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Energy balance in a wind-driven sea

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Abstract
In this paper, we offer the answers to certain questions extremely important for the development of a self-consistent analytical theory for the wind-driven sea. (i) We discuss the separation into 'resonant' and 'slave' harmonics in an ensemble of weakly nonlinear gravity waves on the surface of deep water, and we construct an explicit form of the generation function for canonical transformation that eliminates the slave harmonics. (ii) When two waves compiling a quadruple are short in comparison with two others, we find an asymptotic form for the four-wave coupling coefficient. This result makes it possible to reduce the Hasselmann equation to the nonlinear diffusion equation, whose solution describes the well-known effect of angular spreading of wave spectra on its rear face. (iii) Studying the isotropic Kolmogorov–Zakharov solution of the Hasselmann equation, we find numerically the values of Kolmogorov constants. (iv) We calculate the nonlinear damping of surface waves appearing due to four-wave interaction and compare the damping with the growth rate of the instability of the wave surface induced by the wind. It is found that for all known models of wind input, the nonlinear damping surpasses the instability at least in order of magnitude. This result, supported by numerical simulation of the Hasselmann equation, leads to the conclusion: in a real sea, except for the case of very young waves, four-wave interaction is the dominant process. This statement opens the way for the development of a well-justified analytical theory for the wind-driven sea.

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(Some figures in this article are in colour only in the electronic version.)

1. Introduction

In our opinion, some important theoretical aspects of the physics of the wind-driven sea have not been clarified enough and need to be elucidated. The clarification is necessary for providing an adequate comparison of theory and experiment; without the clarification, the costly and laborious field and laboratory measurements cannot be properly interpreted and understood.

The first question is about the correct definition of wave action $N_k(t)$, which obeys the Hasselmann kinetic equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = S_{nl} + S_{\mathrm{in}} + S_{\mathrm{dis}}, \tag{1.1}$$

augmented by the source and the dissipation terms. How to find the current action spectrum $N_k(t)$ from experimental data? What is measured in the best experiments is the space–time spectrum

$$Q_{k\omega} = \langle |\eta_{k\omega}|^2 \rangle. \tag{1.2}$$

Here $\eta_{k\omega}$ is the Fourier transform of the surface elevation. The most advanced definition of wave action, used in many research papers (see, for example, [1, 2]), is the following:

$$N_k = \frac{2}{\omega_k} \int_0^\infty Q_{k\omega}\, \mathrm{d}\omega. \tag{1.3}$$

Equation (1.3) is certainly correct for waves of very small amplitude in the limit $\mu \to 0$, where μ is the characteristic average steepness of the surface. At a finite steepness, it can be treated as the first term in the expansion

$$N_k = N_0(k) + \mu^2 N_1(k) + \cdots. \tag{1.4}$$

Now $N_0(k)$ is given by equation (1.3), whereas $N_1(k)$ is to be determined. One may assume that this question is not very important because even for the steepest young waves, $\mu^2 \simeq 0.01$, and the accuracy of equation (1.3) is good. However, our preliminary estimates show that the ratio $N_1(k)/N_0(k)$ is a fast growing function of k; thus for spectral tails, the difference between N_k and $N_0(k)$ might be essential.

Now we formulate the inverse problem. Suppose we know N_k. How to find $Q_{k\omega}$?

In the linear approximation, at $\mu \to 0$, the answer is known:

$$Q_{k\omega} = \frac{\omega_k}{2}\left(N_k\,\delta(\omega-\omega_k) + N_{-k}\,\delta(\omega+\omega_{-k})\right). \tag{1.5}$$

What happens if μ is finite? In the neighborhood of $\omega = \omega_k$, we should perform the replacement

$$\delta(\omega-\omega_k) \to \frac{1}{\pi}\,\frac{\Gamma_k}{(\omega-\tilde{\omega}_k)^2+\Gamma_k^2}, \tag{1.6}$$

where $\tilde{\omega}_k = \omega_k + \mu^2\omega_{1k} + \cdots$ is the renormalized frequency and $\Gamma_k \simeq \mu^4\,\tilde{\Gamma}_k + \cdots$ is the effective dissipation due to four-wave processes. As long as μ^2 is small, one may assume that both the shifting of ω_k and the blurring of δ-function are weak effects. However, the quotients ω_{1k}/ω_k and $\tilde{\Gamma}_k/\omega_k$ are growing functions of k; thus for $k \gg k_p$ (k_p is the wave number of a spectral peak), derivation from the simple equation (1.5) could be essential. There is one more important effect. In a real sea, all waves can be separated into two classes: 'resonant waves' with $\omega \sim \omega_k$ and 'slave harmonics' caused by quadratic nonlinearity of primitive dynamic equations. The slave waves do not obey dispersion relations; as a result, their frequency spectrum for the given k is a broad function, not concentrated at $\omega \simeq \omega_k$.

Accurate determination of $N_1(k)$ at given $Q_{k\omega}$ and $Q_{k\omega}$ at given $N(k)$ is possible but is technically a cumbersome problem. In sections 2 and 3 we are taking the first but important steps to obtain their solution. In section 4 we study the axial asymmetric solutions of the equation

$$S_{nl} = 0, \tag{1.7}$$

which has been known since 1966 ([3]; see also [4, 5]). This equation has exactly two powerlike solutions:

$$N_1(k) = c_p\left(\frac{P}{g^2}\right)^{1/3}\frac{1}{k^4}, \tag{1.8}$$

$$N_2(k) = c_q\left(\frac{Q}{g^{3/2}}\right)^{1/2}\frac{1}{k^{23/6}}. \tag{1.9}$$

Equation (1.8) is known as the Zakharov–Filonenko spectrum [4]. Here P is the flux of energy from small wave numbers and Q is the flux of wave action from high wave numbers. The Kolmogorov constants c_p and c_q were not known, but have now been calculated:

$$c_p = 0.219, \qquad c_q = 0.227. \tag{1.10}$$

General isotropic solutions of equation (1.7) depend on two constants P and Q. In section 5, we discuss the general anisotropic solution of this equation. We show that the solution is defined by an arbitrary constant, the flux of wave action from high wave numbers, and an arbitrary function of angle. In the axially symmetric case this function degenerates to the constant P. The general anisotropic solution of equation (1.7) describes the angular spreading of a spectrum growing with frequency. Section 6 is most important from the practical viewpoint. We discuss the balance equation in the universal domain $\omega \gg \omega_p$,

$$S_{nl} + S_{\rm in} + S_{\rm dis} = 0. \tag{1.11}$$

Apparently, in some domain on the k-plane, $S_{\rm in} + S_{\rm dis} > 0$. Suppose that $S_{\rm in} = \gamma(k)\,N_k$. We note that S_{nl} can be presented in the form

$$S_{nl} = F_k - \Gamma_k\,N_k, \tag{1.12}$$

and the nonlinear wave interaction process is predominant if $\Gamma_k \gg \gamma_k$. We show that this condition is satisfied in a majority of realistic cases if the waves are not very young. This means that, as we claimed before, nonlinear wave interaction is the dominant process in the wind-driven sea.

2. What is wave action?

Consider the widely used Hasselmann equation:

$$\frac{\partial N}{\partial t} + \frac{\partial\tilde{\omega}}{\partial\vec{k}}\frac{\partial N}{\partial\vec{r}} = S_{nl}, \tag{2.1}$$

$$\begin{aligned} S_{nl} = \pi g^2 \int & |T_{kk_1,k_2k_3}|^2\,\delta(k+k_1-k_2-k_3) \\ & \times \delta(\omega_k+\omega_{k_1}-\omega_{k_2}-\omega_{k_3}) \\ & \times (N_{k_1}N_{k_2}N_{k_3} + N_kN_{k_2}N_{k_3} \\ & - N_kN_{k_1}N_{k_2} - N_kN_{k_1}N_{k_3})\,{\rm d}k_1\,{\rm d}k_2\,{\rm d}k_3. \end{aligned} \tag{2.2}$$

Here $\omega_k = \sqrt{g\,k\tanh kH}$, H is the depth, $T_{kk_1k_2k_3} = T_{k_1kk_2k_3} = T_{k_2k_3kk_1} = T_{kk_1k_3k_2}$ are the coupling coefficients, and

$$\tilde{\omega}(k) = \omega(k) + 2g\int T_{kk_1,kk_1}\,N_{k_1}\,{\rm d}k_1 \tag{2.3}$$

is the renormalized frequency.

As mentioned earlier, the nonlinear interaction term S_{nl} can be presented in the form

$$S_{nl} = F_k - \Gamma_k\,N_k, \tag{2.4}$$

where

$$\begin{aligned} F_k = \pi g^2 \int & |T_{kk_1k_2k_3}|^2\,\delta(k+k_1-k_2-k_3) \\ & \times \delta(\omega_k+\omega_{k_1}-\omega_{k_2}-\omega_{k_3})N_{k_1}N_{k_2}N_{k_3}\,{\rm d}k_1\,{\rm d}k_2\,{\rm d}k_3. \end{aligned} \tag{2.5}$$

and Γ_k, the dissipation rate due to the presence of four-wave processes, is the following:

$$\begin{aligned} \Gamma_k = \pi g^2 \int & |T_{kk_1,k_2k_3}|^2\,\delta(k+k_1-k_2-k_3) \\ & \times \delta(\omega_k+\omega_{k_1}-\omega_{k_2}-\omega_{k_3}) \\ & \times (N_{k_1}N_{k_2} + N_{k_1}N_{k_3} - N_{k_2}N_{k_3})\,{\rm d}k_1\,{\rm d}k_2\,{\rm d}k_3. \end{aligned} \tag{2.6}$$

One can say that in a real nonlinear sea, the dispersion relation $\omega = \omega_k$ is renormalized and becomes a complex function

$$\omega_k \to \tilde{\omega}_k + \tfrac{1}{2}\,\mathrm{i}\Gamma_k. \tag{2.7}$$

Equations (2.1) and (2.2) are written for the wave action spectrum $N_k(\vec{r}, t)$. What is the exact definition for the wave action? How can $N_k(\vec{r}, t)$ be expressed using the observable measurable quantities? These are not very simple questions.

