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A dynamic equation for water waves in one horizontal dimension

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ABSTRACT

We apply a canonical transformation to a water wave equation to remove cubic nonlinear terms and to drastically simplify fourth-order terms in the Hamiltonian. This transformation explicitly uses the vanishing exact four-wave interaction for water gravity waves for a 2D potential fluid. After transformation, the well-known but cumbersome Zakharov equation is drastically simplified and can be written in X-space in a compact form. This new equation is very suitable for analytical studies and numerical simulations.

1. Introduction

The work described here was motivated by two remarkable facts regarding the hydrodynamics of a one-dimensional free surface:

- We previously showed that the four-wave interaction coefficient vanishes on the resonant manifold [1]

$$k + k_1 = k_2 + k_3,$$
$$\omega_k + \omega_{k_1} = \omega_{k_2} + \omega_{k_3}.$$

- We have also demonstrated that a giant breather that is highly nonlinear exists on the fluid surface in the absence of radiation [2,3]. Moreover, the space–time spectrum of the breather consists of waves propagating in the same direction.

These two facts indicate that fourth-order wave interactions can be drastically simplified by some canonical transformation of the Hamiltonian. Below we show the form of this transformation. The dynamic equation derived using this transformation is very elegant and simple and can easily be generalized for "almost one-dimensional waves.

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2. Zakharov's equation

The one-dimensional potential flow of an ideal incompressible fluid with a free surface in a gravity field is described by the following set of equations:

$$\phi_{xx} + \phi_{zz} = 0 \quad (\phi_z \to 0, z \to -\infty),$$
$$\eta_t + \eta_x \phi_x = \phi_z|_{z=\eta}$$
$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + g\eta = 0|_{z=\eta},$$

where $\eta(x, t)$ is the shape of the surface, $\phi(x, z, t)$ is the potential function for the flow and g is gravitational acceleration. As previously shown [4], the variables $\eta(x, t)$ and $\psi(x, t) = \phi(x, z, t)|_{z=\eta}$ are canonically conjugated and satisfy the equations

$$\frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \eta} \qquad \frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \psi},$$

where $H = K + U$ is the total energy of the fluid with the following kinetic and potential energy terms:

$$K = \frac{1}{2}\int dx \int_{-\infty}^{\eta} v^2 dz \qquad U = \frac{g}{2}\int \eta^2 dx.$$

It is convenient to introduce a normal complex variable a_k:

$$\eta_k = \sqrt{\frac{\omega_k}{2g}}(a_k + a^*_{-k}) \qquad \psi_k = -i\sqrt{\frac{g}{2\omega_k}}(a_k - a^*_{-k}),$$

where $\omega_k = \sqrt{gk}$ is the dispersion law for the gravity waves, and the Fourier transformations $\psi(x) \to \psi_k$ and $\eta(x) \to \eta_k$ are defined as follows:

$$f_k = \frac{1}{\sqrt{2\pi}} \int f(x) e^{-ikx} dx, \qquad f(x) = \frac{1}{\sqrt{2\pi}} \int f_k e^{+ikx} dk.$$

The Hamiltonian can be expanded in an infinite series in powers of a_k [4,5]:

$$H = H_2 + H_3 + H4 + \cdots.$$

This variable a_k satisfies the equation

$$\frac{\partial a_k}{\partial t} + i \frac{\delta H}{\delta a_k^*} = 0,$$

where

$$H_2 = \int \omega_k a_k a_k^* dk,$$

$$\begin{aligned} H_3 &= \int V^k_{k_1 k_2} \{a_k^* a_{k_1} a_{k_2} + a_k a_{k_1}^* a_{k_2}^*\} \delta_{k-k_1-k_2} dk dk_1 dk_2 \\ &\quad + \frac{1}{3} \int U_{k k_1 k_2} \{a_k a_{k_1} a_{k_2} + a_k^* a_{k_1}^* a_{k_2}^*\} \delta_{k+k_1+k_2} dk dk_1 dk_2 \end{aligned}$$

