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Theory of weakly damped free-surface flows: A new formulation based on potential flow solutions

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Abstract

Several theories for weakly damped free-surface flows have been formulated. In this Letter we use the linear approximation to the Navier–Stokes equations to derive a new set of equations for potential flow which include dissipation due to viscosity. A viscous correction is added not only to the irrotational pressure (Bernoulli's equation), but also to the kinematic boundary condition. The nonlinear Schrödinger (NLS) equation that one can derive from the new set of equations to describe the modulations of weakly nonlinear, weakly damped deep-water gravity waves turns out to be the classical damped version of the NLS equation that has been used by many authors without rigorous justification.

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1. Introduction

Even though the irrotational theory of free-surface flows can predict successfully many observed wave phenomena, viscous effects cannot be neglected under certain circumstances. Indeed the question of dissipation in potential flows of fluid with a free surface is a very important one. As stated by Longuet-Higgins [1], it would be convenient to have equations and boundary conditions of comparable simplicity as for undamped free-surface flows. The peculiarity here lies in the fact that the viscous term in the Navier–Stokes (NS) equations is identically equal to zero for a velocity deriving from a potential.

The effect of viscosity on free oscillatory waves on deep water was studied by Boussinesq [2] and Lamb [3], among others. Basset [4] also worked on viscous damping of water waves. It should be pointed out that the famous treatise on hydrodynamics by Lamb has six editions. The paragraphs on wave damping are not present in the first edition (1879) while they are present in the third edition (1906). The authors did not have access to the second edition (1895), so it is possible that Boussinesq and Lamb published similar results at the same time. Lamb derived the decay rate of the linear wave amplitude in two different ways: in §348 of the sixth edition by a dissipation calculation (this is also what Boussinesq did) and in §349 by a direct calculation based on the linearized NS equations. Let α denote the wave amplitude, ν the kinematic viscosity of the fluid and k the wave number of the decaying wave. Lamb showed that

$$\frac{d\alpha}{dt} = -2\nu k^2 \alpha. \qquad (1)$$

Eq. (1) leads to the classical law for viscous decay of waves of amplitude α, namely $\alpha \sim \exp(-2\nu k^2 t)$.

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In order to include dissipation accurately into potential flow solutions, one must somehow take into account vorticity. In the papers by Joseph and Wang [5], Wang and Joseph [6], vorticity was taken into account only in Bernoulli's equation, while only the potential component of the velocity was used in the kinematic condition. Tuck [7], as is commonly done in the ship research community, derived a single linearized free-surface condition with a dissipative term, that combines the kinematic and dynamic boundary conditions. Ruvinsky et al. [8] added the vortical component in the kinematic boundary condition but did not simplify it. Longuet-Higgins [1] simplified the equations of Ruvinsky et al. [8] by introducing a new free surface differing from the classical free surface by the integral over time of the vortical component of the velocity. Below we proceed differently and show that the small vortical component of the velocity plays a role in the kinematic condition as a dissipative term. The new resulting set of equations, which describes potential flow with dissipation, leads to a dispersion relation which corresponds exactly to that of Lamb [3] in the limit of small viscosity.

It should be mentioned that there are several papers where the same artificial dissipation is used in both Bernoulli's equation and in the kinematic boundary condition. By "same" we mean that the authors of these papers all use damping of the type $\partial\gamma/\partial t \to \partial\gamma/\partial t + \tilde{\nu}\gamma$, where γ denotes the quantity to be followed in time and $\tilde{\nu}$ the artificial damping. Such a choice is justified when one is interested in the numerical integration of potential flow equations. One can refer for example to Baker et al. [9], Dyachenko et al. [10,11], Zakharov et al. [12]. This type of dissipation was used not only for the simulation of gravity waves but also for the study of capillary wave turbulence by Pushkarev and Zakharov [13]. The addition of dissipation is sometimes used to satisfy the radiation condition.