Taking a snapshot of the surface from two points, one can get its stereoscopic image and restore the shape of elevation $\eta(\vec{r})$. If we perform nonsymmetric Fourier transform and define

$$\eta_k = \frac{1}{(2\pi)^2}\int \eta(\vec{r})\,\mathrm{e}^{-\mathrm{i}kr}\,\mathrm{d}\vec{r}, \tag{2.8}$$

we can introduce the spatial spectrum

$$Q_k = \langle |\eta_k|^2 \rangle. \tag{2.9}$$

Taking a series of snapshots at consecutive moments of time, one can restore the full space–time spectrum

$$Q_{k\omega} = \langle |\eta_{k\omega}|^2 \rangle. \tag{2.10}$$

Apparently,

$$Q_k = \int_{-\infty}^{\infty} Q_{k\omega}\,\mathrm{d}\omega. \tag{2.11}$$

What is wave action N_k? In some papers and monographs, we can find the following definition:

$$N_k = \frac{Q_k}{\omega_k}. \tag{2.12}$$

This is just carelessness. Spectrum Q_k is an even function, $Q_{-k} = Q_k$, while N_k certainly does not obey this restriction. One can present the spatial spectrum in the form

$$Q_k = \frac{\omega_k}{2}(n_k + n_{-k}), \tag{2.13}$$

where n_k is the wave action. We have deliberately denoted it by a lower case letter, because n_k and N_k are *different* wave actions.

The wave field consists of 'resonant' and 'slave' harmonics. The resonant harmonic with wave vector $\vec{k}$ has a frequency close to the renormalized frequency $\tilde{\omega}_k$. The strongest slave harmonics appear as a result of the interaction of two resonant harmonics. Suppose that they have wave vectors $\vec{k}_1, \vec{k}_2$. In the first order of nonlinearity, they generate four slave harmonics with wave vectors $\vec{p}_1, \vec{p}_2, -\vec{p}_1, -\vec{p}_2$ and frequencies $\Omega_1, \Omega_2, -\Omega_1, -\Omega_2$. Here $\vec{p}_1 = \vec{k}_1 - \vec{k}_2$, $\vec{p}_2 = \vec{k}_1 + \vec{k}_2$, and $\Omega_1 = \omega_1 - \omega_2$, $\Omega_2 = \omega_1 + \omega_2$. There is no definite relationship between the wave vector and frequency for slave harmonics.

Returning to the wave action, let us now explain the difference between n_k and N_k. N_k is the 'refined' wave action that includes resonant harmonics and slave harmonics of higher order only, and n_k is the 'total' wave action that includes both the resonant and the slave harmonics. Apparently, $n_k > N_k$ and is directly connected with the experimentally measurable spatial spectrum by equation (2.13). But n_k does not obey the Hasselmann equation. On the contrary, the 'purified' wave action N_k in principle cannot be measured in any kind of experiment. But exactly this sort of wave action satisfies the Hasselmann equation. As a result, all operational models solve the Hasselmann equation augmented with additional terms: S_{in}, the input from the wind, and S_{dis}, the dissipation due to wave breaking. Hence, the operational models do predict N_k. At the same time, experimentalists can measure n_k only.

At first glance, we see a serious discrepancy; however, nobody pays any attention to it. Why does this happen?

To give an answer, we should estimate the relative difference between n_k and N_k. Let us denote

$$\alpha(k) = \frac{n_k - N_k}{n_k}. \tag{2.14}$$

In a typical observed spectrum of the wind-driven sea, we should separate the spectral area near the peak frequency $\omega \sim \omega_p$ and the tail $\omega \gg \omega_p$. In the energy capacitive spectral band close to ω_p, α is small:

$$\alpha \sim \mu^2.$$

The characteristic steepness μ is defined as

$$\mu^2 \simeq \frac{\omega_p^4}{g^2}\sigma^2,$$

where σ is the total energy of waves. Even for young waves, $\mu^2 \leqslant 0.01$; thus the relative difference between n and N for deep water is not more than 1% and can easily be neglected. However, $\alpha(k)$ is a fast growing function of k. An accurate estimate of the dependence of α on frequency at $\omega \geqslant \omega_p$ is not the subject of this paper. An article on this topic will be submitted for publication soon; however, our preliminary results show that this dependence is fast growing:

$$\alpha \simeq \mu^2 \left(\frac{\omega}{\omega_p}\right)^3. \tag{2.15}$$

As mentioned above, in the area $\omega \sim \omega_p$ one can neglect the difference between n_k and N_k. In this area, we can replace equation (2.9) by

$$Q_k = \frac{\omega_k}{2}(N_k + N_{-k}). \tag{2.16}$$

There is an essential difference between equations (2.13) and (2.16). Because $n_k > 0$ for any k, wave vectors of slave harmonics cover all of the k-plane; thus the determination of n_k from Q_k is impossible in principle. In contrast, in many practical cases, N_k is nonzero only inside the bounded domain G on the k-plane. At the same time, $N_{-k} \neq 0$ inside the domain $\tilde{G}$ only, which is radially symmetric to G. In other words, if the vector $\vec{k}$ belongs to G, the vector $-\vec{k}$ belongs to $\tilde{G}$. Suppose that G and $\tilde{G}$ have no intersection. In this case, in the domain G we have $N_k = 2Q_k/\omega_k$. Despite the presence of factor 2 in (2.13), the integral identity $\int Q_k\,\mathrm{d}k = \int \omega_k N_k\,\mathrm{d}k$ remains the same as if we had used the naive and blatantly incorrect equation (2.12).

In some important cases, domains G and $\tilde{G}$ have intersection. In this case, we face an ambiguity in the determination of N_k from equation (2.16). To overcome this ambiguity, one should use the space–time spectrum $Q_{k,\omega}$ and define

$$n_k = \frac{2}{\omega_k}\int_0^\infty Q(k,\omega)\,\mathrm{d}\omega. \tag{2.17}$$

An equivalent formula is presented in the monograph by Monin and Krasitsky [1] published in Russia in 1985. It was also used by Rosental *et al* [2] at approximately the same time. In this case again

$$\int \omega_k\, n_k\, \mathrm{d}k = \int_{-\infty}^{\infty} Q(k,\omega)\,\mathrm{d}\omega\,\mathrm{d}k. \tag{2.18}$$

Let us note that equations (2.13) and (2.17) account for slave harmonics and can be used in the comparison of spectral tails obtained from the experiment and those obtained from the solution of the Hasselmann equation, both numerical and analytical, with caution. They work out to an accuracy of μ^2 in the neighborhood of the spectral peak, but can lead to essential errors in the area of spectral tails. A preliminary estimation for the accuracy of expression (2.17) will be made in the next section.

3. How to separate resonant and slave harmonics?

To perform an accurate separation into resonant and slave harmonics and find an explicit formula that connects $Q(k,\omega)$ and N_k, one should use Hamiltonian formalism and implement the canonical transformation, excluding cubic terms in the Hamiltonian. This is a cumbersome mathematical procedure. In this section, we will demonstrate how it could be done in the most economical way.

We study the weakly nonlinear waves on the surface of an ideal fluid of infinite depth in an infinite basin. The vertical coordinate is

$$-H < z < \eta(r,t), \qquad r = (x,y), \tag{3.1}$$

the fluid is incompressible, H is the depth of the fluid,

$$\operatorname{div} V = 0 \tag{3.2}$$

and velocity V is a potential field

$$V = \nabla\Phi, \tag{3.3}$$

where potential Φ satisfies the Laplace equation

$$\Delta\Phi = 0 \tag{3.4}$$

under the boundary conditions

$$\Phi|_{z=\eta} = \Psi(r,t), \qquad \Phi_z|_{z=-\infty} = 0. \tag{3.5}$$

The total energy of the fluid, $H = T + U$, has the following terms:

$$T = \frac{1}{2}\int \mathrm{d}\vec{r}\int_{-\infty}^{\eta} (\nabla\Phi)^2\,\mathrm{d}z = \frac{1}{2}\int \Psi\,\Phi_n\,\mathrm{d}S, \tag{3.6}$$

$$U = \frac{1}{2}\,g\int \eta^2\,\mathrm{d}\vec{r}. \tag{3.7}$$

The Dirichlet–Neumann boundary problem (3.4) and (3.5) is uniquely resolved; thus the flow is defined by fixing η and Ψ. This pair of variables is canonical; thus the evolution equations for η and Ψ take the form [6]

$$\frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta\Psi}, \qquad \frac{\partial\Psi}{\partial t} = -\frac{\delta H}{\delta\eta}. \tag{3.8}$$

After non-symmetric Fourier transform,

$$\Psi(r) = \int \Psi(k)\,\mathrm{e}^{\mathrm{i}kr}\,\mathrm{d}k, \qquad \Psi(k) = \frac{1}{(2\pi)^2}\int \Psi(r)\,\mathrm{e}^{-\mathrm{i}kr}\,\mathrm{d}r, \tag{3.9}$$

equation (3.8) reads

$$\frac{\partial\eta}{\partial t} = \frac{\delta\tilde{H}}{\delta\Psi_k^*}, \qquad \frac{\partial\Psi}{\partial t} = -\frac{\delta\tilde{H}}{\delta\eta_k^*}, \tag{3.10}$$

$$\tilde{H} = \frac{1}{4\pi^2}\,H = H_0 + H_1 + H_2 + \cdots. \tag{3.11}$$

In [7–9], it was shown that the Hamiltonian $\tilde{H}$ can be expanded in Taylor series in powers of $k\eta_k$:

$$\begin{aligned}
H_0 &= \frac{1}{2}\int \left\{A_k|\Psi_k|^2 + g\,|\eta_k|^2\right\}\mathrm{d}k, \qquad A_k = k\tan kH,\\
H_1 &= \frac{1}{2}\int L^{(1)}(k_1,k_2)\Psi_{k_1}\Psi_{k_2}\eta_{k_3}\,\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3)\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3,\\
H_2 &= \frac{1}{2}\int L^{(2)}(k_1,k_2,k_3,k_4)\Psi_{k_1}\Psi_{k_2}\eta_{k_3}\eta_{k_4}\\
&\quad\times\delta(k_1+k_2+k_3+k_4)\,\mathrm{d}k_1\,\mathrm{d}k_2\,\eta_{k_3}\eta_{k_4}.
\end{aligned} \tag{3.12}$$

Here

$$\begin{aligned}
L^{(1)}(k_1,k_2) &= -(k_1,k_2) - A_{k_1}A_{k_2}\\
L^{(2)}(k_1,k_2,k_3,k_4) &= \tfrac{1}{2}\,(k_1^2A_2 + k_2^2A_1) + \tfrac{1}{4}A_1A_2(A_{1+3} + A_{2+4}\\
&\quad + A_{1+4} + A_{2+3}).
\end{aligned} \tag{3.13}$$

Now we can introduce normal variables a_k:

$$\begin{aligned}
\eta_k &= \frac{1}{\sqrt{2}}\left(\frac{A_k}{g}\right)^{1/4}(a_k + a_{-k}^*),\\
\Psi_k &= \frac{\mathrm{i}}{\sqrt{2}}\left(\frac{g}{A_k}\right)^{1/4}(a_k - a_{-k}^*).
\end{aligned} \tag{3.14}$$