$$\begin{aligned} V^k_{k_1 k_2} &= \frac{g^{\frac14}}{8\sqrt{\pi}} \left(\left(\frac{k}{k_1 k_2} \right)^{\frac14} L_{k_1 k_2} \right. \\ &\quad \left. - \left(\frac{k_2}{k k_1} \right)^{\frac14} L_{-k k_1} - \left(\frac{k_1}{k k_2} \right)^{\frac14} L_{-k k_2} \right) \end{aligned}$$

$$\begin{aligned} U_{k k_1 k_2} &= \frac{g^{\frac14}}{8\sqrt{\pi}} \left(\left(\frac{k}{k_1 k_2} \right)^{\frac14} L_{k_1 k_2} \right. \\ &\quad \left. + \left(\frac{k_2}{k k_1} \right)^{\frac14} L_{k k_1} + \left(\frac{k_1}{k k_2} \right)^{\frac14} L_{k k_2} \right) \end{aligned}$$

$$L_{k k_1} = (\vec{k}\vec{k}_1) + |k||k_1|.$$

The fourth-order part of Hamiltonian is:

$$\begin{aligned} H_4 &= \frac12 \int W^{k_3 k_4}_{k_1 k_2} a_{k_1}^* a_{k_2}^* a_{k_3} a_{k_4} \delta_{k_1+k_2-k_3-k_4} dk_1 dk_2 dk_3 dk_4 \\ &\quad + \frac13 \int G^{k_4}_{k_1 k_2 k_3} (a_{k_1}^* a_{k_2}^* a_{k_3}^* a_{k_4} + a_{k_1} a_{k_2} a_{k_3} a_{k_4}^*) \\ &\quad \times \delta_{k_1+k_2+k_3-k_4} dk_1 dk_2 dk_3 dk_4 \\ &\quad + \frac{1}{12} \int R_{k_1 k_2 k_3 k_4} (a_{k_1}^* a_{k_2}^* a_{k_3}^* a_{k_4}^* + a_{k_1} a_{k_2} a_{k_3} a_{k_4}) \\ &\quad \times \delta_{k_1+k_2+k_3+k_4} dk_1 dk_2 dk_3 dk_4, \end{aligned}$$

were $W^{k_3 k_4}_{k_1 k_2}$, $G^{k_4}_{k_1 k_2 k_3}$ and $R_{k_1 k_2 k_3 k_4}$ are equal to:

$$\begin{aligned} W^{k_3 k_4}_{k_1 k_2} &= \frac{-1}{32\pi} \left[M^{k_1 k_2}_{-k_3-k_4} + M^{-k_3-k_4}_{k_1 k_2} - M^{k_1-k_3}_{k_2-k_4} - M^{k_2-k_3}_{k_1-k_4} \right. \\ &\quad \left. - M^{k_1-k_4}_{k_2-k_3} - M^{k_2-k_4}_{k_1-k_3} \right] \end{aligned}$$

$$\begin{aligned} G^{k_4}_{k_1 k_2 k_3} &= \frac{-1}{32\pi} \left[M^{k_3-k_4}_{k_1 k_2} + M^{k_2-k_4}_{k_1 k_3} + M^{k_1-k_4}_{k_2 k_3} - M^{k_1 k_2}_{k_3-k_4} \right. \\ &\quad \left. - M^{k_1 k_3}_{k_2-k_4} - M^{k_2 k_3}_{k_1-k_4} \right] \end{aligned}$$

$$\begin{aligned} R_{k_1 k_2 k_3 k_4} &= \frac{-1}{32\pi} \left[M^{k_3 k_4}_{k_1 k_2} + M^{k_2 k_4}_{k_1 k_3} + M^{k_2 k_3}_{k_1 k_4} \right. \\ &\quad \left. + M^{k_1 k_4}_{k_2 k_3} + M^{k_1 k_3}_{k_2 k_4} + M^{k_1 k_2}_{k_3 k_4} \right], \end{aligned}$$

where

$$\begin{aligned} M^{k_3 k_4}_{k_1 k_2} &= |k_1 k_2|^{\frac34} |k_3 k_4|^{\frac14} (|k_1 + k_3| + |k_1 + k_4| \\ &\quad + |k_2 + k_3| + |k_2 + k_4| - 2|k_1| - 2|k_2|). \end{aligned}$$