2. Derivation of the new set of equations in the linear approximation

In order to study water waves, one can use for example the irrotational Euler's equations or the full Navier–Stokes equations. Since the main goal of the Letter is to narrow the gap between these two sets of equations by providing a new system of dissipative equations stated purely in terms of the velocity potential for the irrotational part of the flow, it is natural to state first the water-wave problem in the context of the irrotational Euler's equations and of the full NS equations. Below time is denoted by t, the horizontal coordinates are denoted by x and y, and z denotes the vertical coordinate.

2.1. *Water waves in the framework of the Navier–Stokes equations*

The two-dimensional (2D) flow of a viscous, incompressible fluid is governed by the conservation of mass

$$\nabla\cdot\vec{v} = 0, \tag{2}$$

and by the conservation of momentum

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \nu\Delta\vec{v} + \vec{g}. \tag{3}$$

The vector $\vec{v}(x,z,t) = (u,w)$ is the velocity field, ρ the fluid density, $\vec{g} = (0,-g)$ the acceleration due to gravity, ν the kinematic viscosity and $p(x,z,t)$ the pressure. The free surface $z = \eta(x,t)$ must be found as part of the solution. Two boundary conditions are required. The first one is the kinematic condition, which can be stated as

$$\frac{\partial\eta}{\partial t} + u(x,\eta,t)\frac{\partial\eta}{\partial x} = w(x,\eta,t). \tag{4}$$

The second boundary condition is the dynamic condition which states that the forces must be equal on both sides of the free surface:

$$-(p-p_0)\vec{n} + \underline{\underline{\tau}}\cdot\vec{n} = 0 \quad \text{at } z = \eta(x,t), \tag{5}$$

where $\vec{n}$ is the normal to the free surface and $\underline{\underline{\tau}}$ the viscous part of the stress tensor. The explicit expressions of $\underline{\underline{\tau}}$ and $\vec{n}$ are

$$\underline{\underline{\tau}} = \rho\nu\begin{pmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} & 2\frac{\partial w}{\partial z} \end{pmatrix},$$

$$\vec{n} = \frac{1}{\sqrt{1 + (\frac{\partial\eta}{\partial x})^2}}\begin{pmatrix} -\frac{\partial\eta}{\partial x} \\ 1 \end{pmatrix}.$$

The continuity of tangential stresses obtained by projecting (5) along the tangent to the free surface yields

$$\rho\nu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \rho\nu \times \text{nonlinear terms} = 0 \quad \text{at } z = \eta(x,t), \tag{6}$$

while the continuity of normal stresses reads

$$p - p_0 = 2\rho\nu\frac{\partial w}{\partial z} + \rho\nu \times \text{nonlinear terms} \quad \text{at } z = \eta(x,t). \tag{7}$$

Here surface tension effects have been neglected. Finally, in water of infinite depth, there is no kinematic boundary condition on the bottom. Instead it is replaced by

$$|\vec{v}| \to 0 \quad \text{as } z \to -\infty. \tag{8}$$

To summarize, the water-wave problem in the framework of the NS equations is given by Eqs. (2), (3), (4), (5) and (8).

2.2. *Water waves in the framework of potential flow theory*

In the framework of potential flow theory, the viscous terms are neglected and the flow is assumed to be irrotational. A velocity potential is introduced: $\vec{v} = \nabla\phi$. Therefore the equation for the conservation of mass (2) becomes

$$\Delta\phi = 0. \tag{9}$$

The kinematic boundary condition (4) remains the same but can be expressed in terms of the velocity potential ϕ:

$$\frac{\partial\eta}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x} = \frac{\partial\phi}{\partial z} \quad \text{at } z = \eta(x,t). \tag{10}$$

The dynamic boundary condition (7) on the free surface reduces to the continuity of pressure $p(x,\eta,t)=p_0$. Eq. (6) is no longer relevant. The equation for the conservation of momentum (3) can be integrated. It then leads to Bernoulli's equation, which takes the form

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}|\nabla\phi|^2+gz+\frac{p-p_0}{\rho}=0 \tag{11}$$

everywhere in the fluid.