Normal variables obey the following Hamiltonian equations:

$$\frac{\partial a_k}{\partial t} + \mathrm{i}\frac{\delta H}{\delta a_k^*} = 0. \tag{3.15}$$

All terms in the expansion of Hamiltonian (3.11) must be expressed in terms of a_k:

$$\begin{aligned}
H_0 &= \int \omega_k|a_k|^2\,\mathrm{d}k,\\
H_1 &= \frac{1}{2}\int V^{(1,2)}_{kk_ak_2}(a_ka_{k_1}^*a_{k_2}^* + a_k^*a_{k_1}a_{k_2})\\
&\quad\times\delta(k-k_1-k_2)\,\mathrm{d}k\,\mathrm{d}k_1\,\mathrm{d}k_2\\
&\quad+\frac{1}{6}\int V^{(0,3)}_{kk_ak_2}(a_ka_{k_1}a_{k_2} + a_k^*a_{k_1}^*a_{k_2}^*)\\
&\quad\times\delta(k+k_1+k_2)\,\mathrm{d}k\,\mathrm{d}k_1\,\mathrm{d}k_2.
\end{aligned} \tag{3.16}$$

$$V^{(1,2)}_{kk_1k_2} = \frac{g^{1/4}}{2\sqrt{2}}\left\{\left(\frac{A_k}{A_{k_1}A_{k_2}}\right)^{1/4} L^{(1)}(k_1,k_2) - \left(\frac{A_{k_1}}{A_kA_{k_2}}\right)^{1/4}\right. \\ \left. \times L^{(1)}(-k,k_1) - \left(\frac{A_{k_2}}{A_kA_{k_1}}\right)^{1/4} L^{(1)}(-k,k_2)\right\}, \tag{3.17}$$

$$V^{(0,3)}_{kk_1k_2} = \frac{g^{1/4}}{2\sqrt{2}}\left\{\left(\frac{A_k}{A_{k_1}A_{k_2}}\right)^{1/4} L^{(1)}(k_1,k_2) + \left(\frac{A_{k_1}}{A_kA_{k_2}}\right)^{1/4}\right. \\ \left. \times L^{(1)}(k,k_1) + \left(\frac{A_{k_2}}{A_kA_{k_1}}\right)^{1/4} L^{(1)}(k,k_2)\right\}. \tag{3.18}$$

Now we can define the 'total' or rough action:

$$n_k\,\delta(k-k') = g\langle a_k\,a^*_{k'}\rangle. \tag{3.19}$$

It is clear that the fundamental relation (2.13) is satisfied. Then, we perform the Fourier transform in time

$$a_{k\omega} = \frac{1}{2\pi}\int a(k,t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t \tag{3.20}$$

and introduce

$$n_{k\omega}\,\delta(k-k')\,\delta(\omega-\omega') = g\langle a_{k\omega}\,a^*_{k',\omega'}\rangle. \tag{3.21}$$

The space–time spectrum of elevation is simply

$$Q_{k,\omega} = \frac{\omega_k}{2}(n_{k,\omega}+n_{-k,-\omega}). \tag{3.22}$$

To separate resonant and slave harmonics, we must perform a canonical transformation to new variables, excluding cubic terms in the Hamiltonian. This is a standard procedure known in celestial dynamics down to the nineteenth century. However, in our case, this procedure is rather cumbersome. It was first performed by Krasitski [9]. He found that initial canonical variables a_k transform to new canonical variables b_k, which contain first-order slave harmonics only. Variables a_k are presented by an infinite series as new variables b_k:

$$a_k = b_k + a^{(1)}_k + a^{(2)}_k + a^{(3)}_k. \tag{3.23}$$

He calculated the first two terms in this expansion and found the following expressions:

$$a^{(1)}_k = \int \Gamma^{(1)}(\vec{k},\vec{k}_1,\vec{k}_2)\,b_{k_1}\,b_{k_2}\,\delta(\vec{k}-\vec{k}_1-\vec{k}_2)\,\mathrm{d}k_1\,\mathrm{d}k_2 \\ -2\int \Gamma^{(1)}(\vec{k}_2,\vec{k},\vec{k}_1)\,b^*_{k_1}\,b_{k_2}\,\delta(\vec{k}+\vec{k}_1-\vec{k}_2)\,\mathrm{d}k_1\,\mathrm{d}k_2 \\ +\int \Gamma^{(2)}(\vec{k},\vec{k}_1,\vec{k}_2)\,b^*_{k_1}\,b^*_{k_2}\,\delta(\vec{k}+\vec{k}_1+\vec{k}_2)\,\mathrm{d}k_1\,\mathrm{d}k_2,$$

$$a^{(2)}_k = \int B(\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3)\,b^*_{k_1}\,b_{k_2}\,b_{k_3} \\ \times\delta(\vec{k}+\vec{k}_1-\vec{k}_2-\vec{k}_3)\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3+\cdots, \tag{3.24}$$

where

$$\Gamma^{(1)}(\vec{k},\vec{k}_1,\vec{k}_2) = -\frac{1}{2}\frac{V^{(1,2)}(\vec{k},\vec{k}_1,\vec{k}_2)}{(\omega_k-\omega_{k_1}-\omega_{k_2})}, \\ \Gamma^{(2)}(\vec{k},\vec{k}_1,\vec{k}_2) = -\frac{1}{2}\frac{V^{(0,3)}(\vec{k},\vec{k}_1,\vec{k}_2)}{(\omega_k+\omega_{k_1}+\omega_{k_2})}, \tag{3.25}$$

and

$$B(\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3) = \Gamma^{(1)}(\vec{k}_1,\vec{k}_2,\vec{k}_1-\vec{k}_2)\,\Gamma^{(1)}(\vec{k}_3,\vec{k},\vec{k}_3-\vec{k}) \\ +\Gamma^{(1)}(\vec{k}_1,\vec{k}_3,\vec{k}_1-\vec{k}_3)\,\Gamma^{(1)}(\vec{k}_2,\vec{k},\vec{k}_2-\vec{k}) \\ -\Gamma^{(1)}(\vec{k},\vec{k}_2,\vec{k}-\vec{k}_2)\,\Gamma^{(1)}(\vec{k}_3,\vec{k}_1,\vec{k}_3-\vec{k}_1) \\ -\Gamma^{(1)}(\vec{k}_1,\vec{k}_3,\vec{k}_1-\vec{k}_3)\,\Gamma^{(1)}(\vec{k}_2,\vec{k}_1,\vec{k}_2-\vec{k}_1) \\ -\Gamma^{(1)}(\vec{k}+\vec{k}_1,\vec{k},\vec{k}_1)\,\Gamma^{(1)}(\vec{k}_2+\vec{k}_3,\vec{k}_2,\vec{k}_3) \\ +\Gamma^{(2)}(-\vec{k}-\vec{k}_1,\vec{k},\vec{k}_1)\,\Gamma^{(2)}(-\vec{k}_2-\vec{k}_3,\vec{k}_2,\vec{k}_3). \tag{3.26}$$

In our opinion, Krasitski used a rather long method for the calculation of terms in expansion (3.23). He directly checked the validity of the canonicity condition

$$\{a_k,a_{k'}\} = \int\left\{\frac{\delta a_k}{\delta b_{k''}}\frac{\delta a_{k'}}{\delta b^*_{k''}} - \frac{\delta a_k}{\delta b^*_{k''}}\frac{\delta a_{k'}}{\delta b_{k''}}\right\}\mathrm{d}k'' = 0, \\ \{a_k,a^*_{k'}\} = \int\left\{\frac{\delta a_k}{\delta b_{k''}}\frac{\delta a^*_{k'}}{\delta b^*_{k''}} - \frac{\delta a_k}{\delta b^*_{k''}}\frac{\delta a^*_{k'}}{\delta b_{k''}}\right\}\mathrm{d}k'' = \delta(k-k'). \tag{3.27}$$

Calculation of $a^{(3)}_k$ by this method is just an impossibly complicated task. The canonical transformation can be found using more sophisticated methods; the first one was offered in [7] in 1998. Let us consider that a_k is a solution of the Hamiltonian system

$$\frac{\partial a_k}{\partial\tau} + \mathrm{i}\frac{\delta R}{\delta a^*_k} = 0, \tag{3.28}$$

where τ is 'artificial time' and R is an efficient Hamiltonian:

$$R = \mathrm{i}\int \Gamma^{(1)}_{kk_2k_2}(a^*_ka_{k_1}a_{k_2} - a_ka^*_{k_1}a^*_{k_2}) \\ \times\delta(k-k_1-k_2)\,\mathrm{d}k\,\mathrm{d}k_1\,\mathrm{d}k_2 \\ +\frac{\mathrm{i}}{3}\int \Gamma^{(2)}_{kk_1k_2}(a^*_ka^*_{k_1}a^*_{k_2} - a_ka_{k_1}a_{k_2}) \\ \times\delta(k+k_1+k_2)\,\mathrm{d}k\,\mathrm{d}k_1\,\mathrm{d}k_2. \tag{3.29}$$

Equations (3.28) and (3.29) must be augmented with the initial condition

$$a_k|_{\tau=0} = b_k. \tag{3.30}$$

The needed canonical transformation will be obtained if we put $\tau = 1$. Expanding the solution in Taylor series of τ and putting $\tau = 1$ at the end, we reproduce the result of Krasitski (3.24)–(3.26) in a much more economical way.