Now we can apply canonical transformation from variables a_k to b_k to exclude nonresonant cubic terms and nonresonant fourth-order terms with coefficients $G^{k_4}_{k_1 k_2 k_3}$ and $R_{k_1 k_2 k_3 k_4}$. This transformation up to accuracy $O(b^5)$ has the form [4,6,7]:

$$\begin{aligned} a_k &= b_k + \int \Gamma^k_{k_1 k_2} b_{k_1} b_{k_2} \delta_{k-k_1-k_2} dk_1 dk_2 \\ &\quad - 2 \int \Gamma^{k_2}_{k k_1} b_{k_1}^* b_{k_2} \delta_{k+k_1-k_2} dk_1 dk_2 \\ &\quad + \int \Gamma_{k k_1 k_2} b_{k_1}^* b_{k_2}^* \delta_{k+k_1+k_2} dk_1 dk_2 \\ &\quad + \int B^{k_2 k_3}_{k k_1} b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk_1 dk_2 dk_3 \\ &\quad + \int C^{k_3}_{k k_1 k_2} b_{k_1}^* b_{k_2}^* b_{k_3} \delta_{k+k_1+k_2-k_3} dk_1 dk_2 dk_3 \\ &\quad + \int S_{k k_1 k_2 k_3} b_{k_1}^* b_{k_2}^* b_{k_3}^* \delta_{k+k_1+k_2+k_3} dk_1 dk_2 dk_3, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} B^{k_2 k_3}_{k k_1} &= \Gamma^{k_1}_{k_1-k_2} \Gamma^{k_3}_{k k_3-k} + \Gamma^{k_1}_{k_3 k_1-k_3} \Gamma^{k_2}_{k k_2-k} - \Gamma^{k}_{k_2 k-k_2} \Gamma^{k_3}_{k_1 k_3-k_1} \\ &\quad - \Gamma^{k_1}_{k_3 k_1-k_3} \Gamma^{k_2}_{k_1 k_2-k_1} - \Gamma^{k+k_1}_{k k_1} \Gamma^{k_2+k_3}_{k_2 k_3} \\ &\quad + \Gamma_{-k-k_1 k k_1} \Gamma_{-k_2-k_3 k_2 k_3} + \tilde{B}^{k k_1}_{k_2 k_3}, \end{aligned}$$

$$\Gamma^k_{k_1 k_2} = -\frac{V^k_{k_1 k_2}}{\omega_k - \omega_{k_1} - \omega_{k_2}}, \qquad \Gamma_{k k_1 k_2} = -\frac{U_{k k_1 k_2}}{\omega_k + \omega_{k_1} + \omega_{k_2}}$$

and $\tilde{B}^{k k_1}_{k_2 k_3}$ is an arbitrary function satisfying the following symmetry conditions:

$$\tilde{B}^{k k_1}_{k_2 k_3} = \tilde{B}^{k_1 k}_{k_2 k_3} = \tilde{B}^{k k_1}_{k_3 k_2} = -(\tilde{B}^{k_2 k_3}_{k k_1})^*.$$

Coefficients $C^{k_3}_{k k_1 k_2}$ and $S_{k k_1 k_2 k_3}$ provide vanishing corresponding fourth-order terms in the new Hamiltonian.

