The dominant nonlinear wave interaction in the energy balance of a wind-driven sea

V. E. Zakharov

Citation: Low Temperature Physics **36**, 772 (2010); doi: 10.1063/1.3499239
View online: http://dx.doi.org/10.1063/1.3499239
View Table of Contents: http://scitation.aip.org/content/aip/journal/ltp/36/8?ver=pdfcov
Published by the AIP Publishing

The dominant nonlinear wave interaction in the energy balance of a wind-driven sea

V. E. Zakharov[a)]

Department of Mathematics, University of Arizona, Tucson, USA
(Submitted January 19, 2010)
Fiz. Nizk. Temp. **36**, 971–985 (2010)

Here some aspects of the physics of a wind-driven sea are investigated theoretically. It is demonstrated that an effective four-wave nonlinear interaction plays a leading role in the formation of the spectra of turbulent waves. In particular, this interaction leads to non-linear damping which exceeds standard observations at least by an order of magnitude. The theory developed here is compared with available experimental data. © *2010 American Institute of Physics*.
[doi:10.1063/1.3499239]

I. INTRODUCTION

In this talk we discuss some theoretical aspects of the physics of a wind-driven sea. On our opinion, some important aspects of this theory have not been sufficiently clarified and must be elucidated. Clarification is needed to allow adequate comparison of theory and experiment; otherwise, costly and laborious field and laboratory measurements cannot be properly interpreted and understood.

The first question concerns the correct definition of the wave action $N_k(t)$, which obeys the Hasselmann kinetic equation

$$\frac{dN}{dt} = S_{nl} + S_{\rm in} + S_{\rm dis}, \tag{1.1}$$

augmented by source and dissipation terms. How does one find the current action spectrum $N_k(t)$ from experimental data? In the best experiments the space-time spectrum

$$Q_{k\omega} = \langle |\eta_{k\omega}|^2 \rangle . \tag{1.2}$$

is measured. Here $\eta_{k\omega}$ is the Fourier transform of the surface elevation. The most advanced definition of the wave action, used in many research papers (see, for example Refs. 1 and 2), is

$$N_k = \frac{2}{\omega_k}\int_0^\infty Q_{k\omega} d\omega . \tag{1.3}$$

Equation (1.3) is certainly correct for waves of very small amplitude in the limit $\mu \to 0$, where μ is a characteristic average steepness of the surface. For finite steepness, it can be treated as the first term in the expansion

$$N_k = N_0(k) + \mu^2 N_1(k) + \cdots . \tag{1.4}$$

Now $N_0(k)$ is given by Eq. (1.3), while $N_1(k)$ needs to be determined. One might think that this question is not very important because, even for the steepest young waves, $\mu^2 \simeq 0.01$ and the accuracy of Eq. (1.3) looks good. However, our preliminary estimates show that the ratio $N_1(k)/N_0(k)$ is a rapidly growing function of k; thus, in the spectral tails the difference between N_k and $N_0(k)$ might be essential.

Now we formulate the inverse problem. Suppose we know N_k. How do we find $Q_{k\omega}$?

In the linear approximation, for $\mu \to 0$, the answer is known:

$$Q_{k\omega} = \frac{\omega_k}{2}(N_k \delta(\omega - \omega_k) + N_{-k}\delta(\omega + \omega_{-k})) . \tag{1.5}$$

What happens if μ is finite? In the neighborhood of $\omega = \omega_k$ we should make the replacement

$$\delta(\omega - \omega_k) \to \frac{1}{\pi}\frac{\Gamma_k}{(\omega - \tilde{\omega}_k)^2 + \Gamma_k^2}, \tag{1.6}$$

where $\tilde{\omega}_k = \omega_k + \mu^2 \omega_{1k} + \cdots$ is the renormalized frequency and $\Gamma_k \simeq \mu^4 \tilde{\Gamma}_k + \cdots$ is the effective dissipation owing to four-wave processes. As long as μ^2 is small, regard the shift in ω_k and the blurring of the δ-function as small effects. However, the quotients ω_{1k}/ω_k and $\tilde{\Gamma}_k/\omega_k$ are increasing functions of k; thus, for $k \gg k_p$ (k_p is the wave number of the spectral peak) a derivation from Eq. (1.5) could be essential. There is one more important effect. In a real sea all waves can be separated in two classes: "resonant waves" with $\omega \sim \omega_k$ and "slave harmonics" caused by a quadratic nonlinearity in the primitive dynamic equations. The slave waves do not obey dispersion relations, so their frequency spectrum for a given k is a broad function, not concentrated at $\omega \simeq \omega_k$.

Accurate determination of $N_1(k)$ for given $Q_{k\omega}$ and $Q_{k\omega}$ for given $N(k)$ is possible but it is technically cumbersome. In sections II and III we taking the first but important steps to solve that problem. In section IV we study axially symmetric solutions of the equation

$$S_{nl} = 0, \tag{1.7}$$

which has been known since 1966 (Ref. 3, see also Refs. 4 and 5). This equation has exactly two power-law solutions:

$$N_1(k) = c_p \left(\frac{P}{g^2}\right)^{1/3} \frac{1}{k^4}, \tag{1.8}$$

and

$$N_2(k) = c_q \left(\frac{Q}{g^{3/2}}\right)^{1/2} \frac{1}{k^{23/6}} . \tag{1.9}$$

Equation (1.8) is known as Zakharov–Filonenko spectrum.[4] Here P is the flux of energy from small wave numbers and Q is the flux of wave action from high wave numbers. The Kolmogorov constants c_p and c_q were not known, but now they can be calculated:

$$c_p = 0.219, \quad c_q = 0.227. \tag{1.10}$$

The general, isotropic solutions of Eq. (1.7) depend on the two constants, P and Q. In section V we discuss the general anisotropic solution of this equation. We show that the solution is defined by one arbitrary constant, the flux of wave action from high wave numbers, and one arbitrary function of angle. In the axially symmetric case this function degenerates to the constant P. The general anisotropic solution of Eq. (1.7) describes an angular spreading of the spectrum that increases with frequency. The last section VI, is the most important from a practical standpoint. We discuss the balance equation in the universal domain $\omega \gg \omega_p$,

$$S_{nl} + S_{\text{in}} + S_{\text{dis}} = 0. \tag{1.11}$$

It appears that, in some domain in the k-plane, $S_{\text{in}} + S_{\text{dis}} > 0$. Suppose that $S_{\text{in}} = \gamma(k) N_k$. We notice that S_{nl} can be represented in the form

$$S_{nl} = F_k - \Gamma_k N_k, \tag{1.12}$$

and the nonlinear wave interaction process predominates if $\Gamma_k \gg \gamma_k$. We show that this condition is satisfied in a majority of realistic cases, if the waves are not very young. It means that, as stated above, the nonlinear wave interaction is the dominant process in a wind-driven sea.

II. WHAT IS THE WAVE ACTION?

The widely used Hasselmann equation is

$$\frac{\partial N}{\partial t} + \frac{\partial \tilde{\omega}}{\partial \mathbf{k}} \frac{\partial N}{\partial \mathbf{r}} = S_{nl}, \tag{2.1}$$

with

$$S_{nl} = \pi g^2 \int |T_{kk_1,k_2k_3}|^2 \delta(k + k_1 - k_2 - k_3)\, \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3})(N_{k_1} N_{k_2} N_{k_3} + N_k N_{k_2} N_{k_3} - N_k N_{k_1} N_{k_2} - N_k N_{k_1} N_{k_3})\, dk_1 dk_2 dk_3. \tag{2.2}$$

Here $\omega_k = \sqrt{gk \tanh kH}$, H is the depth, $T_{kk_1k_2k_3} = T_{k_1kk_2k_3} = T_{k_2k_3kk_1} = T_{kk_1k_3k_2}$ are coupling coefficients, and

$$\tilde{\omega}(k) = \omega(k) + 2g \int T_{kk_1,kk_1} N_{k_1} dk_1 \tag{2.3}$$

is the renormalized frequency. As mentioned above, the nonlinear interaction term S_{nl} can be written as

$$S_{nl} = F_k - \Gamma_k N_k, \tag{2.4}$$

where

$$F_k = \pi g^2 \int |T_{kk_1k_2k_3}|^2 \delta(k + k_1 - k_2 - k_3)\, \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) N_{k_1} N_{k_2} N_{k_3} dk_1 dk_2 dk_3 \tag{2.5}$$

and Γ_k, the dissipation rate owing to four-wave processes, is

$$\Gamma_k = \pi g^2 \int |T_{kk_1,k_2k_3}|^2 \delta(k + k_1 - k_2 - k_3)\, \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3})(N_{k_1} N_{k_2} + N_{k_1} N_{k_3} - N_{k_2} N_{k_3}) dk_1 dk_2 dk_3. \tag{2.6}$$

One can say that in a real nonlinear sea the dispersion relation $\omega = \omega_k$ is renormalized and becomes a complex function

$$\omega_k \to \tilde{\omega}_k + \frac{1}{2} i \Gamma_k. \tag{2.7}$$

Equations (2.1) and (2.2) are written for the wave action spectrum $N_k(\mathbf{r}, t)$. What is the exact definition of the wave action? How can $N_k(\mathbf{r}, t)$ be expressed in terms of observable, measurable quantities? These are not such simple questions.

By taking a snapshot of the surface from two points one can get a stereoscopic image and recover the elevation $\eta(\mathbf{r})$. Taking a nonsymmetric Fourier transform and defining

$$\eta_k = \frac{1}{(2\pi)^2} \int \eta(\mathbf{r}) e^{-ikr} d\mathbf{r}, \tag{2.8}$$

we can introduce the spatial spectrum

$$Q_k = \langle |\eta_k|^2 \rangle. \tag{2.9}$$

By taking a series of snapshots at consecutive times, one can restore the full space-time spectrum

$$Q_{k\omega} = \langle |\eta_{k\omega}|^2 \rangle. \tag{2.10}$$

Apparently,

$$Q_k = \int_{-\infty}^{\infty} Q_{k\omega} d\omega. \tag{2.11}$$

What is the wave action N_k? In some papers and monographs we can find the following definition:

$$N_k = \frac{Q_k}{\omega_k}. \tag{2.12}$$

This is a widespread misconception. The spectrum Q_k is an even function, i.e., $Q_{-k} = Q_k$, while N_k certainly does not obey this restriction. One can present the spatial spectrum in the form

$$Q_k = \frac{\omega_k}{2}(n_k + n_{-k}), \tag{2.13}$$

where n_k is the wave action. We have deliberately used a lower case letter for it, because n_k and N_k are different wave actions.