Now we demonstrate another, more traditional method for constructing canonical transformation, which is based on finding the generating function. We present a_k in the form

$$a_k = \frac{1}{\sqrt{2}}(q_k+\mathrm{i}p_k),\qquad q_{-k}=q^*_k,\qquad p_{-k}=p^*_k.$$

The functions q_k, p_k obey the equations

$$\frac{\partial q_k}{\partial t} = \frac{\delta H}{\delta p^*_k},\qquad \frac{\partial p_k}{\partial t} = -\frac{\delta H}{\delta q^*_k}, \tag{3.31}$$

where H is the same Hamiltonian expressed through q_k, p_k. Now

$$H_0 = \frac{1}{2}\int \omega_k(|q_k|^2+|p_k|^2)\,\mathrm{d}k, \tag{3.32}$$

$$H_1 = \frac{1}{2}\int L_{kk_1k_2}\, q_k\, p_{k_1}\, p_{k_2}\,\delta(k+k_1+k_2)\,\mathrm{d}k\,\mathrm{d}k_1\,\mathrm{d}k_2, \tag{3.33}$$

$$L_{kk_1k_2} = \frac{g^{1/4}A_k^{1/4}}{A_{k_1}^{1/4}A_{k_2}^{1/2}}\,L^{(1)}_{k_1k_2}. \tag{3.34}$$

We will perform transformation to new variables R_k, ξ_k using the following generation function (see also [10]):

$$\begin{aligned} S = &\int R_k\, q_k\,\mathrm{d}k + \frac{1}{2}\int A_{kk_1k_2}\, q_k\, q_{k_1}\, R_{k_2}\\ &\times \delta(k+k_1+k_2)\,\mathrm{d}k\,\mathrm{d}k_1\,\mathrm{d}k_2\\ &+\frac{1}{3}\int B_{kk_1k_2}\, R_k\, R_{k_1}\, R_{k_2}\,\delta(k+k_1+k_2)\,\mathrm{d}k\,\mathrm{d}k_1\,\mathrm{d}k_2. \end{aligned} \tag{3.35}$$

The 'old momentum' p_k and the 'new coordinates' ξ_k are expressed as follows:

$$p_k = \frac{\delta S}{\delta q_{-k}} = R_k + \int A_{-k,k_1,k_2}\, q_{k_1}\, R_{k_2}\,\delta(k-k_1-k_2)\,\mathrm{d}k_1\,\mathrm{d}k_2, \tag{3.36}$$

$$\begin{aligned} \xi_k = \frac{\delta S}{\delta R_{-k}} = q_k + \frac{1}{2}&\int A_{k_1,k_2,-k}\, q_{k_1}\, q_{k_2}\,\delta(k-k_1-k_2^*)\,\mathrm{d}k_1\,\mathrm{d}k_2\\ &+\int B_{-k,k_1,k_2}\, R_{k_1}\, R_{k_2}\,\delta(k-k_1-k-2)\,\mathrm{d}k_1\,\mathrm{d}k_2. \end{aligned} \tag{3.37}$$

Apparently $B_{kk_1k_2}$ is symmetric with respect to all permutations and $A_{kk_1k_2} = A_{kk_2k_1}$. To find A, B, we note that in the first approximation

$$\begin{aligned} q_k = \xi_k - \frac{1}{2}&\int A_{k_1,k_2,-k}\,\xi_{k_1}\,\xi_{k_2}\,\delta(k-k_1-k_2)\,\mathrm{d}k_1\,\mathrm{d}k_2\\ &-\int B_{-k,k_1,k_2}\, R_{k_1}\, R_{k_2}\,\delta(k-k_1-k_2)\,\mathrm{d}k_1\,\mathrm{d}k_2 \end{aligned} \tag{3.38}$$

and in equation (3.36) we can replace $q_k \to \xi_k$. Now we plug q_k, p_k into equaton (3.32). In equation (3.33), we can just replace $q_k \to \xi_k$ and $p_k \to R_k$. From the condition for eliminating cubic terms that are proportional to $\xi_k\xi_{k_1}\xi_{k_2}$ and $\xi_k p_{k_1} p_{k_2}$, and the symmetry conditions, we find, after some calculations, the following nice and elegant expressions for A, B:

$$\begin{aligned} A_{kk_1k_2} = &-\frac{1}{4}\left(\frac{L_0+L_1+L_2}{\omega_0+\omega_1+\omega_2}+\frac{L_0+L_1-L_2}{\omega_0+\omega_1-\omega_2}\right)\\ &+\frac{1}{4}\left(\frac{L_0-L_1-L_2}{\omega_0-\omega_1-\omega_2}+\frac{L_1-L_0-L_2}{\omega_1-\omega_0-\omega_2}\right), \end{aligned} \tag{3.39}$$

$$\begin{aligned} B_{kk_1k_2} = &-\frac{1}{4}\left(\frac{L_0+L_1+L_2}{\omega_0+\omega_1+\omega_2}+\frac{L_0-L_1-L_2}{\omega_0-\omega_1-\omega_2}\right)\\ &-\frac{1}{4}\left(\frac{L_1-L_0-L_2}{\omega_1-\omega_0-\omega_2}+\frac{L_2-L_0-L_1}{\omega_2-\omega_0-\omega_1}\right). \end{aligned} \tag{3.40}$$

Here

$$\begin{aligned} &L_0 = L_{kk_1k_2}, \quad L_1 = L_{k_1kk_2}, \quad L_2 = L_{k_2kk_1},\\ &\omega_0 = \omega_k, \quad \omega_1 = \omega_{k_1}, \quad \omega_2 = \omega_{k_2}. \end{aligned} \tag{3.41}$$

To reproduce the results of Krasitski one has to expand old variables q_k, p_k in powers of new variables ξ_k, R_k, and then b_k will be as follows:

$$b_k = \frac{1}{\sqrt{2}}\left(\left(\frac{g}{A_k}\right)^{1/4}\xi_k - \mathrm{i}\left(\frac{A_k}{g}\right)^{1/4}R_k\right). \tag{3.42}$$

New normal variables b_k satisfy Zakharov's equation [6]

$$\frac{\partial b_k}{\partial t} + \mathrm{i}\omega_k\, b_k + \frac{\mathrm{i}}{2}\int T_{kk_1k_2k_3}\, b^*_{k_1}\, b_{k_2}\, b_{k_3}\,\delta_{k+k_1-k_2-k_3}\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3 = 0. \tag{3.43}$$

Here $T_{kk_1k_2k_3}$ is the same as in equation (2.2). An explicit expression for $T_{kk_1k_2k_3}$ is too complicated to be presented here. Note that now we can calculate $n_k = |a_k|^2$ by the use of expansion (3.23). We will assume that triple correlations of new variables are zero

$$\langle b_k b_{k_1} b_{k_2}\rangle = 0, \qquad \langle b^*_k b_{k_1} b_{k_2}\rangle = 0. \tag{3.44}$$

We also use the Gaussian closure for quartic variables

$$\langle b^*_k b^*_{k_1} b_{k_2} b_{k_3}\rangle = N_k\, N_{k_1}(\delta_{k-k_2}\,\delta_{k_1-k_3} + \delta_{k-k_3}\,\delta_{k_1-k_2}). \tag{3.45}$$

Here N_k is the 'refined' action. After some calculations, we find that n_k and N_k are connected by the following relation (it can be found in [8]):

$$\begin{aligned} n_k = N_k + \frac{1}{2}&\int \frac{|V^{(1,2)}(\vec{k},\vec{k}_1,\vec{k}_2)|^2}{(\omega_k-\omega_{k_1}-\omega_{k_2})^2}\\ &\times(N_{k_1}N_{k_2} - N_kN_{k_1} - N_kN_{k_2})\,\delta(\vec{k}-\vec{k}_1-\vec{k}_2)\,\mathrm{d}k_1\,\mathrm{d}k_2\\ &+\frac{1}{2}\int\frac{|V^{(1,2)}(\vec{k},\vec{k}_1,\vec{k}_2)|^2}{(\omega_{k_1}-\omega_k-\omega_{k_2})^2}\\ &\times(N_{k_1}N_{k_2} + N_kN_{k_1} - N_kN_{k_2})\,\delta(\vec{k}_1-\vec{k}-\vec{k}_2)\,\mathrm{d}k_1\,\mathrm{d}k_2\\ &+\frac{1}{2}\int\frac{|V^{(1,2)}(\vec{k}_2,\vec{k},\vec{k}_1)|^2}{(\omega_{k_2}-\omega_k-\omega_{k_1})^2}\\ &\times(N_{k_1}N_{k_2} + N_kN_{k_2} - N_kN_{k_1})\,\delta(\vec{k}_2-\vec{k}-\vec{k}_1)\,\mathrm{d}k_1\,\mathrm{d}k_2\\ &+\frac{1}{2}\int\frac{|V^{(0,3)}(\vec{k},\vec{k}_1,\vec{k}_2)|^2}{(\omega_k+\omega_{k_1}+\omega_{k_2})^2}\\ &\times(N_{k_1}N_{k_2} + N_kN_{k_1} + N_kN_{k_2})\,\delta(\vec{k}+\vec{k}_1+\vec{k}_2)\,\mathrm{d}k_1\,\mathrm{d}k_2. \end{aligned} \tag{3.46}$$

The difference between n_k and N_k,

$$\Delta_k = \frac{n_k - N_k}{N_k},$$

is essential on the surface of shallow water. However, even on the surface of deep water Δ_k is a fast growing function of k.

The relationship between space–time spectra of the 'total' $n_{k\omega}$ and 'purified' $N_{k\omega}$ versions of wave action is not known so far. This is a subject for future research. However, $N_{k\omega}$ can be presented in the form

$$N_{k\omega} = \frac{1}{\pi} \frac{\Gamma_k N_k}{(\omega - \tilde{\omega}_k)^2 + \Gamma_k^2} \tag{3.47}$$

and we can put approximately

$$\begin{aligned} Q_{k\omega} &= \frac{1}{2}\,\omega_k (N_{k\omega} + N_{-k,-\omega}) \\ &= \frac{1}{2\pi} \left\{ \frac{\Gamma_k N_k}{(\omega - \tilde{\omega}_k)^2 + \Gamma_k^2} + \frac{\Gamma_{-k} N_{-k}}{(\omega - \tilde{\omega}_k)^2 + \Gamma_k^2} \right\}. \end{aligned} \tag{3.48}$$

After integration by ω and assuming that $\arctan \Gamma_k/\omega_k \sim \Gamma_k/\omega_k$, one gets the following relationship:

$$N_k = \int_0^\infty N(k, \omega)\, \mathrm{d}\omega + \frac{1}{\pi} \left(\frac{N_k \Gamma_k}{\omega_k} - \frac{N_{-k} \Gamma_{-k}}{\omega_{-k}} \right). \tag{3.49}$$

From (3.48), we see that the identity

$$N_k = \int_0^\infty N(k, \omega)\, \mathrm{d}\omega \tag{3.50}$$

is valid up to a relative accuracy Γ_k/ω_k. The value of this accuracy will be discussed in section 6. Near the spectral peak it is of the order of $4\pi\mu^4$. Identity (2.17) is satisfied with much less accuracy. Even near the spectral peak, the accuracy is of the order of μ^2 and becomes worse at $k \gg k_p$. An explicit expression for $Q(k, \omega)$ through N_k will be the subject of a separate forthcoming paper.

4. Stationary solutions: the isotropic case

In this section, we address the following question: How to solve the stationery kinetic equation

$$S_{nl} \equiv 0\,? \tag{4.1}$$

Formally speaking, this equation has thermodynamically equilibrium solutions

$$N_k = \frac{T}{\omega_k + \mu}, \tag{4.2}$$

where temperature T and μ are constants. It might sound like a paradox, but in fact spectrum (4.2) is not a real solution of equation (4.1). From this moment we discuss only the case of deep water and consider $\omega = \sqrt{gk}$. Also we denote that $k = |\vec{k}|$.