Therefore one can replace the dynamic boundary condition $p(x,\eta,t)=p_0$ by

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}|\nabla\phi|^2+g\eta=0 \quad \text{at } z=\eta(x,t). \tag{12}$$

The bottom boundary condition (8) remains the same but can be expressed in terms of the velocity potential ϕ:

$$|\nabla\phi|\to 0 \quad \text{as } z\to-\infty. \tag{13}$$

To summarize, the water-wave problem in the framework of potential flow theory is given by Eqs. (9), (10), (12) and (13). Eq. (11) can be used a posteriori to compute the pressure everywhere in the fluid.

2.3. New set of equations

In order to clearly show how the new set of equations is derived, we first introduce the correction due to viscosity in the linearized equations for the potential flow of an incompressible fluid with a free surface (the linearization applies to the dynamic boundary condition expressed through Bernoulli's equation and the kinematic boundary condition on the free surface). As in §349 of Lamb [3], we resolve the velocity field into irrotational (curl-free) and solenoidal (divergence-free) component vector fields (Helmholtz decomposition). In other words, the velocity field is considered to be generated by a pair of potentials: a scalar potential and a vector potential. One writes

$$\vec{v}=\nabla\phi+\nabla\times\vec{A}, \tag{14}$$

where $\vec{A}$ is a vector stream function.

The linearized NS equations (3) read

$$\frac{\partial u}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}\right),$$
$$\frac{\partial w}{\partial t}=-g-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2}\right), \tag{15}$$

with the condition of flow incompressibility (2)

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0. \tag{16}$$

Using the Helmholtz decomposition (14) for the velocity yields

$$u(x,z,t)=\frac{\partial\phi}{\partial x}-\frac{\partial A_y}{\partial z},$$
$$w(x,z,t)=\frac{\partial\phi}{\partial z}+\frac{\partial A_y}{\partial x}, \tag{17}$$

since in 2D there is a single component to the vector stream function, which we denote by A_y. After substitution of the decomposition (17) into Eqs. (15) and (16), one notices that the equations are verified provided that the potentials ϕ and $\vec{A}$ satisfy the following equations:

$$\frac{\partial A_y}{\partial t}=\nu\left(\frac{\partial^2 A_y}{\partial x^2}+\frac{\partial^2 A_y}{\partial z^2}\right), \tag{18}$$
$$\frac{\partial\phi}{\partial t}=-\frac{p(x,z,t)}{\rho}-gz+\frac{p_0}{\rho}, \tag{19}$$
$$\Delta\phi=0. \tag{20}$$

To determine the 'normal modes' which are periodic in respect of x with a prescribed wavelength $2\pi/k$, we assume a time-factor $e^{-i\omega t}$ and a space-factor e^{ikx}. The solutions for the potential ϕ and the single component of the vector potential A_y are then

$$\phi(x,z,t)=\phi_0 e^{i(kx-\omega t)}e^{|k|z},$$
$$A_y(x,z,t)=A_0 e^{i(kx-\omega t)}e^{mz}, \tag{21}$$

where

$$m^2=k^2-i\frac{\omega}{\nu}. \tag{22}$$

Let us now write down the boundary conditions along the free surface. The linearized kinematic boundary condition (4) reads

$$\frac{\partial\eta}{\partial t}=w(x,0,t), \tag{23}$$

since the velocity is evaluated at $z=0$ in the linear approximation. It yields

$$\eta(x,t)=\frac{1}{\omega}\big(i|k|\phi_0-kA_0\big)e^{i(kx-\omega t)}. \tag{24}$$

The linearized dynamic boundary conditions (6) and (7) read

$$\rho\nu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)=0 \quad \text{at } z=0,$$
$$p-2\rho\nu\frac{\partial w}{\partial z}=p_0 \quad \text{at } z=0. \tag{25}$$

The boundary conditions (25) together with (24) provide two pieces of information:

(1) the relationship between the potential ϕ_0 and the vector stream function A_0,

$$A_0=\frac{2i|k|k}{m^2+k^2}\phi_0$$
$$=-2\left(\frac{\nu|k|k}{\omega}\right)\left(\frac{1}{1+2i(\nu k^2/\omega)}\right)\phi_0, \tag{26}$$