The wave field consists of "resonant" and "slave" harmonics. The resonant harmonic with wave vector $\mathbf{k}$ has a frequency close to the renormalized frequency $\tilde{\omega}_k$. The strongest slave harmonics are the result of the interaction of two resonant harmonics. Suppose they have wave vectors $\mathbf{k}_1, \mathbf{k}_2$. In the first order of nonlinearity they generate four slave harmonics with wave vectors $\mathbf{p}_1, \mathbf{p}_2, -\mathbf{p}_1, -\mathbf{p}_2$ and frequencies $\Omega_1, \Omega_2, -\Omega_1, -\Omega_2$. Here $\mathbf{p}_1 = \mathbf{k}_1 - \mathbf{k}_2$, $\mathbf{p}_2 = \mathbf{k}_1 + \mathbf{k}_2$, and $\Omega_1 = \omega_1 - \omega_2$, $\Omega_2 = \omega_1 + \omega_2$. There is no definite relationship between the wave vector and the frequency for slave harmonics.

Returning to the wave action, we now explain the difference between n_k and N_k. N_k is the "refined" wave action that includes resonant harmonics and slave harmonics of higher order only, while n_k is the "total" wave action that includes both resonant and all slave harmonics. Apparently, $n_k > N_k$ and is directly related to the experimentally measurable spatial spectrum by Eq. (2.13). However, n_k *does not obey* the Hasselmann equation. On the other hand, the "purified" wave action N_k cannot *in principle* be measured in any kind of experiment. But exactly this sort of wave action satisfies the Hasselmann equation. As a result, all operational models solve the Hasselmann equation augmented with additional terms: S_{in}, the input from wind, and S_{dis}, the dissipation due to wave breaking. Hence, operational models do predict N_k. At the same time, experimentalists can only measure n_k.

At first glance we see a serious discrepancy; however, nobody pays any attention. Why does this happen?

To answer this, we should estimate the relative difference between n_k and N_k. Let us use the notation

$$\alpha(k) = \frac{n_k - N_k}{n_k}. \tag{2.14}$$

In a typical observed spectrum of a wind-driven sea, we should distinguish the spectral areas near the peak frequency $\omega \sim \omega_p$ and in the tail $\omega \gg \omega_p$. In the energy spectral band close to ω_p, α is small:

$$\alpha \sim \mu^2.$$

The characteristic steepness μ is defined as

$$\mu^2 \simeq \frac{\omega_p^4}{g^2}\sigma^2,$$

where σ is the total energy (density) of the waves. Even for young waves $\mu^2 \leqslant 0.01$; thus, the relative difference between n and N for deep water is no more than one percent and can easily be neglected. $\alpha(k)$, however, is a rapidly growing function of k. An accurate estimate of the dependence of α on frequency for $\omega \geqslant \omega_p$ is beyond the scope of this article. The article on this topic will be submitted for publication soon, but our preliminary results show that this dependence increases very rapidly, with

$$\alpha \simeq \mu^2 \left(\frac{\omega}{\omega_p}\right)^3. \tag{2.15}$$

As mentioned above, for $\omega \sim \omega_p$ one can neglect the difference between n_k and N_k. In this region we can replace Eq. (2.9) by

$$Q_k = \frac{\omega_k}{2}(N_k + N_{-k}). \tag{2.16}$$

There is an essential difference between Eqs. (2.13) and (2.16). Because $n_k > 0$ for all k, the wave vectors of the slave harmonics cover the entire k-plane; thus, determining n_k from Q_k is impossible in principle. On the other hand, in many practical cases N_k is nonzero only inside the bounded domain G in the k-plane. At the same time $N_{-k} \neq 0$ only inside the domain $\tilde{G}$, which is radially symmetric to G. In other words, if the vector $\mathbf{k}$ belongs to G, the vector $-\mathbf{k}$ belongs to $\tilde{G}$. Suppose that G and $\tilde{G}$ do not overlap. Then, in the domain G we have $N_k = 2Q_k/\omega_k$. In spite of the factor 2 in Eq. (2.13), the integral identity $\int Q_k dk = \int \omega_k N_k dk$ is the same as if we had used the naive and blatantly incorrect Eq. (2.12).

In some important cases domains G and $\tilde{G}$ intersect. Then we face some ambiguity in determining N_k from Eq. (2.16). To overcome this ambiguity one should use the space-time spectrum $Q_{k,\omega}$ and define

$$n_k = \frac{2}{\omega_k}\int_0^\infty Q(k,\omega)d\omega. \tag{2.17}$$

An equivalent formula is given in the monograph of Monin and Krasitsky[1] printed in Russia in 1985. It was also used by Rosental *et al.*[2] at approximately the same time. In this case, again,

$$\int \omega_k n_k dk = \int_{-\infty}^\infty Q(k,\omega)d\omega dk. \tag{2.18}$$

Note that Eqs. (2.13) and (2.17) include the slave harmonics and can be used for comparing experimental spectral tails with the solutions of the Hasselmann equation, both numerical and analytical, but only with caution. They work up to an accuracy of μ^2 in the neighborhood of a spectral peak, but can lead to major errors in the spectral tails. A preliminary estimate of the accuracy of Eq. (2.17) will be made in the next section.

III. HOW TO SEPARATE RESONANT AND SLAVE HARMONICS?

For accurate separation of resonant and slave harmonics and finding an explicit formula that connects $Q(k,\omega)$ and N_k, one should use a Hamiltonian formalism and implement a canonical transformation, excluding cubic terms in the Hamiltonian. This is a cumbersome mathematical procedure. In this section we demonstrate a more economical way of doing this. We study weakly nonlinear waves on the surface of an ideal fluid of infinite depth in an infinite basin. The vertical coordinate is

$$-H < z < \eta(r,t), \quad r = (x,y), \tag{3.1}$$

the fluid is incompressible, H is the depth of fluid,

$$\text{div } V = 0, \tag{3.2}$$

and, the velocity V is a potential field, i.e.,

$$V = \nabla\Phi, \tag{3.3}$$

where the potential Φ satisfies the Laplace equation

$$\Delta\Phi = 0 \tag{3.4}$$

with the boundary conditions

$$\Phi|_{z=\eta} = \Psi(r,t), \quad \Phi_z|_{z=-\infty} = 0. \tag{3.5}$$

The total energy of the fluid, $H = T + U$, has the following terms:

$$T = \frac{1}{2}\int d\mathbf{r}\int_{-\infty}^{\eta}(\nabla\Phi)^2 dz = \frac{1}{2}\int \Psi\Phi_n dS, \tag{3.6}$$

and

$$U = \frac{1}{2}g\int \eta^2 d\mathbf{r}. \tag{3.7}$$

The Dirichlet–Neumann boundary value problem (3.4) and (3.5) is uniquely solved; thus the flow is defined by fixing η and Ψ. This pair of variables is canonical; thus, the equations for the evolution of η, Ψ take the form:[6]

$$\frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \Psi}, \quad \frac{\partial \Psi}{\partial t} = -\frac{\delta H}{\delta \eta}. \tag{3.8}$$

After taking the non-symmetric Fourier transform,

$$\Psi(r) = \int \Psi(k) e^{ikr} dk, \quad \Psi(k) = \frac{1}{(2\pi)^2} \int \Psi(r) e^{-ikr} dr. \tag{3.9}$$

Equation (3.8) becomes

$$\frac{\partial \eta}{\partial t} = \frac{\delta \widetilde{H}}{\delta \Psi_k^*}, \quad \frac{\partial \Psi}{\partial t} = -\frac{\delta \widetilde{H}}{\delta \eta_k^*}, \tag{3.10}$$

with

$$\widetilde{H} = \frac{1}{4\pi^2} H = H_0 + H_1 + H_2 + \cdots \tag{3.11}$$

It has been shown[7–9] that the Hamiltonian $\widetilde{H}$ can be expanded in a Taylor series in powers of $k\eta_k$:

$$H_0 = \frac{1}{2} \int \{A_k |\Psi_k|^2 + g|\eta_k|^2\} dk, \quad A_k = k \tan kH$$

$$H_1 = \frac{1}{2} \int L^{(1)}(k_1,k_2) \Psi_{k_1} \Psi_{k_2} \eta_{k_3} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) dk_1 dk_2 dk_3.$$

$$H_2 = \frac{1}{2} \int L^{(2)}(k_1,k_2,k_3,k_4) \Psi_{k_1} \Psi_{k_2} \eta_{k_3} \eta_{k_4} \delta(k_1 + k_2 + k_3 + k_4) dk_1 dk_2 \eta_{k_3} \eta_{k_4}. \tag{3.12}$$

Here

$$L^{(1)}(k_1,k_2) = -(k_1,k_2) - A_{k_1} A_{k_2},$$

$$L^{(2)}(k_1,k_2,k_3,k_4) = \frac{1}{2}(k_1^2 A_2 + k_2^2 A_1) + \frac{1}{4} A_1 A_2 (A_{1+3} + A_{2+4} + A_{1+4} + A_{2+3}). \tag{3.13}$$

Now we introduce the normal variables a_k:

$$\eta_k = \frac{1}{\sqrt{2}} \left(\frac{A_k}{g}\right)^{1/4} (a_k + a_{-k}^*),$$

$$\Psi_k = \frac{i}{\sqrt{2}} \left(\frac{g}{A_k}\right)^{1/4} (a_k - a_{-k}^*). \tag{3.14}$$

Normal variables obey the following Hamiltonian equations:

$$\frac{\partial a_k}{\partial t} + i \frac{\delta H}{\delta a_k^*} = 0. \tag{3.15}$$