To justify this statement, we note that in two particular cases, $\mu = 0$ and $T = c\mu$, $\mu \to \infty$, solution (4.2) takes the form

$$\begin{aligned} N &= \frac{T}{\omega_k} = \frac{T}{\sqrt{g}}\, k^{-1/2}, \\ N &= c. \end{aligned} \tag{4.3}$$

Both these solutions are isotropic powerlike functions

$$N_k = k^{-x} \tag{4.4}$$

with particular values $x = 1/2$ and 0. Let us study the general powerlike solution of equation (4.1). By plugging equation (4.4) into equation (4.1) we find that each particular term in S_{nl} is diverging, but in different terms the divergence can be cancelled; thus there is a 'window of opportunity' for the exponent x. As a result,

$$S_{nl} = g^{3/2}\, k^{-3x+19/2}\, F(x). \tag{4.5}$$

Here $F(x)$ is a dimensionless function, defined inside the interval $x_1 < x < x_2$. The edges of the window, x_1 and x_2, are to be determined. Outside the 'window of opportunity', at $x < x_1$ and $x > x_2$, $F(x) = \infty$. Thus all admitted values of x must be posed between x_1 and x_2.

Let the quadruplet of waves be formed of wave vectors satisfying resonant conditions

$$\begin{aligned} \vec{k}_1 + \vec{k}_2 &= \vec{k}_3 + \vec{k}_4, \\ \omega_{k_1} + \omega_{k_2} &= \omega_{k_3} + \omega_{k_4}. \end{aligned} \tag{4.6}$$

Suppose that $|k_1| \ll |k|$. The three-wave resonant condition,

$$\vec{k} = \vec{k}_2 + \vec{k}_3, \quad \omega_k = \omega_{k_2} + \omega_{k_3}, \tag{4.7}$$

cannot be satisfied; thus one of the vectors $\vec{k}_2$, $\vec{k}_3$ must be small. If $|k_3| \ll |k_2|$, then

$$\begin{aligned} \vec{k}_2 &= \vec{k} + \vec{k}_1 - \vec{k}_3, \\ \omega(k_2) &= \sqrt{gk} \left(1 + \frac{1}{2} \frac{(k, \vec{k}_1 - \vec{k}_3)}{k^2} + \cdots \right). \end{aligned} \tag{4.8}$$

In the first approximation by a small parameter $|k_1|/|k|$, one can put $\omega(k_2) = \omega(k)$, $\omega(k_1) = \omega(k_3)$ and $|k_3| \simeq |k_1|$. In other words, vectors $\vec{k}_1$, $\vec{k}_3$ are small and have approximately the same length k_1. If vector k is directed along the axis x, the coupling coefficient $T_{kk_1k_2k_3}$ depends on four parameters k, k_1, θ_1, θ_3. Here θ_1, θ_3 are angles between $\vec{k}_1$, $\vec{k}_3$ and $\vec{k}$. Recalling that $k_1 \ll k$, we calculate the coupling coefficient in this asymptotic domain. A tedious calculation presented in [11] leads to the following compact result:

$$\begin{aligned} T_{kk_1k_2k_3} &\simeq \tfrac{1}{2} k\, k_1^2\, T_{\theta_1,\theta_3}, \\ T_{\theta_1,\theta_2} &= 2(\cos\theta_1 + \cos\theta_3) - \sin(\theta_1 - \theta_3)(\sin\theta_1 - \sin\theta_3). \end{aligned} \tag{4.9}$$

On the diagonal $k_3 = k_1$, $\theta_3 = \theta_1$, we get the following very simple expression, published in 2003 in [29]:

$$T_{kk_1} \simeq 2k_1^2 k \cos\theta_1. \tag{4.10}$$

Suppose that the spectrum is separated into the low-frequency component $N_0(k)$ and the high-frequency component $N_1(k)$. We assume that $N_1 \ll N_0$ and take into account the interaction between N_0 and N_1 only. One can see that N_1 satisfies the linear diffusion equation

$$\frac{\partial}{\partial t} N_1 = \frac{\partial}{\partial k_i} D_{ij}\, k^2 \frac{\partial}{\partial k_j} N_1, \tag{4.11}$$

where D_{ij} is the tensor of diffusion coefficients,

$$\begin{aligned} D_{ij} &= 2\pi g^{3/2} \int_0^\infty \mathrm{d}q\, q^{17/2} \int_0^{2\pi} \mathrm{d}\theta_1 \\ &\quad \times \int_0^{2\pi} \mathrm{d}\theta_3 |T(\theta_1, \theta_3)|^2\, p_i\, p_j\, N(\theta, q) N(\theta_3, q), \end{aligned} \tag{4.12}$$

$$p_1 = \cos\theta_1 - \cos\theta_3, \quad p_2 = \sin\theta_1 - \sin\theta_3.$$

If the spectrum is isotropic and does not depend on angle θ, we get the further simplification:

$$D_{ij} = D\,\delta_{ij}, \quad D = \frac{5}{8}\pi^3 g^{3/2} \int_0^\infty q^{17/2} N^2(q)\,\mathrm{d}q. \qquad (4.13)$$

The diffusion coefficient D diverges at $k \to 0$ if $x > 19/4$. Thus, $x_2 = 19/4$.

Let us find the behavior of the function $F(x)$ near $x = x_2$. In the isotopic case, equation (3.9) reads as

$$\frac{\partial N_1}{\partial t} = \frac{D}{k}\frac{\partial}{\partial k} k^3 \frac{\partial}{\partial k} N_1. \qquad (4.14)$$

If $k \to 19/4$, we get the following estimate:

$$F(x) = \frac{19}{4}\cdot\frac{11}{4}\cdot\frac{5\pi^3}{16}\,\frac{1}{19/4 - x} \simeq \frac{126.4}{19/4 - x}. \qquad (4.15)$$

To find x_1, the lower end of the window, we should study the influence of short waves on the long ones. Let us suppose that $|k_1|, |k_2| \gg k$. In the first approximation, $|k_3| = |k|$, and the resonant interaction S_{nl} can be separated into two groups of terms: $S_{nl} = S_{nl}^{(1)} + S_{nl}^{(2)}$. For $S_{nl}^{(1)}$ the integrand includes the product $N_{k_1} N_{k_2}$. If we put $k_1 = k_2$, we get the following expression for the low-frequency tail of the spectrum:

$$S_{nl}^{(1)} = 2\pi g^2 \int |T_{kk_1,k_1,k_3}|^2\, \delta(\omega - \omega_{k_3})\,(N_{k_3} - N_k)\, N_{k_1}^2\, \mathrm{d}k_1. \qquad (4.16)$$

Note that if $|k_1| \gg |k|$, then $|T_{kk_1,k_1,k_3}|^2 \simeq k_1^2$, and the integrand in (4.16) is proportional to $k_1^2\, N_{k_1}^2$. If $x < 2$, the integral diverges.

The group of terms linear with respect to the high-frequency tail of the spectrum is more complicated:

$$\begin{aligned} S_{nl}^{(2)} = 2\pi g^2 N_k &\int |T_{kk_1k_2k_3}|^2 N_{k_3}(N_{k_1} - N_{k_2}) \\ &\times \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \\ &\times \delta(k + k_1 - k_2 - k_3)\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3. \end{aligned} \qquad (4.17)$$

We can perform the expansion

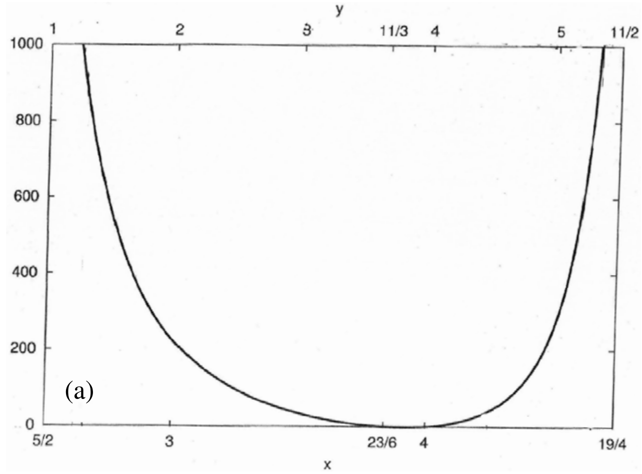
$$N_{k_1} - N_{k_3} = p_i \frac{\partial N}{\partial k_{1i}}, \qquad p_i = (k - k_3)_i. \qquad (4.18)$$

In the general anisotropic case, the integrand is proportional to $k_1^2(p\nabla N_{k_1})$ and the divergence occurs if $x = x_1 = 3$. However, in the isotropic case, this term, the most divergent one, is canceled after integration by angles. In this case, we should study quadratic terms in the expansion of the integrand in powers of parameter $(P, k_1)/k_1^2$. The most aggressive term appears from the expansion of δ-function on frequencies $\delta(\omega_{k_1} - \omega_{k_1+p} + \omega_k - \omega_{k_3})$. Performing integration by angles, we end up with the equation

$$\begin{aligned} &\frac{\partial N_k}{\partial t} = q\, k^7 N_k \frac{\partial N}{\partial k}, \\ &q = \frac{25}{16}\pi^3 g^{3/2} E = \frac{25}{8}\pi^3 g^{3/2} \int_0^\infty k^{3/2} N_k\, \mathrm{d}k. \end{aligned} \qquad (4.19)$$

Here E is the total energy. Thus, in the isotropic case, $x_1 = 5/2$, and we get, for the function $F(x)$, the following estimate:

$$F = \frac{5}{2}\,\frac{25}{8}\,\pi^3 \frac{1}{(5/2) - x} = \frac{241.86}{(5/2) - x}. \qquad (4.20)$$



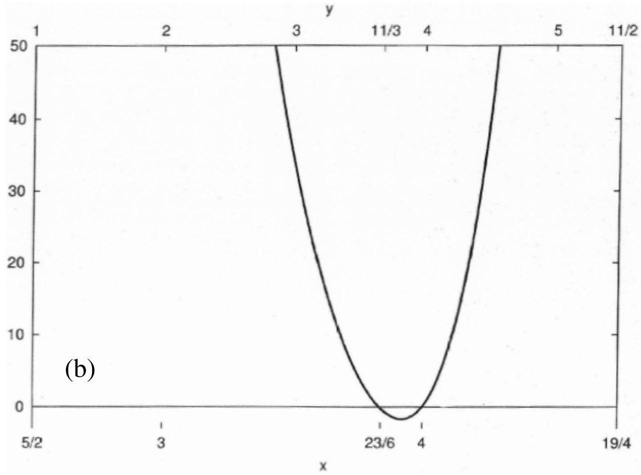


Figure 1. (a) A plot of the function $F(x)$. (b) A zoom of the function $F(x)$ in the vertical coordinate.

