t - \frac{x}{v_g}\right)e^{i(k_0x - \omega_0 t)},$$
$$\omega_0 = \sqrt{gk_0}, \quad v_g = \frac{\omega_0}{2k_0}, \tag{22}$$

where capital $C(x, t)$ is a slowly varying function. Plugging (22) into the motion Eq. (18), and keeping in the nonlinear part of the equation term with no more the first time derivative, one can derive the equation

$$\frac{\partial}{\partial x}C + \frac{i}{g}\frac{\partial^2}{\partial t^2}C + \frac{2i\omega_0^5}{g^3}|C|^2C = \frac{4\omega_0^4}{g^3}\left[4|C|^2\dot{C} + \frac{3}{2}C^2\dot{C}^* + iC\hat{\omega}|C|^2\right]. \quad (23)$$

This is Dysthe equation for spatial Cauchy problem. Dropping the small corrections, namely the right-hand side, we end up just with the nonlinear Schrödinger equation. So, we have now both full dynamical equation (18) and envelope approximation (23). The Hamiltonian of the NLSE is the following:

$$H = \frac{1}{g}\int\left[|\dot{C}|^2 - \frac{\omega_0^5}{g^2}|C|^4\right]dt$$

and the equation of motion is:

$$\frac{\partial}{\partial x}C = i\frac{\partial H}{\partial C^*}.$$

6. CONCLUSIONS

The spatial compact equation is the most convenient tool for comparison of the theory of nonlinear gravity waves on deep water and their experimental study in laboratory wave tanks. It can be easily solved numerically by the use of spectral code. We plan to present the results of our numerical simulations shortly.

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