All terms in the expansion of Hamiltonian (3.11) must be expressed in terms of the a_k:

$$H_0 = \int \omega_k |a_k|^2 dk,$$

$$H_1 = \frac{1}{2} \int V_{kk_dk_2}^{(1,2)} (a_k a_{k_1}^* a_{k_2}^* + a_k^* a_{k_1} a_{k_2}) \delta(k - k_1 - k_2) dk dk_1 dk_2 + \frac{1}{6} \int V_{kk_dk_2}^{(0,3)} (a_k a_{k_1} a_{k_2} + a_k^* a_{k_1}^* a_{k_2}^*) \delta(k + k_1 + k_2) dk dk_1 dk_2, \tag{3.16}$$

$$V_{kk_1k_2}^{(1,2)} = \frac{g^{1/4}}{2\sqrt{2}} \left\{ \left(\frac{A_k}{A_{k_1} A_{k_2}}\right)^{1/4} L^{(1)}(k_1,k_2) - \left(\frac{A_{k_1}}{A_k A_{k_2}}\right)^{1/4} \times L^{(1)}(-k,k_1) - \left(\frac{A_{k_2}}{A_k A_{k_1}}\right)^{1/4} L^{(1)}(-k,k_2) \right\}, \tag{3.17}$$

$$V_{kk_1k_2}^{(0,3)} = \frac{g^{1/4}}{2\sqrt{2}} \left\{ \left(\frac{A_k}{A_{k_1} A_{k_2}}\right)^{1/4} L^{(1)}(k_1,k_2) + \left(\frac{A_{k_1}}{A_k A_{k_2}}\right)^{1/4} \times L^{(1)}(k,k_1) + \left(\frac{A_{k_2}}{A_k A_{k_1}}\right)^{1/4} L^{(1)}(k,k_2) \right\}. \tag{3.18}$$

Now we can define the "total" or rough action:

$$n_k \delta(k - k') = g \langle a_k a_{k'}^* \rangle. \tag{3.19}$$

It is clear that fundamental relation (2.13) is satisfied. Now we take the temporal Fourier transform

$$a_{k\omega} = \frac{1}{2\pi} \int a(k,t) e^{-i\omega t} dt \tag{3.20}$$

and introduce

$$n_{k\omega} \delta(k - k') \delta(\omega - \omega') = g \langle a_{k\omega} a_{k',\omega'}^* \rangle. \tag{3.21}$$

The space-time spectrum of the elevation is simply

$$Q_{k,\omega} = \frac{\omega_k}{2} (n_{k,\omega} + n_{-k,-\omega}). \tag{3.22}$$

To separate the resonant and slave harmonics we must perform a canonical transformation to new variables, excluding cubic terms in the Hamiltonian. This is a standard procedure known in celestial dynamics since the nineteenth century. In our case, however, this procedure is rather cumbersome. It was first done by Krasitski.[9] He transformed the initial canonical variables a_k to new canonical variables b_k, which contain first order slave harmonics only. The variables a_k are represented by infinite series in the new variables b_k:

$$a_k = b_k + a_k^{(1)} + a_k^{(2)} + a_k^{(3)}. \tag{3.23}$$

He calculated first two terms in this expansion and found the following expressions:

$$a_k^{(1)} = \int \Gamma^{(1)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) b_{k_1} b_{k_2} \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) dk_1 dk_2 - 2 \int \Gamma^{(1)}(\mathbf{k}_2,\mathbf{k},\mathbf{k}_1) b_{k_1}^* b_{k_2} \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2) dk_1 dk_2 + \int \Gamma^{(2)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) b_{k_1}^* b_{k_2}^* \delta(\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2) dk_1 dk_2,$$

$$a_k^{(2)} = \int B(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) b_{k_1}^* b_{k_2} b_{k_3} \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2 - \mathbf{k}_3) dk_1 dk_2 dk_3 + \cdots \quad (3.24)$$

where

$$\Gamma^{(1)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) = -\frac{1}{2}\frac{V^{(1,2)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)}{(\omega_k-\omega_{k_1}-\omega_{k_2})},$$

$$\Gamma^{(2)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) = -\frac{1}{2}\frac{V^{(0,3)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)}{(\omega_k+\omega_{k_1}+\omega_{k_2})}, \quad (3.25)$$

and

$$\begin{aligned} B(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = {} & \Gamma^{(1)}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1-\mathbf{k}_2)\Gamma^{(1)}(\mathbf{k}_3,\mathbf{k},\mathbf{k}_3-\mathbf{k}) \\ & + \Gamma^{(1)}(\mathbf{k}_1,\mathbf{k}_3,\mathbf{k}_1-\mathbf{k}_3)\Gamma^{(1)}(\mathbf{k}_2,\mathbf{k},\mathbf{k}_2-\mathbf{k}) \\ & - \Gamma^{(1)}(\mathbf{k},\mathbf{k}_2,\mathbf{k}-\mathbf{k}_2)\Gamma^{(1)}(\mathbf{k}_3,\mathbf{k}_1,\mathbf{k}_3-\mathbf{k}_1) \\ & - \Gamma^{(1)}(\mathbf{k}_1,\mathbf{k}_3,\mathbf{k}_1-\mathbf{k}_3)\Gamma^{(1)}(\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2-\mathbf{k}_1) \\ & - \Gamma^{(1)}(\mathbf{k}+\mathbf{k}_1,\mathbf{k},\mathbf{k}_1)\Gamma^{(1)}(\mathbf{k}_2+\mathbf{k}_3,\mathbf{k}_2,\mathbf{k}_3) \\ & + \Gamma^{(2)}(-\mathbf{k}-\mathbf{k}_1,\mathbf{k},\mathbf{k}_1)\Gamma^{(2)}(-\mathbf{k}_2 - \mathbf{k}_3,\mathbf{k}_2,\mathbf{k}_3). \end{aligned} \quad (3.26)$$

On our opinion, Krasitski used a rather long way for calculation of terms in expansion (3.23). He directly checked the validity of canonicity condition

$$\{a_k,a_{k'}\} = \int \left\{ \frac{\delta a_k}{\delta b_{k''}}\frac{\delta a_{k'}}{\delta_{k''}^*} - \frac{\delta a_k}{\delta b_{k''}^*}\frac{\delta a_{k'}}{\delta b_{k''}} \right\} dk'' = 0,$$

$$\{a_k,a_{k'}^*\} = \int \left\{ \frac{\delta a_k}{\delta b_{k''}}\frac{\delta a_{k'}^*}{\delta_{k''}^*} - \frac{\delta a_k}{\delta b_{k''}^*}\frac{\delta a_{k'}^*}{\delta b_{k''}} \right\} dk'' = \delta(k-k'). \quad (3.27)$$

Calculating $a_k^{(3)}$ by this method is an impossibly complicated task. The canonical transformation can be found using more sophisticated methods. The first one was offered[7] in 1998. Suppose that a_k is a solution of Hamiltonian system

$$\frac{\partial a_k}{\partial \tau} + i\frac{\delta R}{\delta a_k^*} = 0 \quad (3.28)$$

where τ is an "artificial time" and R is an effective Hamiltonian given by

$$R = i\int \Gamma_{kk_2k_2}^{(1)}(a_k^* a_{k_1} a_{k_2} - a_k a_{k_1}^* a_{k_2}^*)\delta(k-k_1-k_2)dkdk_1dk_2 + \frac{i}{3}\int \Gamma_{kk_1k_2}^{(2)}(a_k^* a_{k_1}^* a_{k_2}^* - a_k a_{k_1} a_{k_2})\delta(k+k_1 + k_2)dkdk_1dk_2. \quad (3.29)$$

Equations (3.28) and (3.29) must be supplemented by the initial condition

$$a_k|_{\tau=0} = b_k. \quad (3.30)$$

The needed canonical transformation is obtained on setting $\tau=1$. Expanding the solution in a Taylor series in τ and setting $\tau=1$ at the end, we reproduce the result of Krasitski (3.24)–(3.26) in a much more economical way. Now we demonstrate another, more traditional way for constructing the canonical transformation based on finding a generating function. We represent a_k in the form

$$a_k = \frac{1}{\sqrt{2}}(q_k + ip_k), \quad q_{-k} = q_k^*, \quad p_{-k} = p_k^*.$$

The functions q_k, p_k obey the equations

$$\frac{\partial q_k}{\partial t} = \frac{\delta H}{\delta p_k^*}, \quad \frac{\partial p_k}{\partial t} = -\frac{\delta H}{\delta q_k^*}, \quad (3.31)$$

where H is the same Hamiltonian expressed in terms of q_k, p_k. Now

$$H_0 = \frac{1}{2}\int \omega_k(|q_k|^2 + |p_k|^2)dk, \quad (3.32)$$

$$H_1 = \frac{1}{2}\int L_{kk_1k_2} q_k p_{k_1} p_{k_2}\delta(k+k_1+k_2)dkdk_1dk_2, \quad (3.33)$$

$$L_{kk-1k_2} = \frac{g^{1/4}A_k^{1/4}}{A_{k-1}^{1/4}A_{k_2}^{1/2}}L_{k_1k_2}^{(1)}. \quad (3.34)$$

We now transform to new variables R_k, ξ_k using the following generating function (see Ref. 10, as well):

$$S = \int R_k q_k dk + \frac{1}{2}\int A_{kk_1k_2} q_k q_{k_1} R_{k_2}\delta(k+k_1 + k_2)dkdk_1dk_2 + \frac{1}{3}\int B_{kk_1k_2} R_k R_{k_1} R_{k_2}\delta(k+k_1 + k_2)dkdk_1dk_2. \quad (3.35)$$

The "old momentum" p_k and "new coordinates" ξ_k are given by

$$p_k = \frac{\delta S}{\delta q_{-k}} = R_k + \int A_{-k,k_1,k_2} q_{k_1} R_{k_2}\delta(k-k_1-k_2)dk_1dk_2, \quad (3.36)$$

and

$$\xi_k = \frac{\delta S}{\delta R_{-k}} = q_k + \frac{1}{2}\int A_{k_1,k_2,-k} q_{k_1} q_{k_2}\delta(k-k_1-k_2^*)dk_1dk_2 + \int B_{-k,k_1,k_2} R_{k_1} R_{k_2}\delta(k-k_1-k-2)dk_1dk_2. \quad (3.37)$$