In figure 1(a) is presented a plot of the function $F(x)$ for the isotropic case, which we calculated numerically. One can see that in the interval $x_1 < x < x_2$, the function $F(x)$ has exactly two zeros at

$$x = y_1 = 4, \quad x = y_2 = \tfrac{23}{6}. \qquad (4.21)$$

To prove this result, let us consider that the spectra are isotropic, and present the conservation laws of energy and wave action in the differential form:

$$\frac{\partial I_k}{\partial t} = 2\pi k\omega_k \frac{\partial N_k}{\partial t} = -\frac{\partial P}{\partial k}, \qquad (4.22)$$

$$P = 2\pi \int_0^k k\omega_k S_{nl}\,\mathrm{d}k, \qquad (4.23)$$

$$2\pi k \frac{\partial N_k}{\partial t} = \frac{\partial Q}{b}, \qquad (4.24)$$

$$Q = 2\pi \int_0^k k S_{nl}\,\mathrm{d}k. \qquad (4.25)$$

Here, P is the flux of energy directed to high wave numbers, while Q is the flux of wave action directed to small wave numbers. The equations

$$P = P_0 = \text{const}, \quad Q = Q_0 = \text{const} \qquad (4.26)$$

apparently are solutions of the stationary equation $S_{nl} = 0$. We will look for the solution in the powerlike form $N = \lambda k^{-x}$; then equations (4.23) and (4.25) read as

$$P_0 = 2\pi g^2 \lambda^3 \frac{F(x)}{3(x-4)} k^{-3(x-4)}, \qquad (4.27)$$

$$Q_0 = -2\pi g^{3/2} \lambda^3 \frac{F(x)}{3(x-26/3)} k^{-3(x-26/3)}. \qquad (4.28)$$

One can see that P_0 and Q_0 are finite only if $F(4) = 0$ and $F(26/3) = 0$ and if $F'(4) > 0$ and $F'(26/3) < 0$. We conclude that equation $S_{nl} = 0$ has the following solutions:

$$N_k^{(1)} = c_p \left(\frac{P_0}{g^2}\right)^{1/3} \frac{1}{k^4}, \qquad (4.29)$$

$$N_k^{(2)} = c_q \left(\frac{Q_0}{g^{3/2}}\right)^{1/3} \frac{1}{k^{23/6}}. \qquad (4.30)$$

Here, c_p, c_q are dimensionless Kolmogorov constants

$$c_p = \left(\frac{3}{2\pi\, F'(4)}\right)^{1/3}, \quad c_q = \left(\frac{3}{2\pi\, |F'(23/6)|}\right)^{1/3}.$$

In figure 1(b) is presented a zoom of the function $F(x)$ in the vertical coordinate. The numerics gives $F'(4) = 45.2$ and $F'(23/6) = -40.4$. In the area of zeros, $F(x)$ can be approximated by a parabola,

$$F(x) \simeq 256.8(x - 23/6)(x - 4). \qquad (4.31)$$

Let us note that

$$F(9/2) = 85.6; \qquad (4.32)$$

thus, we obtain

$$c_p = 0.219, \quad c_q = 0.227, \qquad (4.33)$$

and find that both the Kolmogorov constants are numerically small.

In the isotropic case, the energy spectrum $F(\omega)$ can be expressed through N_k,

$$F(\omega)\, d\omega = 2\pi \omega_k N_k\, k\, dk, \qquad (4.34)$$

and the energy spectrum corresponding to solution (4.29) has the following form, called the Zakharov–Filonenko spectrum:

$$F^{(1)}(\omega) = 4\pi c_p \left(\frac{P}{g^2}\right)^{1/3} \frac{g^2}{\omega^4}. \qquad (4.35)$$

This spectrum was found to be a solution of the equation $S_{nl} = 0$ [3].

For the spatial spectrum

$$I_k\, dk = 2\pi \omega_k N(k)\, k\, dk, \qquad (4.36)$$

solution (4.30) transforms to

$$I_k^{(1)} = 2\pi\, c_p \left(\frac{P}{g^2}\right)^{1/3} \frac{g^{1/2}}{k^{5/2}} \simeq k^{-2.5}. \qquad (4.37)$$

Spectra (4.29), (4.35) and (4.37) are realized if we have a source of energy that is concentrated at a small wave number and generates the amount of energy P in one unit of time. For spectrum (4.30), first reported by Zakharov in 1966 [3],

$$I_k^{(2)} = 2\pi\, c_q\, Q^{1/3} k^{-7/3} \simeq 2\pi\, c_q\, Q^{1/3} k^{2.33}, \qquad (4.38)$$

$$F^{(2)}(\omega) = 4\pi\, c_q\, Q^{1/3} \frac{g^{4/3}}{\omega^{11/3}}. \qquad (4.39)$$

Spectra (4.30) and (4.38) can be realized in the case of a source of wave action in the high wave number area.

The described spectra exhaust all powerlike isotropic solutions of the stationary kinetic equation $S_{nl} = 0$. It is important to stress that thermodynamical solutions $N = \text{const}$ and $N = c/k^{1/2}$ are not the solutions of this equation, because their exponents $x = 0$ and $x = 1/2$ are far below the lower end of the 'window of possibility' $x_1 = 5/2$. This fact means that thermodynamics has nothing in common with the theory of the wind-driven sea.

Solutions (4.29) and (4.30) are not unique stationary solutions of $S_{nl} = 0$. The general isotropic solution describes the situation when both the energy source at small wave numbers and the wave action source exist simultaneously and have the following form:

$$N_k^{(3)} = c_p \left(\frac{P}{g^2}\right)^{1/3} \frac{1}{k^4} L\left(\frac{g^{1/2} Q\, k^{1/2}}{P}\right). \qquad (4.40)$$

Here L is an unknown function of one variable,

$$L \to 1 \quad \text{at } k \to 0, \quad L(\xi) \to \frac{c_q}{c_p} \xi^{1/3} \quad \text{at } k \to \infty. \quad (4.41)$$

Let us note that if there is no flux of wave action from infinity, we must set $Q = 0$. Under this constraint, the general isotropic solution is the Zakharov–Filonenko spectrum (4.29), parameterized by a single arbitrary constant P, which is a flux of energy to $k \to \infty$.

Frequency spectra with tails in the form $F(\omega) \simeq \omega^{-4}$ were observed in numerous field experiments [11–16] and were obtained in numerical experiments as well [17–19]. Spatial spectra with asymptotics $I_k \simeq k^{5/2}$ were also observed in many experiments [20–22]. A more careful study of the experimental results shows that in a majority of cases, the spectral area right behind the spectral peak can be better approximated by the tail $\omega^{-11/3}$ in the frequency spectrum and by the tail $k^{-7/3}$ in the spatial spectrum.This is especially clear from experiments of Huang *et al* [20]. Figure 2 taken from [20] demonstrates the coexistence of both types of Kolmogorov–Zakharov (KZ) spectra.

5. Stationary solutions: the anisotropic case

To study the anisotropic solutions of equation (4.1), we introduce the polar coordinates on the k-plane and put $k^2 = \omega/g$. Thereafter, we will use the notation

$$N(\omega, \phi)\, d\omega\, d\phi = N(\vec{k})\, d\vec{k},$$
$$N(\omega, \phi) = \frac{2\omega^3}{g^2} N(\vec{k}). \qquad (5.1)$$

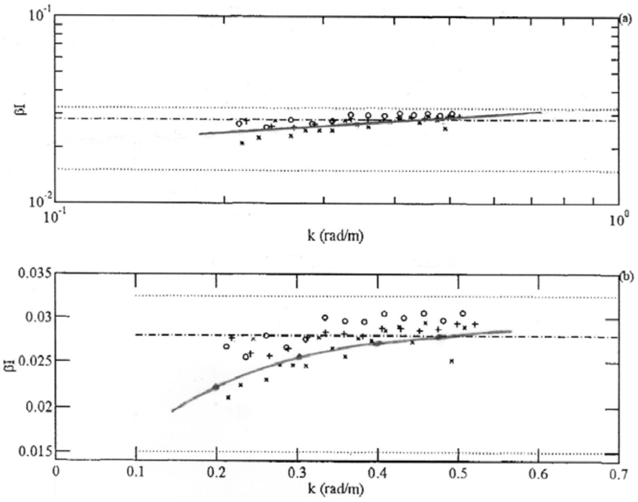


Figure 2. Dimensionless wave number spectral coefficient β_i plotted on logarithmic scales (a) and linear scales (b), taken from [20]. Here crosses represent the omnidirectional (averaged by angles) spectrum and dots correspond to $\xi(k) = 2\beta_I\, u_*\, g^{-0.5}\, k^{-2.5}$. The solid line in (a) and solid curve in (b) correspond to $\xi(k) \simeq k^{-7/3}$.

In the spatially homogeneous case, $N(\omega, \phi)$ satisfies the equation

$$\frac{\delta N(\omega,\phi)}{\partial t} = S_{nl}(\omega,\phi). \tag{5.2}$$

In new variables:

$$\begin{aligned}
S_{nl}(\omega,\phi) &= 2\pi g^2 \int |T_{\omega,\omega_1,\omega_2,\omega_3}|^2\, \delta(\omega+\omega_1-\omega_2-\omega_3)\\
&\times \delta(\omega^2\cos\phi + \omega_1^2\cos\phi_1 - \omega_2^2\cos\phi_2 - \omega_3^2\cos\phi_3)\\
&\times \delta(\omega^2\sin\phi + \omega_1^2\sin\phi_1 - \omega_2^2\sin\phi_2 - \omega_3^2\sin\phi_2)\\
&\times \big\{\omega^3\, N(\omega_1,\phi_1)\, N(\omega_2,\phi_2)\, N(\omega_3,\phi_3)\\
&+ \omega_1^3\, N(\omega,\phi)\, N(\omega_2,\phi_2)\, N(\omega_3,\phi_3)\\
&- \omega_2^3\, N(\omega,\phi)\, N(\omega_1,\phi_1)\, N(\omega_3,\phi_3)\\
&- \omega_3^2\, N(\omega,\phi)\, N(\omega_1,\phi_1)\, N(\omega_2,\phi_2)\big\}\\
&\times \mathrm{d}\omega_1\, \mathrm{d}\omega_2\, \mathrm{d}\omega_3\, \mathrm{d}\phi_1\, \mathrm{d}\phi_2\, \mathrm{d}\phi_3.
\end{aligned} \tag{5.3}$$