After transformation of (2.1) the Hamiltonian takes the following form:

$$\begin{aligned} H &= \int \omega_k b_k b_k^* dk \\ &\quad + \frac12 \int (T_{k k_1, k_2 k_3} - (\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \tilde{B}^{k_2 k_3}_{k k_1}) \\ &\quad \times b_k^* b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk dk_1 dk_2 dk_3 + \cdots. \end{aligned} \tag{2.2}$$

If $\tilde{B}^{k_2 k_3}_{k k_1} = 0$, Eq. (2.2) is known as Zakharov's equation. Here $T_{k k_1, k_2 k_3}$ satisfies the symmetry conditions:

$$T_{k k_1, k_2 k_3} = T_{k_1 k, k_2 k_3} = T_{k k_1, k_3 k_2} = T_{k_2 k_3 k k_1}$$

and has the form:

$$\begin{aligned} T_{k k_1, k_2 k_3} &= W_{k_1 k, k_2 k_3} - V_{k k_2 k-k_2} V_{k_3 k_1 k_3-k_1} \left[\frac{1}{\omega_{k_2} + \omega_{k-k_2} - \omega_k} \right. \\ &\quad \left. + \frac{1}{\omega_{k_1} + \omega_{k_3-k_1} - \omega_{k_3}} \right] - V_{k_1 k_2 k_1-k_2} V_{k_3 k k_3-k} \\ &\quad \times \left[\frac{1}{\omega_{k_2} + \omega_{k_1-k_2} - \omega_{k_1}} + \frac{1}{\omega_k + \omega_{k_3-k} - \omega_{k_3}} \right] \\ &\quad - V_{k k_3 k-k_3} V_{k_2 k_1 k_2-k_1} \left[\frac{1}{\omega_{k_3} + \omega_{k-k_3} - \omega_k} \right. \\ &\quad \left. + \frac{1}{\omega_{k_1} + \omega_{k_2-k_1} - \omega_{k_2}} \right] - V_{k_1 k_3 k_1-k_3} V_{k_2 k k_2-k} \\ &\quad \times \left[\frac{1}{\omega_{k_3} + \omega_{k_1-k_3} - \omega_{k_1}} + \frac{1}{\omega_k + \omega_{k_2-k} - \omega_{k_2}} \right] \end{aligned}$$

$$- V_{k+k_1kk_1}V_{k_2+k_3k_2k_3}\left[\frac{1}{\omega_{k+k_1}-\omega_k-\omega_{k_1}}\right.$$
$$\left.+\frac{1}{\omega_{k_2+k_3}-\omega_{k_2}-\omega_{k_3}}\right] - U_{-k-k_1kk_1}U_{-k_2-k_3k_2k_3}$$
$$\times\left[\frac{1}{\omega_{k+k_1}+\omega_k+\omega_{k_1}}+\frac{1}{\omega_{k_2+k_3}+\omega_{k_2}+\omega_{k_3}}\right]. \quad (2.3)$$

Now we address the key point of this article: appropriate choice of $\tilde{B}^{kk_1}_{k_2k_3}$ in (2.2) is based on two items:

(1) The coefficient T_{kk_1,k_2k_3} is identically equal to zero for the resonant manifold [1]:

$$k+k_1 = k_2+k_3,$$
$$\omega_k+\omega_{k_1} = \omega_{k_2}+\omega_{k_3}, \quad (2.4)$$

with nontrivial solution:

$$k = a(1+\zeta)^2,$$
$$k_1 = a(1+\zeta)^2\zeta^2,$$
$$k_2 = -a\zeta^2,$$
$$k_3 = a(1+\zeta+\zeta^2)^2, \quad (2.5)$$

where $0<\zeta<1$ and $a>0$. This is the only nontrivial solution for (2.4).

(2) We also consider waves moving in the same direction, which allows us to consider only positive wavenumbers k. This assumption came from numerical simulations [2,3].