Apparently $B_{kk_1k_2}$ is symmetric with respect to all permutations and $A_{kk_1k_2}=A_{kk_2k_1}$. To find A, B we notice that in the first approximation

$$q_k = \xi_k - \frac{1}{2}\int A_{k_1,k_2,-k}\xi_{k_1}\xi_{k_2}\delta(k-k_1-k_2)dk_1dk_2 - \int B_{-k,k_1,k_2} R_{k_1} R_{k_2}\delta(k-k_1-k_2)dk_1dk_2. \quad (3.38)$$

and in Eq. (3.36) we can make the substitution $q_k \to \xi_k$. Now we plug q_k, p_k into Eq. (3.32). In Eq. (3.33) we can just make the substitutions $q_k \to \xi_k$ and $p_k \to R_k$. From the condition of eliminating cubic terms that proportional to $\xi_k\xi_{k_1}\xi_{k_2}$ and $\xi_k p_{k_1} p_{k_2}$, and the symmetry conditions, we find after some calculations the following nice and elegant expressions for A and:

$$A_{kk_1k_2} = -\frac{1}{4}\left(\frac{L_0+L_1+L_2}{\omega_0+\omega_1+\omega_2} + \frac{L_0+L_1-L_2}{\omega_0+\omega_1-\omega_2}\right) + \frac{1}{4}\left(\frac{L_0-L_1-L_2}{\omega_0-\omega_1-\omega_2} + \frac{L_1-L_0-L_2}{\omega_1-\omega_0-\omega_2}\right), \quad (3.39)$$

$$B_{kk_1k_2} = -\frac{1}{4}\left(\frac{L_0+L_1+L_2}{\omega_0+\omega_1+\omega_2} + \frac{L_0-L_1-L_2}{\omega_0-\omega_1-\omega_2}\right) - \frac{1}{4}\left(\frac{L_1-L_0-L_2}{\omega_1-\omega_0-\omega_2} + \frac{L_2-L_0-L_1}{\omega_2-\omega_0-\omega_1}\right). \quad (3.40)$$

Here

$$L_0 = L_{kk_1k_2}, \quad L_1 = L_{k_1kk_2}, \quad L_2 = L_{k_2kk_1},$$

$$\omega_0 = \omega_k, \quad \omega_1 = \omega_{k_1}, \quad \omega_2 = \omega_{k_2}. \quad (3.41)$$

To reproduce the results of Krasitski one has to expand the old variables q_k, p_k in powers of the new variables ξ_k, R_k; then take b_k in the following form:

$$b_k = \frac{1}{\sqrt{2}}\left(\left(\frac{g}{A_k}\right)^{1/4}\xi_k - i\left(\frac{A_k}{g}\right)^{1/4}R_k\right). \quad (3.42)$$

The new normal variables b_k satisfy Zakharov's equation[6]

$$\frac{\partial b_k}{\partial t} + i\omega_k b_k + \frac{i}{2}\int T_{kk_1k_2k_3} b^*_{k_1} b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk_1 dk_2 dk_3 = 0. \quad (3.43)$$

Here $T_{kk_1k_2k_3}$ is the same as in Eq. (2.2). An explicit expression for $T_{kk_1k_2k_3}$ is too complicated to be presented here. Notice that now we can calculate $n_k=|a_k|^2$ by using the expansion (3.23). We will assume that triple correlations of the new variables are zero, i.e.,

$$\langle b_k b_{k_1} b_{k_2}\rangle = 0, \quad \langle b^*_k b_{k_1} b_{k_2}\rangle = 0 \quad (3.44)$$

We use also Gaussian closure for quartic variables

$$\langle b^*_k b^*_{k_1} b_{k_2} b_{k_3}\rangle = N_k N_{k_1}(\delta_{k-k_2}\delta_{k_1-k_3} + \delta_{k-k_3}\delta_{k_1-k_2}). \quad (3.45)$$

Here N_k is the "refined" action. After some calculations we find that n_k and N_k are connected by the following equation (it can be found in Ref. 8):

$$n_k = N_k + \frac{1}{2}\int \frac{|V^{(1,2)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)|^2}{(\omega_k-\omega_{k_1}-\omega_{k_2})^2}(N_{k_1}N_{k_2} - N_k N_{k_1} - N_k N_{k_2})\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)dk_1 dk_2$$
$$+ \frac{1}{2}\int \frac{|V^{(1,2)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)|^2}{(\omega_{k_1}-\omega_k-\omega_{k_2})^2}(N_{k_1}N_{k_2} + N_k N_{k_1} - N_k N_{k_2})\delta(\mathbf{k}_1-\mathbf{k}-\mathbf{k}_2)dk_1 dk_2$$
$$+ \frac{1}{2}\int \frac{|V^{(1,2)}(\mathbf{k}_2,\mathbf{k},\mathbf{k}_1)|^2}{(\omega_{k_2}-\omega_k-\omega_{k_1})^2}(N_{k_1}N_{k_2} + N_k N_{k_2} - N_k N_{k_1})\delta(\mathbf{k}_2-\mathbf{k}-\mathbf{k}_1)dk_1 dk_2$$
$$+ \frac{1}{2}\int \frac{|V^{(0,3)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)|^2}{(\omega_k+\omega_{k_1}+\omega_{k_2})^2}(N_{k_1}N_{k_2} + N_k N_{k_1} + N_k N_{k_2})\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)dk_1 dk_2. \quad (3.46)$$

The difference between n_k and N_k,

$$\Delta_k = \frac{n_k - N_k}{N_k},$$

is important in shallow water. However, even in deep water Δk is a fast growing function of k.

The relation between the space-time spectra of the "total" $n_{k\omega}$ and "purified" $N_{k\omega}$ versions of the wave action is not known so far. This is a subject for future research. However, $N_{k\omega}$ can be written as

$$N_{k\omega} = \frac{1}{\pi}\frac{\Gamma_k N_k}{(\omega-\tilde{\omega}_k)^2 + \Gamma_k^2} \quad (3.47)$$

and we can approximately set

$$Q_{k\omega} = \frac{1}{2}\omega_k(N_{k\omega} + N_{-k,-\omega}) = \frac{1}{2\pi}\left\{\frac{\Gamma_k N_k}{(\omega-\tilde{\omega}_k)^2 + \Gamma_k^2} + \frac{\Gamma_{-k}N_{-k}}{(\omega-\tilde{\omega}_k)^2 + \Gamma_k^2}\right\}. \quad (3.48)$$

After integrating over ω and taking $\arctan \Gamma_k/\omega_k \sim \Gamma_k/\omega_k$, we get

$$N_k = \int_0^\infty N(k,\omega)d\omega + \frac{1}{\pi}\left(\frac{N_k\Gamma_k}{\omega_k} - \frac{N_{-k}\Gamma_{-k}}{\omega_{-k}}\right). \quad (3.49)$$

From Eq. (3.48) we see that identity

$$N_k = \int_0^\infty N(k,\omega)d\omega \quad (3.50)$$

is valid up to a relative accuracy of Γ_k/ω_k. This value for the accuracy will be discussed in section VI. Near the spectral peak it is of order $4\pi\mu^4$. The identity (2.17) is satisfied with much less accuracy. Even near the spectral peak, the accuracy is of order μ^2 and it becomes worse for $k \gg k_F$. An explicit expression for $Q(k,\omega)$ in terms of N_k will be the subject of a separate article.

IV. STATIONARY SOLUTIONS OF KINETIC EQUATION: ISOTROPIC CASE

In this section we address the question of how to solve the stationary kinetic equation

$$S_{nl} = 0? \quad (4.1)$$

Formally speaking, this equation has the thermodynamically equilibrium solutions

$$N_k = \frac{T}{\omega_k + \mu}, \quad (4.2)$$

where the temperature T and μ are constants. It might sound like a paradox, but in fact, the spectrum (4.2) in not a real solution of equation (4.1) because here we are only discussing the case of deep water and assume that $\omega=\sqrt{gk}$. Also we write $k=|\mathbf{k}|$.

To justify this statement we notice that in two particular cases, $\mu=0$ and $T=c_\mu$, $\mu\to\infty$, the solution (4.2) has the form

$$N=\frac{T}{\omega_k}=\frac{T}{\sqrt{g}}k^{-1/2}, \quad N=c. \tag{4.3}$$

Both these solutions are isotropic power-law functions

$$N_k=k^{-x} \tag{4.4}$$

with particular values of $x=1/2,0$. Let us study the general power-law solution of (4.1). By plugging (4.4) into (4.1) we find that each particular term in S_{nl} diverges, but the divergences in different terms may add out, so there is a "window of opportunity" for the exponent x. As a result,

$$S_{nl}=g^{3/2}k^{-3x+19/2}F(x). \tag{4.5}$$

Here $F(x)$ is a dimensionless function, defined inside interval $x_1<x<x_2$. The edges of the window, x_1 and x_2, are to be determined. Outside the "window of opportunity," at $x<x_1$ and $x>x_2$, $F(x)=\infty$. Thus, all admitted values of x must be lie between x_1 and x_2.

Let the quadruplet of waves be formed of wave vectors satisfying the resonance conditions

$$\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}_4,$$

$$\omega_{k_1}+\omega_{k_2}=\omega_{k_3}+\omega_{k_4}. \tag{4.6}$$

Suppose that $|k_1|\ll|k|$. The three-wave resonance condition,

$$\mathbf{k}=\mathbf{k}_2+\mathbf{k}_3, \quad \omega_k=\omega_{k_2}+\omega_{k_3}, \tag{4.7}$$

cannot be satisfied, thus one of vectors $\mathbf{k}_2$, $\mathbf{k}_5$ must be small. If $|k_3|\ll|k_2|$, then

$$\mathbf{k}_2=\mathbf{k}+\mathbf{k}_1-\mathbf{k}_3,$$

$$\omega(k_2)=\sqrt{gk}\left(1+\frac{1}{2}\frac{(k,\mathbf{k}_1-\mathbf{k}_3)}{k^2}+\cdots\right). \tag{4.8}$$

In the first approximation with a small parameter $|k_1|/|k|$, one can set $\omega(k_2)=\omega(k)$, $\omega(k_1)=\omega(k_3)$ and $|k_3|\simeq|k_1|$. In other words, the vectors $\mathbf{k}_1,\mathbf{k}_3$ are small and have approximately the same length k_1. If the vector $\mathbf{k}$ is directed along the x axis, the coupling coefficient $T_{kk_1k_2k_3}$ depends on the four parameters k,k_1,θ_1,θ_3. Here θ_1,θ_3 are angles between $\mathbf{k}_1$, $\mathbf{k}_3$ and $\mathbf{k}$. Remembering that $k_1\ll k$, we calculate the coupling coefficient in this asymptotic domain. A tedious calculation[11] yields the following compact result:

$$T_{kk_1k_2k_3}\simeq\frac{1}{2}kk_1^2T_{\theta_1,\theta_3},$$

$$T_{\theta_1,\theta_2}=2(\cos\theta_1+\cos\theta_3)-\sin(\theta_1-\theta_3)(\sin\theta_1-\sin\theta_3). \tag{4.9}$$

On the diagonal $k_3=k_1$, $\theta_3=\theta$ we get a very simple expression published in 2003:[29]

$$T_{kk_1}\simeq 2k_1^2k\cos\theta_1. \tag{4.10}$$

Suppose that spectrum is separated into a low-frequency component $N_0(k)$ and a high-frequency component $N_1(k)$. We assume that $N_1\ll N_0$ and take into account only the interaction between N_0 and N_1. One can see that N_1 satisfies the linear diffusion equation

$$\frac{\partial}{\partial t}N_1=\frac{\partial}{\partial k_i}D_{ij}k^2\frac{\partial}{\partial k_j}N_1, \tag{4.11}$$

where Dij is the diffusion tensor,

$$D_{ij}=2\pi g^{3/2}\int_0^\infty dqq^{17/2}\int_0^{2\pi}d\theta_1$$
$$\times\int_0^{2\pi}d\theta_3|T(\theta_1,\theta_3)|^2p_ip_jN(\theta,q)N(\theta_3,q), \tag{4.12}$$

with

$$p_1=\cos\theta_1-\cos\theta_3, \quad p_2=\sin\theta_1-\sin\theta_3.$$

If the spectrum is isotropic and does not depend on the angle θ, we get the further simplification:

$$D_{ij}=D\delta_{ij}, \quad D=\frac{5}{8}\pi^3g^{3/2}\int_0^\infty q^{17/2}N^2(q)dq. \tag{4.13}$$

The diffusion coefficient D diverges at $k\to0$, if $x>19/4$. Thus, $x_2=19/4$.