Exactly this form of S_{nl} is used for numerical simulation of the Hasselmann equation. Suppose that $N(\omega,\phi) = \omega^{-z}$ is the isotropic spectrum. Then,

$$S_{nl} = \frac{\omega^{-3z+13}}{4g^4}\, F\left(\frac{z+3}{2}\right) = \frac{G(z)}{g^4}\, \omega^{-3z+13}, \tag{5.4}$$

where $F(x)$ is defined by equation (4.5). Now the 'window of opportunity' is $2 < z < 13/2$. Zeros of $G(z)$ are posed at $z_1 = 5$ and $z_2 = 14/3$, and near these zeros, $G(z)$ can be presented as a parabola,

$$G(z) \simeq 16.05(z-5)(z-14/3). \tag{5.5}$$

To make the motion constants more conspicuous, we introduce the elliptic differential operator

$$L\, f(\omega,\phi) = \left(\frac{\partial^2}{\partial\omega^2} + \frac{2}{\omega^2}\frac{\partial^2}{\partial\phi^2}\right) f(\omega,\phi) \tag{5.6}$$

with the following parameters: $0 < \omega < \infty$, $0 < \phi < 2\pi$. The equation

$$L\, G = \delta(\omega-\omega')\, \delta(\phi-\phi') \tag{5.7}$$

with the boundary conditions

$$G|_{\omega\to 0} = 0, \quad G_{\omega\to\infty} < \infty, \quad G(2\pi) = G(0)$$

can be resolved as

$$\begin{aligned}
G(\omega,\omega',\phi-\phi') &= \frac{1}{4\pi}\sqrt{\omega\omega'} \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}n(\phi-\phi')}\\
&\times \left[\left(\frac{\omega}{\omega'}\right)^{\Delta_n} \Theta(\omega'-\omega) + \left(\frac{\omega'}{\omega}\right)^{\Delta_n} \Theta(\omega-\omega')\right],
\end{aligned} \tag{5.8}$$

where $\Delta_n = 1/2\sqrt{1+8n^2}$. Now we present S_{nl} in the form

$$A(\omega,\phi) = \int_0^\infty \mathrm{d}\omega' \int_0^{2\pi} \mathrm{d}\phi'\, G(\omega,\omega',\phi-\phi')\, S_{nl}(\omega',\phi'). \tag{5.9}$$

Note that $A(\omega,\phi)$ is a regular integral operator and suppose that $N(\omega,\phi) = \omega^{-z}$. Then

$$\begin{aligned}
A[\omega^{-z}] &= \frac{\omega^{-3z+15}}{g^4}\, H(z),\\
H(z) &= \frac{G(z)}{9(z-5)(z-14/3)}.
\end{aligned} \tag{5.10}$$

The function $H(z)$ is positive and has no zeros. If $G(z)$ is presented by a parabola (5.5), $H(z)$ is just a constant:

$$H(z) = H_0 = 16.05/9 = 1.83. \tag{5.11}$$

This fact leads us to a bold idea. If we assume that

$$A = \frac{H_0}{g^4}\, \omega^{15}\, N^3, \tag{5.12}$$

the nonlinear term S_{nl} turns into the elliptic operator:

$$S_{nl} = \frac{H_0}{g^4}\left(\frac{\partial^2}{\partial\omega^2} + \frac{2}{\omega^2}\frac{\partial^2}{\partial\phi^2}\right)\omega^{15}\, N^3. \tag{5.13}$$

This is the so-called 'diffusion approximation', introduced in [23]. Being very simple, it grasps the basic features of the wind-driven sea theory. We will refer mostly to this model, bearing in mind that the real case (5.9) does not differ much from it, at least qualitatively.

Let us integrate equation (5.2) by angles. We get

$$\frac{\partial N(\omega,t)}{\partial t} = \frac{\partial Q}{\partial\omega}. \tag{5.14}$$

Here $N(\omega,t) = \int_0^{2\pi} N(\omega,\phi)\, \mathrm{d}\phi$. Then

$$B(\omega,t) = \frac{g}{2\omega}\int_0^{2\pi} \cos\phi\, N(\omega,\phi)\, \mathrm{d}\phi, \tag{5.15}$$

and the flux of wave action is

$$Q = \frac{\partial K}{\partial\omega}, \quad K = \int_0^{2\pi} A(\omega,\phi)\, \mathrm{d}\phi. \tag{5.16}$$

After multiplying equation (5.14) by ω, one obtains the equation

$$\frac{\partial F(\omega, t)}{\partial t} + \frac{\partial P}{\partial \omega} = 0, \tag{5.17}$$

where $P = K - \omega \partial K/\partial \omega$ is the flux of energy.

Let us introduce now the following definitions: the integrated by angle spectral density of momentum

$$M_x(\omega, t) = \frac{\omega^2}{g} \int_0^{2\pi} \cos\phi \, B(\omega, \phi) \, \mathrm{d}\phi, \tag{5.18}$$

the quantity

$$C_x(\omega, t) = \frac{\omega}{2g} \int_0^{2\pi} \cos^2\phi \, N(\omega, \phi) \, \mathrm{d}\phi, \tag{5.19}$$

and the flux of momentum

$$R_x = \int_0^{2\pi} \cos\phi \left(\omega A - \frac{\omega^2}{2} \frac{\partial A}{\partial \omega} \right) \mathrm{d}\phi. \tag{5.20}$$

All these quantities are connected by the equation

$$\frac{\partial M_x}{\partial t} + \frac{\partial R_x}{\partial \omega} = 0. \tag{5.21}$$

Equations (5.14), (5.17) and (5.21) are averaged by angle balance equations for the basic conservative quantities.

Now we can return to the question formulated above. How many solutions do the stationary kinetic equations (1.5) and (4.1) have? Note that we simplified it to the linear equation

$$\left(\frac{\partial^2}{\partial \omega^2} + \frac{2}{\omega^2} \frac{\partial^2}{\partial \phi^2} \right) A = 0. \tag{5.22}$$

In particular, the kinetic equation has the anisotropic KZ solution

$$A = \frac{1}{2\pi} \left\{ P + \omega Q + \frac{R_x}{\omega} \cos\phi \right\}, \tag{5.23}$$

where P and R_x are fluxes of energy and momentum at $\omega \to \infty$ and Q is the flux of wave action directed to small wave numbers. In the general case, equation (5.23) is a nonlinear integral equation; however, in the diffusion approximation the KZ solution can be found in the explicit form

$$N(\omega, \phi) = \frac{1}{(2\pi H_0)^{1/3}} \frac{g^{4/3}}{\omega^5} \left(P + \omega Q + \frac{R_x}{\omega} \cos\phi \right)^{1/3}. \tag{5.24}$$

By comparing with equations (4.35) and (4.38), one can easily find that, in this case,

$$c_p = c_q = \frac{1}{2(2\pi H_0)^{1/3}} = 0.223, \qquad H_0 = 1.83.$$

This is exactly the arithmetic mean between the values of Kolmogorov constants given by equation (3.31).

By multiplying equation (5.24) by $2\pi\omega$, we get the general KZ spectrum in the diffusion approximation:

$$F(\omega) = 2.78 \frac{g^{4/3}}{\omega^4} \left(P + \omega Q + \frac{R_x}{\omega} \cos\phi \right)^{1/3}. \tag{5.25}$$

We must be sure that in the isotropic case $R_x = 0$, the expression

$$F(\omega) = 2.78 \frac{g^{4/3}}{\omega^4} (P + \omega Q)^{1/3} \tag{5.26}$$

approximates the generic KZ spectrum with accuracy up to a few per cent.

If we somehow know the value of $A(\omega, \phi)$ on the circle $\omega = \omega_0$, we can solve the external and internal Dirichlet boundary problems for equation (5.22) with the boundary condition $A(\omega, \phi) < \infty$ at $\omega \to \infty$. Suppose that

$$\begin{aligned} A(\omega, \phi) &= A_0(\phi) \\ &= A_0 + \frac{A_1}{\omega} \cos\phi \\ &\quad + \sum_{n=2}^{\infty} A_n \left(\frac{\omega_0}{\omega} \right)^{-1/2 + \sqrt{1/4 + 4n^2}} \cos n\phi. \end{aligned} \tag{5.27}$$

The first two terms in equation (5.27) present the KZ spectrum with $Q = 0$, $P = 2\pi A_n$, $R_x = 2\pi\omega_0 A_1$. The next terms describe the fast stabilization of any arbitrary solution to the KZ spectrum at $\omega/\omega_0 \to \infty$. The first additional term in (5.27) decays as $(\omega_0/\omega)^{3.53} \cos 2\phi$.

This stabilization to the KZ spectrum is actually the 'angular spreading' of wind-driven wave spectra that is usually observed in field experiments (see, for instance, [12]). If $Q = 0$, the general KZ solution (5.25) at $\omega \to 0$ is the following spectrum:

$$F(\omega) \to \frac{2.78}{\omega^4} g^{4/3} p^{1/3} \left(1 + \frac{1}{3} \frac{R_x}{P\omega} \cos\phi + \cdots \right). \tag{5.28}$$

Similar results were predicted by Kontorovich and Kats [30] and Balk [31].

From equation (5.27), one can see that $A(\omega, \phi)$ is parametrized by the function of one variable, $A_0(\phi)$. In the presence of flux of action Q from infinity, one should add to equation (5.27) an additional term Q_ω. Thus, in the general case, the uncertainty for the determination of A consists of a function that has one variable and one constant. We implicitly assume that the mapping $N \to A$ is uniquely inversible. This fact has not been proven, but it is plausible.

6. Damping due to nonlinear interaction

How must we compare S_{nl} and S_{in}?

In this section, we show that S_{nl} is the leading term in the balance equation (1.11). In fact, the forcing terms S_{in} and S_{dis} are not sufficiently accurately known; thus it is reasonable to accept the simplest models of both terms assuming that they are proportional to the action spectrum:

$$S_{\mathrm{in}} = \gamma_{\mathrm{in}}(k)\, N(k), \tag{6.1}$$

$$S_{\mathrm{dis}} = -\gamma_{\mathrm{dis}}(k)\, N(k). \tag{6.2}$$

Hence

$$\gamma(k) = \gamma_{\mathrm{in}}(k) - \gamma_{\mathrm{dis}}(k). \tag{6.3}$$

In reality, $\gamma_{\rm dis}(k)$ depends strongly on the overall steepness μ. So far, let us note that the balance kinetic equation (1.24) can be written in the form

$$S_{nl} + \gamma(k)\, N_k = 0 \tag{6.4}$$

and present the S_{nl} term as

$$S_{nl} = F_k - \Gamma_k\, N_k. \tag{6.5}$$

The definitions of Γ_k and F_k are given by equations (2.5) and (2.6).