This fact and observation allow a drastic simplification of the Hamiltonian. We can remove the cumbersome expression for T_{kk_1,k_2k_3} in (2.2) and retain only its diagonal part, which corresponds to trivial four-wave scattering:

$$k_2 = k_1, \qquad k_3 = k, \qquad \text{or} \qquad k_2 = k, \qquad k_3 = k_1. \quad (2.6)$$

This is equal to

$$T_{kk_1} = T^{kk_1}_{kk_1} = \frac{1}{4\pi}|k||k_1|\left(|k+k_1|-|k-k_1|\right). \quad (2.7)$$

Using this diagonal part, we can construct the following function (with tilde):

$$\tilde{T}^{kk_1}_{k_2k_3} = \left[\frac{1}{2}(T_{kk_2}+T_{kk_3}+T_{k_1k_2}+T_{k_1k_3})\right. \quad (2.8)$$

$$\left.-\frac{1}{4}(T_{kk}+T_{k_1k_1}+T_{k_2k_2}+T_{k_3k_3})\right]\theta(k)\theta(k_1)\theta(k_2)\theta(k_3), \quad (2.9)$$

$$\theta(x) = \begin{cases} 0, & \text{if } x<0;\\ 1, & \text{if } x>0.\end{cases}$$

Consider the case in which all the waves move in the same direction. This means that all k values have the same sign. Thus, let $k_i>0$, and

$b \simeq e^{i(kx-\omega t)}$.

Then $\tilde{T}^{kk_1}_{k_2k_3}$ can be significantly simplified, and the modulus for $|k+k_1|$, $|k|$ and $|k_1|$ can be dropped. Now

$$T_{kk_1} = \frac{1}{4\pi}kk_1\left(k+k_1-|k-k_1|\right).$$

Simple calculations yield

$$\tilde{T}^{kk_1}_{k_2k_3} = \left[-\frac{1}{8\pi}(kk_2|k-k_2|+kk_3|k-k_3|+k_1k_2|k_1-k_2|\right.$$
$$\left.+k_1k_3|k_1-k_3|)+\frac{1}{8\pi}(kk_1(k+k_1)+k_2k_3(k_2+k_3))\right]$$
$$\times\theta(k)\theta(k_1)\theta(k_2)\theta(k_3). \quad (2.10)$$

3. Compact equation

We choose $\tilde{B}^{kk_1}_{k_2k_3}$ as follows:

$$\tilde{B}^{kk_1}_{k_2k_3} = \frac{T^{kk_1}_{k_2k_3}-\tilde{T}^{kk_1}_{k_2k_3}}{\omega_k+\omega_{k_1}-\omega_{k_2}-\omega_{k_3}}. \quad (3.11)$$

This makes the four-wave coefficient in (2.2) equal to $\tilde{T}^{kk_1}_{k_2k_3}$. Note that Eq. (3.11) for $\tilde{B}^{kk_1}_{k_2k_3}$ has no singularity for the resonance manifold (2.4) because $T^{kk_1}_{k_2k_3}$ vanishes on that manifold. Using the following relations for $\hat{K}$ and the space derivative,

$$kb^*_k \Leftrightarrow i\frac{\partial}{\partial x}b^*(x), \quad (3.12)$$

$$kb_k \Leftrightarrow -i\frac{\partial}{\partial x}b(x), \quad (3.13)$$

$$|k-k_2|b^*_kb_{k_2} \Leftrightarrow \hat{K}(|b(x)|^2), \quad (3.14)$$

the Hamiltonian can easily be written in X-space:

$$\mathcal{H} = \int b^*(g\hat{K})^{1/2}b\,dx + \frac{i}{16}$$
$$\times\int\left[b^{*2}\frac{\partial}{\partial x}(b'^2) - b^2\frac{\partial}{\partial x}(b^{*\prime 2})\right]dx$$
$$-\frac{1}{4}\int |b|^2\cdot\hat{K}(|b'|^2)dx, \quad (3.15)$$

where $b' = \frac{\partial}{\partial x}b$. After integrating by parts, the Hamiltonian takes the form:

$$\mathcal{H} = \int b^*(g\hat{K})^{1/2}b\,dx + \frac{1}{4}\int |b'|^2$$
$$\times\left[\frac{i}{2}(bb'^* - b^*b') - \hat{K}|b|^2\right]dx. \quad (3.16)$$