Let us find how the function $F(x)$ behaves near $x=x_2$. In the isotopic case, Eq. (3.9) becomes

$$\frac{\partial N_1}{\partial t}=\frac{D}{k}\frac{\partial}{\partial k}k^3\frac{\partial}{\partial k}N_1. \tag{4.14}$$

As $k\to19/4$, we get the following estimate:

$$F(x)=\frac{19}{4}\frac{11}{4}\frac{5\pi^3}{16}\frac{1}{19/4-x}\simeq\frac{126.4}{19/4-x} \tag{4.15}$$

To find x_1, the lower end of the window, we should study the influence of short waves on long waves. Suppose that $|k_1|,|k_2|\gg k$. In the first approximation $|k_3|=|k|$, and the resonant interaction S_{nl} can be separated into two groups of terms: $S_{nl}=S_{nl}^{(1)}+S_{nl}^{(2)}$. For $S_{nl}^{(2)}$ the integrand includes the product $N_{k_1}N_{k_2}$. If we set $k_1=k_2$, we get the following expression for the low-frequency tail

$$S_{nl}^{(1)}=2\pi g^2\int|T_{kk_1,k_1,k_3}|^2\delta(\omega-\omega_{k_3})(N_{k_3}-N_k)N_{k_1}^2dk_1. \tag{4.16}$$

Notice, that if $|k_1|\gg|k|$, then $|T_{kk_1,k_1,k_3}|^2\simeq k_1^2$, and the integrand in Eq. (4.16) is proportional to $k_1^2N_{k_1}^2$. If $x<2$, the integral diverges.

The group of terms that are linear in the high-frequency tail of the spectrum is more complicated:

$$S_{nl}^{(2)}=2\pi g^2N_k\int|T_{kk_1k_2k_3}|^2N_{k_3}(N_{k_1}-N_{k_2})\delta(\omega_k+\omega_{k_1}-\omega_{k_2}$$
$$-\omega_{k_3})\delta(k+k_1-k_2-k_3)dk_1dk_2dk_3. \tag{4.17}$$

We can perform the expansion

$$N_{k_1}-N_{k_3}=p_i\frac{\partial N}{\partial k_{1i}}, \quad p_i=(k-k_3)_i. \tag{4.18}$$

In the general anisotropic case the integrand is proportional to $k_1^2(p\nabla N_{k_1})$ and a divergence occurs if $x=x_1-3$. However,

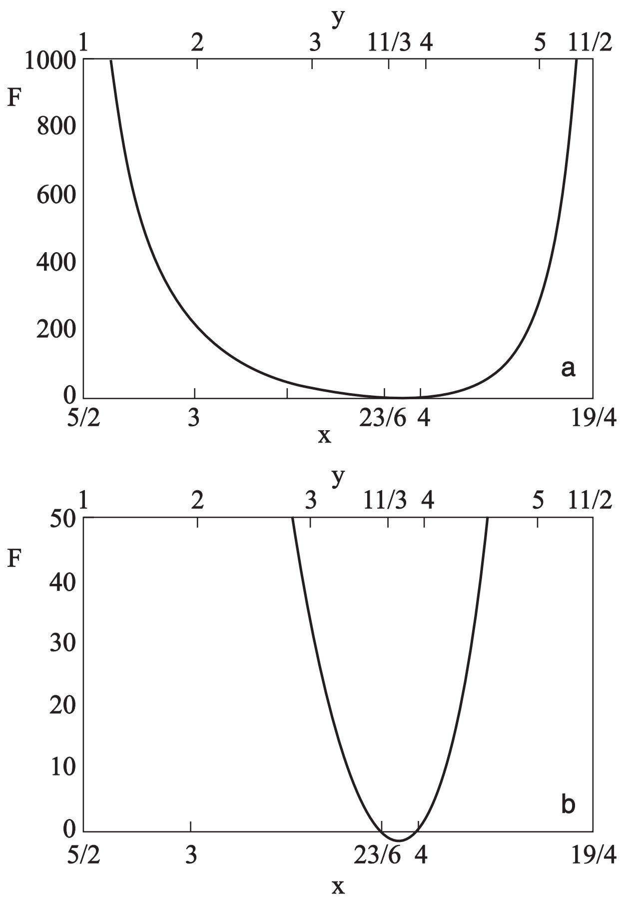

FIG. 1. A plot of the function F(x) (a). A plot of F(x) with a magnified vertical scale (b).

in the isotropic case this term, the most divergent one, is cancelled after integration with respect to the angles. In this case, we should examine the quadratic terms in the expansion of the integrand in powers of the parameter $(pk_1)/k_1^2$. The leading term arises from the expansion of the δ-function of the frequencies $\delta(\omega_{k_1}-\omega_{k_1-p}+\omega_k-\omega_{k_3})$. Integrating over the angles, we end up with the equation

$$\frac{\partial N_k}{\partial t}=qk^7N_k\frac{\partial N}{\partial k}, \tag{4.19}$$

where

$$q=\frac{25}{16}\pi^3g^{3/2}E=\frac{25}{8}\pi^3g^{3/2}\int_0^\infty k^{3/2}N_k dk.$$

Here E is the total energy. In the isotropic case, therefore, $x_1=5/2$ and for $F(x)$ we obtain the following estimate:

$$F=\frac{5}{2}\frac{25}{8}\pi^3\frac{1}{5/2-x}=\frac{241.86}{5/2-x}. \tag{4.20}$$

Figure 1a is a plot of $F(x)$ in the isotropic case that we calculated numerically. One can see that in the interval $x_1<x<x_2$ $F(x)$ has exactly two zeros at

$$x=y_1=4, \quad x=y_2=\frac{23}{6}. \tag{4.21}$$

To prove this result, let us assume that the spectra are isotropic and obey the differential conservation laws for energy and wave action:

$$\frac{\partial I_k}{\partial t}=2\pi k\omega_k\frac{\partial N_k}{\partial t}=-\frac{\partial P}{\partial k}, \tag{4.22}$$

$$P=2\pi\int_0^k k\omega_k S_{nl}dk, \tag{4.23}$$

$$2\pi k\frac{\partial N_k}{\partial t}=\frac{\partial Q}{\partial k}, \tag{4.24}$$

$$Q=2\pi\int_0^k kS_{nl}dk. \tag{4.25}$$

Here P is the flux of energy directed to high wave numbers, while Q is the flux of wave action directed to small wave numbers. The equations

$$P=P_0=\text{const}, \quad Q=Q_0=\text{const} \tag{4.26}$$

appear to be solutions of the stationary equation $S_{nl}=0$. We seek a solution of the power-law form $N=\lambda k^{-x}$; then Eqs. (4.23) and (4.25) become

$$P_0=2\pi g^2\lambda^3\frac{F(x)}{3(x-4)}k^{-3(x-4)}, \tag{4.27}$$

$$Q_0=-2\pi g^{3/2}\lambda^3\frac{F(x)}{3(x-26/3)}k^{-3(x-26/3)}. \tag{4.28}$$

One can see that P_0 and Q_0 are finite only if $F(4)=0$ and $F(26/3)=0$, and if $F'(4)>0$ and $F'(26/3)<0$. We conclude that the equation $S_{nl}=0$ has the following solutions:

$$N_k^{(1)}=c_p\left(\frac{P_0}{g^2}\right)^{1/3}\frac{1}{k^4}, \tag{4.29}$$

$$N_k^{(2)}=c_q\left(\frac{Q_0}{g^{3/2}}\right)^{1/3}\frac{1}{k^{23/6}}. \tag{4.30}$$

Here c_p and c_q are the dimensionless Kolmogorov constants

$$c_p=\left(\frac{3}{2\pi F'(4)}\right)^{1/3}, \quad c_q=\left(\frac{3}{2\pi|F'(23/6)|}\right)^{1/3}.$$

Figure 1b is a plot of $F(x)$ with a magnified vertical axis. The calculations yield $F'(4)=45.2$ and $F'(23/6)=-40.4$. Near the zeros $F(x)$ can be approximated by the parabola,

$$F(x)\simeq 256.8(x-23/6)(x-4). \tag{4.31}$$

Note that

$$F(9/2)=85.6 \tag{4.32}$$

thus, we get

$$c_p=0.219, \quad c_q=0.227, \tag{4.33}$$

and can see that both Kolmogorov constants are numerically small.