The solution of stationary equation (6.4) is the following:

$$N_k = \frac{F_k}{\Gamma_k - \gamma_k}. \tag{6.6}$$

A positive solution exists if $\Gamma_k > \gamma_k$. The term Γ_k can be treated as the nonlinear damping that appears due to four-wave interaction. This damping has a very powerful effect. A 'naive' dimensional consideration gives

$$\Gamma_k \simeq \frac{4\pi g^2}{\omega_k}\, k^{10}\, N_k^2; \tag{6.7}$$

however, this estimate works only if $k \simeq k_p$, with k_p being the wave number of the spectral maximum.

Let $k \gg k_p$. Now for Γ_k one gets

$$\Gamma_k = 2\pi g^2 \int |T_{kk_1,kk_3}|^2\, \delta(\omega_{k_1} - \omega_{k_3})\, N_{k_1} N_{k_3}\, \mathrm{d}k_1 \mathrm{d}k_2. \tag{6.8}$$

The main source of Γ_k is the interaction between long and short waves. To estimate integral (2.6) more accurately, we assume that the spectrum of long waves is narrow in angle, $N(k_1, \theta_1) = \tilde{N}(k_1)\, \delta(\theta_1)$. Long waves propagate along the axis x and $\vec{k}$ is the wave vector of the short wave propagating in the direction θ. For the coupling coefficient we must put $T_{kk_1,k_2,k_3} \simeq 2k_1^2 k \cos\theta$. Then

$$\Gamma_k = 8\pi g^{3/2}\, k^2\, \cos^2\theta \int_0^\infty k_1^{13/2}\, \tilde{N}^2(k_1)\, \mathrm{d}k_1. \tag{6.9}$$

Even for the most mildly decaying KZ spectrum, $N_k \simeq k^{-23/6}$, the integrand behaves like $k_1^{-7/6}$ and the integral diverges. For steeper KZ spectra, the divergence is stronger.

Let us estimate Γ_k for the case of a 'mature sea', when the spectrum can be taken in the form

$$N_k \simeq \frac{3}{2}\, \frac{E}{\sqrt{g}}\, \frac{k_p^{3/2}}{k^4}\, \theta(k - k_p). \tag{6.10}$$

Here E is the total energy. By plugging (6.10) into (6.9), one gets the equation

$$\Gamma_\omega = 36\,\pi\,\omega \left(\frac{\omega}{\omega_p}\right)^3 \mu_p^4\, \cos^2\theta, \tag{6.11}$$

which includes a huge enhancing factor: $36\pi \simeq 113.04$. For a very modest value of steepness, $\mu_p \simeq 0.05$, we get

$$\Gamma_\omega \simeq 7.06 \times 10^{-4} \omega \left(\frac{\omega}{\omega_p}\right)^3 \cos^2\theta. \tag{6.12}$$

In the isotropic case, to find Γ_k for $\omega/\omega_p \gg 1$ we need to perform a simple integration over angles that yields

$$\int_0^{2\pi}\int_0^{2\pi} T^2_{\theta_1,\theta_2}\, \mathrm{d}\theta_1\, \mathrm{d}\theta_2 = \frac{5}{2}(2\pi)^2;$$

thus instead of equation (6.11) we get

$$\Gamma_k = 5\pi g^{3/2} k^2 \int_0^\infty k_1^{13/2} \tilde{N}(k_1)^2\, \mathrm{d}k_1 \tag{6.13}$$

or

$$\Gamma_\omega = \frac{45\pi}{2} g^{3/2} \omega \left(\frac{\omega}{\omega_p}\right)^3 \mu_p^4. \tag{6.14}$$

Finally, assuming that

$$N_{k_p} \simeq \frac{3}{2}\frac{E}{\sqrt{g} k_p^{5/2}},$$

we get from equation (6.8) the following estimate for $\Gamma_p = \Gamma|_{k=kp}$:

$$\Gamma_p \simeq 9\pi\,\omega_p \mu_p^4. \tag{6.15}$$

Even in this case, we have a pretty high enhancing factor: $9\pi \simeq 28.26$. In fact, in all known models, Γ_k surpasses $\tilde{\gamma}_k$ at least in order of magnitude even for these very smooth waves.

In the presence of peakedness

$$\Gamma_p \simeq \Lambda\, \omega_p \mu_p^4. \tag{6.16}$$

Here $\Lambda \simeq 4\pi\omega_p/\delta\omega$ is the enhancing factor due to peakedness. If $\Lambda\mu_p^2 \sim 1$, then Γ_p is associated with the maximal growth of modulational instability for a monochromatic wave: $\Gamma_p \simeq \gamma_{\rm mod} \sim \omega_p \mu_p^2$. If $\Lambda \sim 1/\mu_p^2$, the nonlinearity becomes so strong that the weak-turbulent statistical approach is not applicable. This is quite a realistic possibility. Suppose that $\mu_p \simeq 0.11$ and $\omega_p/\delta\omega \simeq 5$. Then $\Lambda\mu_p^2 \sim 0.76$ and the weak-turbulent description is hardly correct. In the case of strong nonlinearity the wind-driven sea generates freak waves (see [24, 25]). The very fact of their existence as a common phenomenon is implicit proof of S_{nl} domination in the energy balance.

Note that Γ_k diverges for KZ spectra. However, it does not hurt the spectra existence, because in the full kinetic equation the divergence of Γ_k is canceled by divergence of F_k. Indeed, if we consider the contribution of small wave numbers in integral (2.5), we end up with the following expression:

$$F_k = 2\pi g^2\, N_k \int |T_{kk_1,kk_3}|^2\, \delta(\omega_{k_1} - \omega_{k_3})\, N_{k_1} N_{k_3}\, \mathrm{d}k_1 \mathrm{d}k_3 \simeq N_k\, \Gamma_k. \tag{6.17}$$

Neglecting γ_k, equation (4.1) is satisfied automatically.

The results obtained in this section show that the four-wave nonlinear interaction has a very strong effect. The strong turbulence of the near-surface air boundary layer makes the development of a reliable theory for air–water interaction, including a well-justified analytical calculation of γ_k, an extremely difficult task. Making field and laboratory measurements of γ_k is also difficult, and the scatter in the determination of γ_k is itself of the order of γ_k. Anyway, a comparison of the above calculated Γ_k with experimental

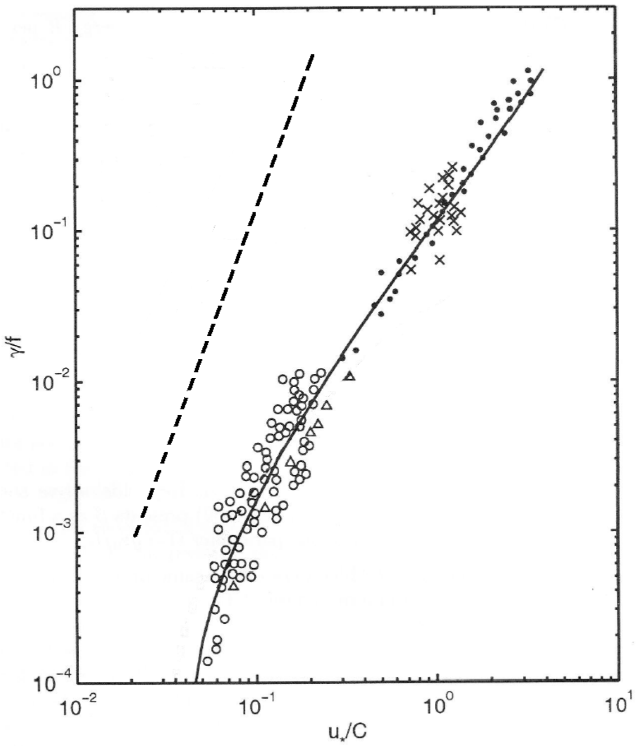


Figure 3. Comparison of the experimental data on the wind-induced growth rate $2\pi\gamma_{in}(\omega)/\omega$ taken from [26] and the damping due to four-wave interactions $2\pi\Gamma(\omega)/\omega$, calculated for the narrow in angle spectrum at $\mu \simeq 0.05$ using equation (6.11) (dashed line).

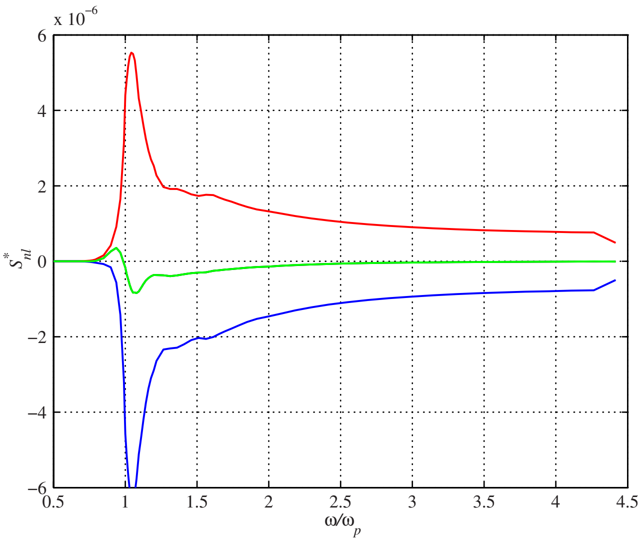


Figure 4. Split of the nonlinear interaction term S_{nl} (central curve) into F_k (upper curve) and $\Gamma_k N_k$ (lower curve).

data on γ_k shows that Γ_k surpasses γ_k at least by an order of magnitude. This fact is demonstrated in figure 3, where experimental data taken from [26] are presented.

As a result, we can conclude that S_{nl} is the leading term in the balance equation (1.11) and that the rear face of the spectrum is described by the solution of equation (4.1), which has a rich family of solutions. In particular, this equation describes the angular spreading.

In figure 4, we demonstrate that for the nonlinear interaction term $S_{nl} = F_k - \Gamma_k N_k$, the magnitudes of constituents F_k and Γ_{kN_k} essentially exceed their difference. They are one order higher than the magnitude of S_{nl}

The dominance of S_{nl} was not apparent until now for two reasons. Firstly, it is not correct to compare S_{nl} and $S_{\rm in}$; instead, one should compare Γ_k and γ_k. Secondly, the widely accepted models for $S_{\rm dis}$ essentially overestimate the dissipation due to white capping. As a result, the dominance of S_{nl} is masked. We offer an alternative model for $S_{\rm dis}$, which will be published in a forthcoming article [27]. Preliminary results obtained in this direction are given in [28].

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