The corresponding equation of motion is:

$$i\frac{\partial b}{\partial t} = (g\hat{K})^{1/2}b + \frac{i}{8}\left[b^*\frac{\partial}{\partial x}(b'^2) - \frac{\partial}{\partial x}\left(b^{*\prime}\frac{\partial}{\partial x}b^2\right)\right]$$
$$-\frac{1}{4}\left[b\cdot\hat{K}(|b'|^2) - \frac{\partial}{\partial x}(b'\hat{K}(|b|^2))\right]. \quad (3.17)$$

4. Some solutions

4.1. Monochromatic wave

A monochromatic wave

$$b(x) = B_0e^{i(k_0x-\omega_0t)} \quad (4.18)$$

is the simplest solution of (3.17). Indeed, substituting (4.18) into Eq. (3.17) yields the following relation:

$$\omega_0 = \omega_{k_0} + \frac{1}{2}k_0^3|B_0|^2. \quad (4.19)$$

Recalling the transformation from a_k to b_k, we can see that for waves with small amplitude ($a_k \simeq b_k$)

$$|B_0|^2 = \frac{\omega_{k_0}}{k_0}\eta_0^2$$

and relation (4.19) coincides with the well-known Stokes correction to the frequency due to finite wave amplitude.

$$\omega_0 = \omega_{k_0}\left(1+\frac{1}{2}k_0^2|\eta_0|^2\right). \quad (4.20)$$

4.2. Modulational instability of a monochromatic wave

We consider a perturbation to the solution

$$b = B_0 e^{i(k_0 x - \omega_0 t)},$$

where

$$B_0 = \frac{1}{\sqrt{2\pi}} \int b_{k_0} e^{i(k_0 x - kx)} dx$$

and

$$\omega_0 = \omega_{k_0} + \frac{1}{2}|B_0|^2 k_0^3, \qquad \frac{1}{4\pi}|b_{k_0}|^2 k_0^3 = \frac{1}{2} T_{k_0 k_0}^{k_0 k_0} |b_{k_0}|^2.$$

The perturbed solution has the following form:

$$b \Rightarrow (b_{k_0} + \delta b_{k_0+k} e^{-i\Omega_k t} + \delta b_{k_0-k} e^{-i\Omega_{-k} t}) e^{-i\omega_0 t} \tag{4.21}$$

with the following condition:

$$\Omega_k = -\Omega_{-k}.$$

Substituting the perturbed solution (4.21) into the equation

$$i\dot{b}_k = \omega_k b_k + \frac{1}{2} \int \tilde{T}_{k_2 k_3}^{k k_1} b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk_1 dk_2 dk_3,$$

we obtain the sum of two independent equations:

$$\Big[i\delta\dot{b}_{k_0+k} + (\omega_0 + \Omega_k)\delta b_{k_0+k} = \omega_{k_0+k}\delta b_{k_0+k}$$
$$+ \tilde{T}_{k_0+k k_0}^{k_0+k k_0} |b_{k_0}|^2 \delta b_{k_0+k} + \frac{1}{2}\tilde{T}_{k_0 k_0}^{k_0+k k_0-k} b_{k_0}^2 \delta b_{k_0-k}^* \Big]$$
$$\times e^{-i\omega_0 t - i\Omega_k t} + \Big[i\delta\dot{b}_{k_0-k} + (\omega_0 + \Omega_{-k})\delta b_{k_0-k} = \omega_{k_0-k}\delta b_{k_0-k}$$
$$+ \tilde{T}_{k_0-k k_0}^{k_0-k k_0} |b_{k_0}|^2 \delta b_{k_0-k} + \frac{1}{2}\tilde{T}_{k_0 k_0}^{k_0-k k_0+k} b_{k_0}^2 \delta b_{k_0+k}^* \Big] e^{-i\omega_0 t - i\Omega_{-k} t}.$$