In the isotropic case, the energy spectrum $F(\omega)$ can be written in terms of N_k as

$$F(\omega)d\omega = 2\pi\omega_k N_k k dk, \tag{4.34}$$

and the energy spectrum corresponding to the solution (4.29) has the following form, known as the Zakharov–Filonenko spectrum:

$$F^{(1)}(\omega) = 4\pi c_p \left(\frac{P}{g^2}\right)^{1/3} \frac{g^2}{\omega^4}. \tag{4.35}$$

This spectrum was found as a solution of the equation $S_{nl} = 0$.[3] For the spatial spectrum

$$I_k dk = 2\pi\omega_k N(k) k dk, \tag{4.36}$$

the solution (4.30) transforms to

$$I_k^{(1)} = 2\pi c_p \left(\frac{P}{g^2}\right)^{1/3} \frac{g^{1/2}}{k^{5/2}} \simeq k^{-2.5}. \tag{4.37}$$

The spectra (4.29), (4.35), and (4.37) are realized if we have a source of energy that is concentrated at small wave numbers and generates an amount of energy P per unit time. For the spectrum (4.30), first reported by Zakharov in 1966,[3]

$$I_k^{(2)} = 2\pi c_q Q^{1/3} k^{-7/3} \simeq 2\pi c_q Q^{1/3} k^{2.33}, \tag{4.38}$$

and

$$F^{(2)}(\omega) = 4\pi c_q Q^{1/3} \frac{g^{4/3}}{\omega^{11/3}}. \tag{4.39}$$

The spectra (4.30) and (4.38) can be realized with a source of wave action operating at high wave numbers.

The spectra described here exhaust all the possible power-law isotropic solutions of the stationary kinetic equation $S_{nl}=0$. It is important to stress that the thermodynamic solutions $N=\text{const}$ and $N=c/k^{1/2}$ are not the solutions of this equation, because their exponents $x=0$ and $x=1/2$ lie far below the lower end of the "window of possibility" $x_1 = 5/2$. This means that thermodynamics has nothing in common with the theory of a wind-driven sea. The solutions (4.29) and (4.30) are not unique stationary solutions of $S_{nl} = 0$. The general isotropic solution describes a situation when an energy source at small wave numbers and a wave action source both exist simultaneously and have the following form:

$$N_k^{(3)} = c_p \left(\frac{P}{g^2}\right)^{1/3} \frac{1}{k^4} L\left(\frac{g^{1/2} Q k^{1/2}}{P}\right). \tag{4.40}$$

Here L is an unknown function of one variable,

$$L \to 1 \ at\ k \to 0, \quad L(\xi) \to \frac{c_q}{c_p} \xi^{1/3} \ at\ k \to \infty. \tag{4.41}$$

Note that if there is no flux of wave action from infinity, we must set $Q=0$. Under this constraint, the general isotropic solution is the Zakharov–Filonenko spectrum (4.29), parametrized by a single arbitrary constant P, which is the flux of energy to $k \to \infty$.

Frequency spectra with tails in the form $F(\omega) \simeq \omega^{-4}$ have been observed in numerous field experiments[11–16] and obtained in numerical simulations, as well.[17–19] Spatial spectra with asymptotes $I_k \simeq k^{5/2}$ have also been observed in many

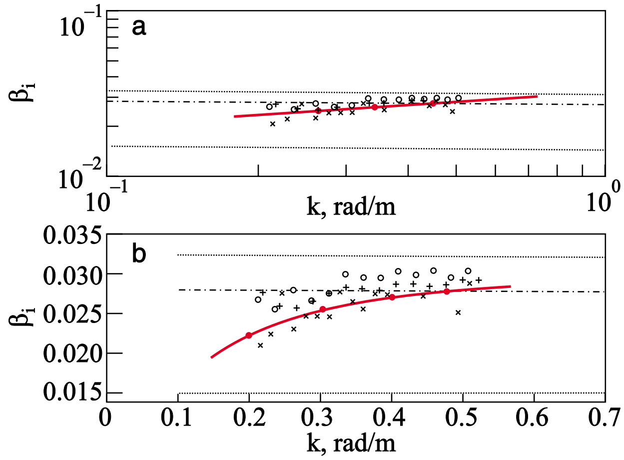

FIG. 2. The dimensionless wave number spectral coefficient β_i[20] plotted on logarithmic axes (a) and semi-log axes (b). Here the crosses represent the omnidirectional (angle averaged) spectrum and the dots correspond to $\xi(k) = 2\beta_I u_* g^{0.5} k^{-2.5}$. The solid line in (a) and solid curve in (b) correspond to $\xi(k) \simeq k^{7/3}$.

experiments.[20–22] A more careful study of the experimental results shows that in the majority of cases the spectral area right behind the spectral peak can be better approximated by a tail $\omega^{-11/3}$ in frequency spectrum and by a tail $k^{-7/3}$ in the spatial spectrum. This shows up especially clearly in the experiments by Huang *et al.*[20] Figure 2, which is taken from that article demonstrates the coexistence of both types of Kolmogorov–Zakharov (KZ) spectra.

V. STATIONARY SOLUTIONS OF KINETIC EQUATION: ANISOTROPIC CASE

In order to study the anisotropic solutions of Eq. (4.1), we introduce polar coordinates on the k-plane and set $k^2 = \omega/g$. Thereafter we shall use notation

$$N(\omega,\phi) d\omega d\phi = N(\mathbf{k}) d\mathbf{k},$$

$$N(\omega,\phi) = \frac{2\omega^3}{g^2} N(\mathbf{k}). \tag{5.1}$$

In the spatially homogenous case, $N(\omega,\phi)$ obeys the equation

$$\frac{\delta N(\omega,\phi)}{\partial t} = S_{nl}(\omega,\phi). \tag{5.2}$$

In the new variables:

$$\begin{aligned} S_{nl}(\omega,\phi) = 2\pi g^2 \int & |T_{\omega,\omega_1,\omega_2,\omega_3}|^2 \delta(\omega + \omega_1 - \omega_2 \\ & - \omega_3)\delta(\omega^2 \cos\phi + \omega_1^2 \cos\phi_1 - \omega_2^2 \cos\phi_2 \\ & - \omega_3^2 \cos\phi_3)\delta(\omega^2 \sin\phi + \omega_1^2 \sin\phi_1 \\ & - \omega_2^2 \sin\phi_2 - \omega_3^2 \sin\phi_2) \\ & \times\{\omega^3 N(\omega_1,\phi_1)N(\omega_2,\phi_2)N(\omega_3,\phi_3) \\ & + \omega_1^3 N(\omega,\phi)N(\omega_2,\phi_2)N(\omega_3,\phi_3) \\ & - \omega_2^3 N(\omega,\phi)N(\omega_1,\phi_1)N(\omega_3,\phi_3) \\ & - \omega_3^3 N(\omega,\phi)N(\omega_1,\phi_1)N(\omega_2,\phi_2)\} \end{aligned}$$

$$\times d\omega_1 d\omega_2 d\omega_3 d\phi_1 d\phi_2 d\phi_3. \quad (5.3)$$

This very form of S_{nl} was used in a numerical simulation of the Hasselmann equation. Suppose that $N(\omega,\phi)=\omega^{-z}$ is an isotropic spectrum. Then

$$S_{nl} = \frac{\omega^{-3z+13}}{4g^4} F\left(\frac{z+3}{2}\right) = \frac{G(z)}{g^4}\omega^{-3z+13}, \quad (5.4)$$

where $F(x)$ is defined by Eq. (4.5). Now the "window of opportunity" is: $2<z<13/2$. The zeros of $G(z)$ appear at $z_1=5$ and $z_2=14/3$ and near these zeros $G(z)$ can be represented by the parabola,

$$G(z) \simeq 16.05(z-5)(z-14/3). \quad (5.5)$$

To make the constants of motion more conspicuous, we introduce the elliptic differential operator

$$Lf(\omega,\phi) = \left(\frac{\partial^2}{\partial\omega^2} + \frac{2}{\omega^2}\frac{\partial^2}{\partial\phi^2}\right) f(\omega,\phi) \quad (5.6)$$

with the following parameters: $0<\omega<\infty$, $0<\phi<2\pi$. The equation

$$LG = \delta(\omega-\omega')\delta(\phi-\phi') \quad (5.7)$$

with the boundary conditions

$$G|_{\omega\to 0} = 0, \quad G_{\omega\to\infty} < \infty, \quad G(2\pi) = G(0),$$

can be resolved as

$$G(\omega,\omega',\phi-\phi') = \frac{1}{4\pi}\sqrt{\omega\omega'}\sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} \times \left[\left(\frac{\omega}{\omega'}\right)^{\Delta_n}\Theta(\omega'-\omega) + \left(\frac{\omega'}{\omega}\right)^{\Delta_n}\Theta(\omega-\omega')\right], \quad (5.8)$$

where $\Delta_n = 1/2\sqrt{1+8n^2}$. Now we represent S_{nl} in the form

$$A(\omega,\phi) = \int_0^\infty d\omega' \int_0^{2\pi} d\phi' G(\omega,\omega',\phi-\phi') S_{nl}(\omega',\phi'). \quad (5.9)$$

Notice that $A(\omega,\phi)$ is a regular integral operator and suppose that $N(\omega,\phi)=\omega^{-z}$. Then,

$$A[\omega^{-z}] = \frac{\omega^{-3z+15}}{g^4} H(z),$$

$$H(z) = \frac{G(z)}{9(z-5)(z-14/3)}. \quad (5.10)$$

$H(z)$ is positive and has no zeros. If $G(z)$ is represented by the parabola (5.5), $H(z)$ is just a constant:

$$H(z) = H_0 = 16.05/9 = 1.83. \quad (5.11)$$

This fact leads to a bold idea. If we assume that

$$A = \frac{H_0}{g^4}\omega^{15} N^3, \quad (5.12)$$

then the nonlinear term S_{nl} turns into the elliptic operator:

$$S_{nl} = \frac{H_0}{g^4}\left(\frac{\partial^2}{\partial\omega^2} + \frac{2}{\omega^2}\frac{\partial^2}{\partial\phi^2}\right)\omega^{15}N^3. \quad (5.13)$$

This is the so-called "diffusion approximation."[23] Being very simple, this approximation encompasses the basic features of the theory of a wind-driven sea. We will refer mostly to this model, keeping in mind that the real case, Eq. (5.9), does not differ much from it, at least qualitatively.

Let us integrate Eq. (5.2) over angles. This yields

$$\frac{\partial N(\omega,t)}{\partial t} = \frac{\partial Q}{\partial\omega}. \quad (5.14)$$

Here

$$B(\omega,t) = \frac{g}{2\omega}\int_0^{2\pi} \cos\phi N(\omega,\phi)d\phi, \quad (5.15)$$

and the flux of the wave action is

$$Q = \frac{\partial K}{\partial\omega}, \quad K = \int_0^{2\pi} A(\omega,\phi)d\phi. \quad (5.16)$$

After multiplying Eq. (5.14) by ω we obtain the equation

$$\frac{\partial F(\omega,t)}{\partial t} + \frac{\partial P}{\partial\omega} = 0, \quad (5.17)$$

where $P=K-\omega\partial K/\partial\omega$ is the energy flux.