Expressions for $\tilde{T}_{k_0+k k_0}^{k_0+k k_0}$ and $T_{k_0 k_0}^{k_0-k k_0+k}$ can be easily obtained from (2.10):

$$\tilde{T}_{k_0+k k_0}^{k_0+k k_0} = \frac{k_0^3}{4\pi} + \frac{k_0(3k_0 - |k|)}{4\pi}k + \frac{k_0}{4\pi}(k_0^2 - k_0|k| + k^2), \tag{4.22}$$

$$T_{k_0 k_0}^{k_0-k k_0+k} = \frac{k_0}{2\pi}\left(k_0^2 - k_0|k| - \frac{k^2}{2}\right). \tag{4.23}$$

Considering even and odd powers of k, we can see that

$$\Omega_k = \frac{\omega_{k_0+k} - \omega_{k_0-k}}{2} + \frac{|B_0|^2}{2}(3k_0 - |k|)k.$$

We denote

$$d(k) = \frac{\omega_{k_0+k} - 2\omega_{k_0} + \omega_{k_0-k}}{2}.$$

Then

$$i\delta\dot{b}_{k_0+k} = d(k)\delta b_{k_0+k} + \frac{|B_0|^2 k_0}{2}(k_0^2 - k_0|k| + k^2)\delta b_{k_0+k}$$
$$+ \frac{B_0^2 k_0}{2}\left(k_0^2 - k_0|k| - \frac{k^2}{2}\right)\delta b_{k_0-k}^*.$$

Suppose δb_{k_0+k} grows according to

$$\delta b_{k_0+k} \Rightarrow \delta b_{k_0+k} e^{\gamma_k t}.$$

Then we can easily obtain the following formula for γ_k:

$$\gamma_k^2 = \left[-d(k) - \frac{3|B_0|^2}{4}k_0 k^2\right]$$
$$\times \left[d(k) + |B_0|^2 k_0 \left(k_0 - \frac{|k|}{2}\right)^2\right]. \tag{4.24}$$

If we introduce the steepness of the carrier wave $\omega_{k_0}\mu^2 = |B_0|^2 k_0^2$ and approximate $d(k)$ as

$$d(k) \simeq -\frac{1}{8}\omega_{k_0}'' k^2 = -\frac{1}{8}\omega_{k_0}\frac{k^2}{k_0^2},$$

then the growth rate is

$$\gamma_k^2 = \frac{1}{8}\frac{\omega_{k_0}^2}{k_0^4}(1 - \mathbf{6}\mu^2)k^2\left[\mu^2\left(k_0 - \frac{|\mathbf{k}|}{\mathbf{2}}\right)^2 - \frac{k^2}{8}\right].$$

The difference between this formula and the well-known expression derived from the nonlinear Schrodinger equation is highlighted by the two terms in bold.

4.3. Breathers

An equation for the amplitude of the wave train can easily be derived from (3.17). We introduce envelope $B(x,t)$ so that

$$b(x,t) = B(x,t)e^{i(k_0 x - \omega_0 t)}. \tag{4.25}$$

$B(x,t)$ is also a normal Hamiltonian variable and the Hamiltonian takes the form:

$$\mathcal{H} = \int B^*(\hat{\omega}_{k_0+k} - \omega_{k_0})B dx + \frac{1}{4}\int |B' + ik_0 B|^2$$
$$\times \left[\frac{i}{2}(B(B'^* - ik_0 B^*) - B^*(B' + ik_0 B)) - \hat{K}|B|^2\right] dx,$$

where the operator $\hat{\omega}_{k_0+k}$ acts in k-space as $\sqrt{g(k_0+k)}$.