Let us introduce now the following definitions: the angle-integrated spectral density of the momentum,

$$M_x(\omega,t) = \frac{\omega^2}{g}\int_0^{2\pi} \cos\phi B(\omega,\phi)d\phi, \quad (5.18)$$

the quantity

$$C_x(\omega,t) = \frac{\omega}{2g}\int_0^{2\pi} \cos^2\phi N(\omega,\phi)d\phi, \quad (5.19)$$

and the momentum flux

$$R_x = \int_0^{2\pi} \cos\phi\left(\omega A - \frac{\omega^2}{2}\frac{\partial A}{\partial\omega}\right)d\phi. \quad (5.20)$$

These quantities are all coupled by the equation

$$\frac{\partial M_x}{\partial t} + \frac{\partial R_x}{\partial\omega} = 0. \quad (5.21)$$

Equations (5.14), (5.17), and (5.21) are the angle-averaged balance equations for the basic conserved quantities. Now we can return to the question formulated above. How many solutions does the stationary kinetic equation (1.5) and (4.1) have? Notice that we simplified it to the linear equation

$$\left(\frac{\partial^2}{\partial\omega^2} + \frac{2}{\omega^2}\frac{\partial^2}{\partial\phi^2}\right)A = 0. \quad (5.22)$$

In particular, the kinetic equation has anisotropic KZ solution

$$A = \frac{1}{2\pi}\left\{P + \omega Q + \frac{R_x}{\omega}\cos\phi\right\}, \quad (5.23)$$

where P and Rx are the fluxes of energy and momentum as $\omega\to\infty$ and Q is the flux of wave action directed to small wave numbers. In general, Eq. (5.23) is a nonlinear integral equation; however in the diffusion approximation the KZ solution can be found explicitly as

$$N(\omega,\phi)=\frac{1}{(2\pi H_0)^{1/3}}\frac{g^{4/3}}{\omega^5}\left(P+\omega Q+\frac{R_x}{\omega}\cos\phi\right)^{1/3}. \tag{5.24}$$

By comparison with Eqs. (4.35) and (4.38) we easily find that in this case

$$c_p=c_q=\frac{1}{2(2\pi H_0)^{1/3}}=0.223,\quad H_0=1.83.$$

This is exactly the arithmetic mean of the Kolmogorov constants given by Eq. (3.31). Multiplying Eq. (5.24) by $2\pi\omega$ yields the general KZ spectrum in the diffusion approximation:

$$F(\omega)=2.78\frac{g^{4/3}}{\omega^4}\left(P+\omega Q+\frac{R_x}{\omega}\cos\phi\right)^{1/3}. \tag{5.25}$$

We must make sure that in the isotropic case, $Rx=0$, the expression

$$F(\omega)=2.78\frac{g^{4/3}}{\omega^4}(P+\omega Q)^{1/3} \tag{5.26}$$

approximates the generic KZ spectrum to within a few percent.

If we know the value of $A(\omega,\varphi)$ on the circle $\omega=\omega_0$, we can solve the external and internal Dirichlet boundary problems for Eq. (5.22) with boundary condition $A(\omega,\phi)<\infty$ at $\omega\to\infty$. Suppose that

$$A(\omega,\phi)=A_0(\phi)=A_0+\frac{A_1}{\omega}\cos\phi+\sum_{n=2}^{\infty}A_n\left(\frac{\omega_0}{\omega}\right)^{-1/2+\sqrt{1/4+4n^2}}\cos n\phi. \tag{5.27}$$

First two terms in (5.27) represent the KZ spectrum with $Q=0$, $P=2\pi A_n$, $R_x=2\pi\omega_0A_1$. The next terms describe the fast stabilization of any arbitrary solution to the KZ spectrum as $\omega/\omega_0\to\infty$. The first additional term in Eq. (5.27) decays as $(\omega_0/\omega)^{3.53}\cos 2\phi$.

This stabilization to the KZ spectrum is actually the "angular spreading" of wind-driven wave spectra that is usually observed in field experiments (see, for instance, Ref. 12). If $Q=0$, the general KZ solution (5.25) at $\omega\to 0$ is the following spectrum:

$$F(\omega)\to\frac{2.78}{\omega^4}g^{4/3}p^{1/3}\left(1+\frac{1}{3}\frac{R_x}{P\omega}\cos\phi+\cdots\right). \tag{5.28}$$

Similar results were predicted by Kontorovich and Kats[30] and Balk.[31]

From Eq. (5.27) one can see that $A(\omega,\varphi)$ is parametrized by a function of one variable, $A_0(\varphi)$. In the presence of a flux of action Q from infinity, Eq. (5.27) should be supplemented by an additional term Q_ω. Thus in the general case, the freedom in determining A involves a function that has one variable and one constant. We implicitly assume that the mapping $N\to A$ is uniquely invertible. This has not been proved, but is very plausible.

VI. DAMPING DUE TO NONLINEAR INTERACTION

How should we compare S_{nl} and $S_{\rm in}$? In this section we show that S_{nl} is the leading term in the balance equation (1.11). In fact, the forcing terms $S_{\rm in}$ and $S_{\rm dis}$ are not known well enough, so it is reasonable to accept the simplest models of both terms assuming that they are proportional to the action spectrum:

$$S_{\rm in}=\gamma_{\rm in}(k)N(k), \tag{6.1}$$

$$S_{\rm dis}=-\gamma_{\rm dis}(k)N(k). \tag{6.2}$$

Hence,

$$\gamma(k)=\gamma_{\rm in}(k)-\gamma_{\rm dis}(k). \tag{6.3}$$

In reality $\gamma_{\rm dis}(k)$ depends dramatically on the overall steepness μ. The balance kinetic equation (1.24) can be written in the form

$$S_{nl}+\gamma(k)N_k=0, \tag{6.4}$$

and the S_{nl} term can be represented as

$$S_{nl}=F_k-\Gamma_kN_k. \tag{6.5}$$

The definitions of Γ_k and F_k are given by Eqs. (2.5) and (2.6). The solution of the stationary equation (6.4)is

$$N_k=\frac{F_k}{\Gamma_k-\gamma_k}. \tag{6.6}$$

A positive solution exists if $\Gamma_k>\gamma_k$. The term Γ_k can be treated as the nonlinear damping owing to the four-wave interaction. This damping has a very powerful effect. A "naive" dimensional consideration gives

$$\Gamma_k\simeq\frac{4\pi g^2}{\omega_k}k^{10}N_k^2, \tag{6.7}$$

however, this estimate works only if $k\simeq k_p$, where k_p is the wave number of the spectral maximum.

Let $k\gg k_p$. Now for Γ_k one gets

$$\Gamma_k=2\pi g^2\int|T_{kk_1,kk_3}|^2\delta(\omega_{k_1}-\omega_{k_3})N_{k_1}N_{k_3}dk_1dk_2. \tag{6.8}$$

The main source of Γ_k is the interaction of long and short waves. To estimate the integral (2.6) more accurately, we assume that the spectrum of long waves is narrow in angle, with $N(k_1,\theta_1)=\tilde{N}(k_1)\delta(\theta_1)$. Long waves propagate along the x axis and **k** is the wave vector of short waves propagating in direction θ. For the coupling coefficient we must put set $T_{kk_1,k_2,k_3}\simeq 2k_1^2k\cos\theta$. Then

$$\Gamma_k=8\pi g^{3/2}k^2\cos^2\theta\int_0^\infty k_1^{13/2}\tilde{N}^2(k_1)dk_1. \tag{6.9}$$

Even for the most slowly decaying KZ spectrum, $N_k\simeq k^{-23/6}$, the integrand behaves as $k_1^{-7/6}$ and the integral diverges. For steeper KZ spectra the divergence is stronger.

Let us estimate Γ_k for the case of a "mature sea," when the spectrum can be taken in the form

$$N_k \simeq \frac{3}{2}\frac{E}{\sqrt{g}}\frac{k_p^{3/2}}{k^4}\theta(k-k_p). \quad (6.10)$$

Here E is the total energy. By plugging Eq. (6.10) to Eq. (6.9) one gets

$$\Gamma_\omega = 36\pi\omega\left(\frac{\omega}{\omega_p}\right)^3 \mu_p^4 \cos^2\theta, \quad (6.11)$$

which includes a huge enhancement factor: $36\pi \simeq 113.04$. For a very modest steepness, $\mu_p \simeq 0.05$, we get

$$\Gamma_\omega \simeq 7.06 \cdot 10^{-4}\omega\left(\frac{\omega}{\omega_p}\right)^3 \cos^2\theta. \quad (6.12)$$

In the isotropic case, to find Γ_k for $\omega/\omega_p \gg 1$ we need to take a simple integral over angles that yields

$$\int_0^{2\pi}\int_0^{2\pi} T^2_{\theta_1,\theta_2} d\theta_1 d\theta_2 = \frac{5}{2}(2\pi)^2,$$

so that, instead of Eq. (6.11), we get

$$\Gamma_k = 5\pi g^{3/2}k^2\int_0^\infty k_1^{13/2}\tilde{N}(k_1)^2 dk_1 \quad (6.13)$$

or

$$\Gamma_\omega = \frac{45\pi}{2}g^{3/2}\omega\left(\frac{\omega}{\omega_p}\right)^3\mu_p^4. \quad (6.14)$$

Finally, assuming that

$$N_{k_p} \simeq \frac{3}{2}\frac{E}{\sqrt{g}k_p^{5/2}},$$

we get from Eq. (6.8) the following estimate for $\Gamma_p = \Gamma|_{k=kp}$:

$$\Gamma_p \simeq 9\pi\omega_p\mu_p^4. \quad (6.15)$$

Even in this case we have a fairly high enhancement factor: $9\pi \simeq 28.26$. In fact in all known models Γ_k surpasses $\tilde{\gamma}_k$ by at least an order of magnitude, even for these very smooth waves.

In the presence of peakedness

$$\Gamma_p \simeq \Lambda\omega_p\mu_p^4. \quad (6.16)$$

Here $\Lambda \simeq 4\pi\omega_p/\delta\omega$ is the enhancement factor owing to peakedness. If $\Lambda\mu_p^2 \sim 1$, then Γ_p is associated with maximal growth of the modulational instability for monochromatic waves: $\Gamma_p \simeq \gamma_{\rm mod} \sim \omega_p\mu_p^2$. If $\Lambda \sim 1/\mu_p^2$, the nonlinearity becomes so strong that the weak-turbulent statistical approach is no longer applicable. This is s quite realistic situation. Suppose that $\mu_p \simeq 0.11$ and $\omega_p/\delta\omega \simeq 5$. Then $\Lambda\mu_p^2 \sim 0.76$ and the weak turbulence model is hardly correct. In the situation of strong nonlinearity a wind-driven sea generates freak waves (see Refs. 24 and 25). The very fact of their existence as a common phenomenon is an implicit proof that S_{nl} dominates the energy balance.

Note that Γ_k diverges for KZ spectra. However, this does endanger the existence of the spectra because in the full kinetic equation the divergence in Γ_k is cancelled by the divergence in F_k. Indeed, if we consider the contribution of small wave-numbers to the integral (2.5), we end up with

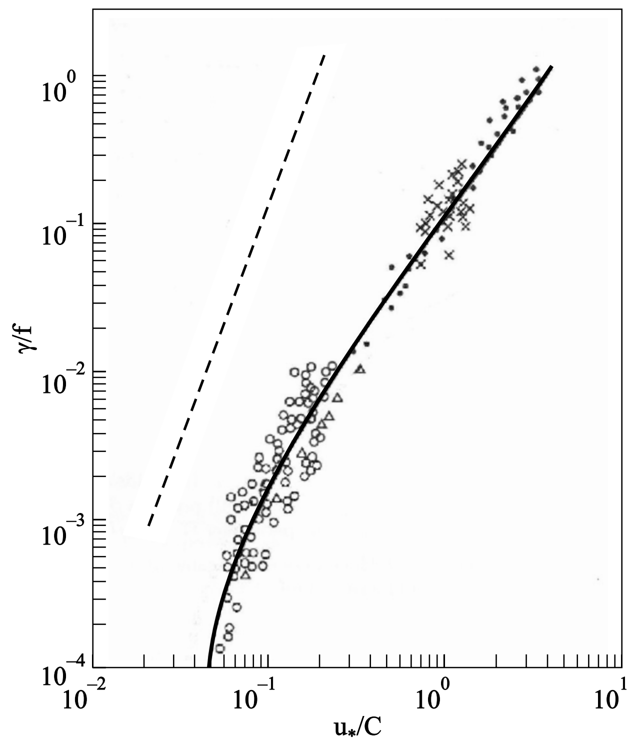

FIG. 3. Comparison of experimental data for the wind-induced growth rate $2\pi\gamma_{\rm in}(\omega)/\omega$ taken from Ref. 26 and the damping due to four-wave interactions $2\pi\Gamma(\omega)/\omega$, calculated for a narrow-angle spectrum with $\mu \simeq 0.05$ using Eq. (6.11) (dashed line).

$$F_k = 2\pi g^2 N_k \int |T_{kk_1,kk_3}|^2\delta(\omega_{k_1}-\omega_{k_3})N_{k_1}N_{k_3}dk_1dk_3 \simeq N_k\Gamma_k. \quad (6.17)$$

When γ_k is neglected, Eq. (4.1) is satisfied automatically.

The results obtained in this section show that the four-wave nonlinear interaction is a very strong effect. Strong turbulence of near-surface air boundary layer makes the development of a reliable theory of the air-water interaction, including a well-justified analytical calculation of γ_k, an extremely difficult task. Field and laboratory measurements of γ_k are difficult, and the scatter in the determination of γ_k is on the order of γ_k, itself. In any case, comparison of the Γ_k calculated above with experimental data on γ_k shows that Γ_k surpasses γ_k by at least an order of magnitude. This fact is illustrated in Fig. 3, where the experimental data are taken from Ref. 26.

As a result, we can conclude that S_{nl} is the leading term in the balance equation (1.11) and that the spectrum is described by solving Eq. (4.1), which has a rich family of solutions. In particular, this equation describes angular spreading.

In Fig. 4 we illustrate the fact that, for the nonlinear interaction term $S_{nl}=F_k-\Gamma_k N_k$, the magnitudes of the constituents, F_k and $\Gamma_k N_k$, essentially exceed their difference. Each is an order greater than S_{nl}!

The dominance of S_{nl} has not been apparent until now for two reasons. First, it is not correct to compare S_{nl} and $S_{\rm in}$; instead one should compare Γ_k and γ_k. Second, the widely accepted models for $S_{\rm dis}$ essentially overestimate the dissipation due to white capping. As a result, the dominance of S_{nl} is

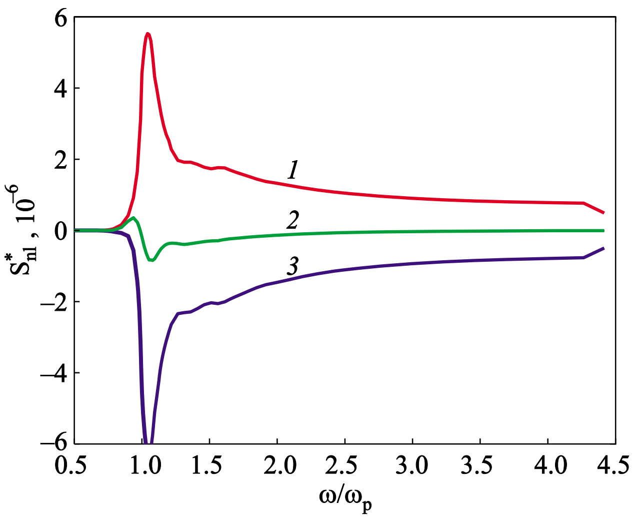

FIG. 4. The splitting of the nonlinear interaction term S_{nl} (2) into Fk (1) and $\Gamma_k N_k$ (3).

masked. We offer an alternative model for S_{dis} which will be published in a forthcoming article.[27] Preliminary results obtained in this direction were reported on ICNAAM-2009, Crete, Rethimno, September 2009.[28]

The author thanks Vladimir Geogjaev and Sergei Badulin for permission to include the numerical results presented in Figs. 1 and 4 of this talk. Details of these simulations will be published soon.

[a]Email: zakharov@math.arizona.edu

[1]A. S. Monin and V. P. Krasitskii, *Phenomena on Ocean Surface*, Gidrometizdat, Leningrad (1985) (in Russian).
[2]W. Rosental and F. Ziemer, in *Directional Ocean Wave Spectra*, Robert C. Beal (ed.), The John Hopkins Univ. Press, Baltimore (1991).
[3]V. E. Zakharov, *Some Questions on the Theory of Nonlinear Waves on the Surface of Fluid*, PhD *Thesis*, Budker Institute for Nuclear Physics, Novosibirsk (1966) (in Russian).
[4]V. E. Zakharov and N. N. Filonenko, *The Energy Spectrum for Stochastic Oscillations of a Fluid Surface*, Doc. Akad. Nauk SSSR **170**, 1292 (1966); V. E. Zakharov and N. N. Filonenko, J. Appl. Mech. Tech. Phys. **4**, 506 (1967).
[5]M. M. Zaslavski and V. E. Zakharov, Izv. Atm. Ocean. Phys. **18**, 747 (1982).
[6]V. E. Zakharov, J. Appl. Mech. Tech. Phys. **9**, 86 (1968).
[7]V. E. Zakharov, Am. Math. Soc. Transl. **182**, 167 (1998).
[8]V. E. Zakharov, Eur. J. Mech. B/Fluids **18**, 327 (1999).
[9]V. P. Krasitskii, Sov. Phys. JETP **98**, 1644 (1990).
[10]V. E. Zakharov and P. Lushnikov, Physica D **203**, 9 (2005).
[11]Y. Toba, J. Oceanogr. Soc. Jpn. **29**, 209 (1973).
[12]M. A. Donelan, J. Hamilton, and W. H. Hui, Philos. Trans. R. Soc. London, Ser. A **315**, 509 (1985).
[13]K. K. Kahma, Finn. Mar. Res. **253**, 52 (1986).
[14]O. M. Phillips, J. Fluid Mech. **156**, 505 (1985).
[15]G. Z. Forristall, J. Geophys. Res. **86**, 8075 (1981).
[16]P. C. Liu, J. Phys. Oceanogr. **1**, 249 (1981).
[17]A. I. Dyachenko, A. O. Korotkevich, and V. E. Zakharov, Phys. Rev. Lett. **92**, 134501 (2004).
[18]S. V. Nazarenko, J. Stat. Mech. L02002 (2006).
[19]S. Yu. Annenkov and V. I. Shrira, Phys. Rev. Lett. **96**, 204501 (2006).
[20]P. A. Hwang, W. C. Wang, E. J. Walsh, W. B. Krabill, and R. N. Swift, J. Phys. Oceanogr. **30**, 2753 (2000).
[21]C. E. Long and D. T. Resio, J. Geophys. Res. **112**, C05001 (2007).
[22]D. T. Resio, C. E. Long, and C. L. Vincent, J. Geophys. Res. **109**, C01018 (2004).
[23]V. Zakharov and A. Pushkarev, Nonlinear Processes Geophys. **6**, 1 (1999).
[24]V. E. Zakharov, A. I. Dyachenko, and A. O. Prokofiev, Eur. J. Mech. B/Fluids **25**, 677 (2006).
[25]A. I. Dyachenko and V. E. Zakharov, JETP Lett. **88**, 356 (2008).
[26]S. J. Komen, L. Cavaleri, M. Donelan, K. Hasselmann, S. Hasselmann, and P. A. E. M. Janssen, *Dynamics and Modelling of Ocean Waves*, Cambridge University Press (1994).
[27]V. E. Zakharov, A. O. Korotkevich, and A. O. Prokofiev, *On Dissipation Function of Ocean Waves due to Whitecapping* (*to be published*).
[28]V. E. Zakharov, A. O. Korotkevich, and A. O. Prokofiev, *Report on* ICNAAM-2009, Crete, Rethimno (2009).
[29]A. Pushkarev, D. Resio, and V. Zakharov, Physica D **184**, 29 (2003).
[30]A. V. Kats and V. M. Kontorovich, Sov. Phys. JETP Lett. **14**, 265 (1971).
[31]A. M. Balk, Physica D **139**, 137 (2000).

This article was published in English in the original Russian journal. Reproduced here with stylistic changes by AIP.