It is convenient to introduce the following operator $\hat{S}$:

$$\hat{S}B = \frac{\partial}{\partial x}B + ik_0 B.$$

Then the Hamiltonian can be written as:

$$\mathcal{H} = \int B^*(\hat{\omega}_{k_0+k} - \omega_{k_0})B dx + \frac{1}{4}\int |\hat{S}B|^2$$
$$\times \left[\frac{i}{2}(B(\hat{S}B)^* - B^*\hat{S}B) - \hat{K}|B|^2\right] dx. \tag{4.26}$$

A variation of the Hamiltonian (4.26) dynamic equation for $B(x,t)$ is:

$$i\dot{B} = (\hat{\omega}_{k_0+k} - \omega_{k_0})B - \frac{1}{4}\hat{S}\left[\left(\frac{i}{2}(B(\hat{S}B)^* - B^*\hat{S}B) - \hat{K}|B|^2\right)\hat{S}B\right]$$
$$- \frac{i}{8}(\hat{S}(|\hat{S}B|^2 B) + |\hat{S}B|^2\hat{S}B) - \frac{1}{4}B * \hat{K}|\hat{S}B|^2. \tag{4.27}$$

The breather is the solution of (4.27) in the following form:

$$B(x,t) = B(x - Vt)e^{-i\Omega t}. \tag{4.28}$$

It satisfies the following equation:

$$-iVB + \Omega B = (\hat{\omega}_{k_0+k} - \omega_{k_0})B$$
$$- \frac{1}{4}\hat{S}\left[\left(\frac{i}{2}(B(\hat{S}B)^* - B^*\hat{S}B)\hat{K}|B|^2\right)\hat{S}B\right]$$
$$- \frac{i}{8}(\hat{S}(|\hat{S}B|^2 B) + |\hat{S}B|^2\hat{S}B) - \frac{1}{4}B * \hat{K}|\hat{S}B|^2. \tag{4.29}$$

V is close to the linear group velocity. The solution to (4.29) can be found numerically and is a generalization of the well-known soliton solution for the nonlinear Schrodinger equation. Indeed, for very small steepness we can neglect the derivative in the nonlinear terms, set $V = \omega_{k_0}'$ and obtain a stationary NLS equation:

$$\Omega B = -\omega_{k_0}'' B'' + \frac{1}{2}k_0^3|B|^2 B. \tag{4.30}$$

Taking into account first derivatives and the operator $\hat{K}$ in the nonlinear term, we can easily obtain the Dysthe equation.

5. Conclusion

A simple equation describing the evolution of 1D water waves is derived based on the important property of a vanishing four-wave interaction for gravity water waves. This property allows drastic simplification of the well-known Zakharov equation for water waves, which is very cumbersome. When written in X-space instead of K-space, the equation allows further analytical and numerical study. The simple Hamiltonian obtained after canonical transformation raises a question about the integrability of equations for the potential flow of a fluid in a gravity field. This question remains open.

This new equation can be generalized for "almost 2D waves" or "almost 3D fluid". Waves that are slightly inhomogeneous in the transverse direction can be considered in the spirit of the Kadomtsev–Petviashvili equation for the Korteweg–de-Vries equation: the frequency ω_k can be treated as 2D, ω_{k_x,k_y}, while leaving the coefficient $\tilde{T}^{kk_1}_{k_2k_3}$ in (2.10) not dependent on y. b now depends on both x and y:

$$\mathcal{H} = \int b^* \hat{\omega}_{k_x,k_y} b dxdy + \frac{1}{4}\int |b'_x|^2 \times \left[\frac{i}{2}(bb'^*_x - b^* b'_x) - \hat{K}_x |b|^2\right] dxdy. \quad (5.31)$$

For the pure 1D case, this is much more applicable than the nonlinear Schrodinger equation or the Dysthe equation. Note that it does not contain multiple integrations in Fourier space and is written in coordinate space.

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