(2) the dispersion relation $\omega(k)$

$$\left(2-\frac{i\omega}{\nu k^2}\right)^2+\frac{g}{\nu^2|k|^3}=4\left(1-\frac{i\omega}{\nu k^2}\right)^{\frac{1}{2}}. \tag{27}$$

Let us now consider the limit of small viscosity. Lamb [3] introduces the dimensionless number $\theta=\nu k^2/|\omega|$. For small θ (in other words when $\nu k^2\ll|\omega|$), we expand (27) in powers of θ. Neglecting terms that are $o(\theta)$, one derives the following approximation for ω:

$$\omega = \pm\sqrt{g|k|} - 2i\nu k^2. \tag{28}$$

In this limit, it is clear that the vortical component of the velocity is much less than the potential component of the velocity since Eq. (26) leads to

$$\frac{|A_0|}{|\phi_0|} \approx 2\frac{\nu k^2}{|\omega|} \ll 1.$$

So far in this section, every line of our calculation can be found in §349 of Lamb [3] and we refer to Lamb for a closer look at the character of the motion affected by viscosity. In particular he calculates the vorticity and the penetration depth of the vortical component of the liquid velocity. Eq. (28) leads to the classical law (1) for viscous decay of free waves. Under the same limit ($\nu k^2 \ll |\omega|$), Eq. (26) can be written as follows:

$$\frac{\partial^2 A_y}{\partial x\partial t} = 2\nu\frac{\partial^3\phi}{\partial x^2\partial z} \quad \text{on the free surface at } z=0. \tag{29}$$

This equation was derived by Ruvinsky et al. [8].

So far, we have only described the viscous solution to the linear NS equations with small viscosity and, as just said, it is a well-known solution. But at this point a natural question arises: *Can we describe this flow by using only the potential part of the velocity?*

First, let us look at the kinematic condition (23) and explicitly separate the potential and vortical components of the velocities in it:

$$\frac{\partial\eta}{\partial t} = \frac{\partial\phi}{\partial z} + \frac{\partial A_y}{\partial x} \quad \text{at } z=0. \tag{30}$$

The vortical part of the velocity $\partial A_y/\partial x$ can be written in different ways, but here we want to express it in terms of η. (Note that we consider linear equations and can therefore freely express one function through another.) To do this, one can use the solution for η (24) with the relation (26) between the constants ϕ_0 and A_0. Thus, the following relation can be easily derived:

$$\begin{aligned}\left.\frac{\partial A_y}{\partial x}\right|_{z=0} &= -2i|k|^3\left(\frac{\nu}{\omega}\right)\frac{\phi_0}{1+2i(\nu k^2/\omega)}e^{i(kx-\omega t)}\\ &= -2k^2\nu\eta(x,t),\end{aligned}$$

that is

$$\left.\frac{\partial A_y}{\partial x}\right|_{z=0} = 2\nu\frac{\partial^2\eta}{\partial x^2} \quad \text{on the free surface.} \tag{31}$$

Eq. (31) is one of the main results of our Letter. The kinematic boundary condition can now be written without the vortical component of the velocity:

$$\frac{\partial\eta}{\partial t} = \frac{\partial\phi}{\partial z} + 2\nu\frac{\partial^2\eta}{\partial x^2} \quad \text{at } z=0. \tag{32}$$

Let us now turn to the equation for the potential component of the velocity. The dissipative correction to this equation is simply Bernoulli's equation, except that the pressure must be replaced by

$$\begin{aligned}p &= 2\rho\nu\left.\frac{\partial w}{\partial z}\right|_{z=0} + p_0\\ &= p_0 + 2\rho\nu\left.\left(\frac{\partial^2\phi}{\partial z^2} + \frac{\partial^2 A_y}{\partial x\partial z}\right)\right|_{z=0}.\end{aligned} \tag{33}$$

Note that the vortical component in the pressure (33) is $o(\nu)$ and can be neglected. So, we can write down the linearized Bernoulli equation (11) with dissipation

$$\frac{\partial\phi}{\partial t} + g\eta = -2\nu\frac{\partial^2\phi}{\partial z^2} \quad \text{at } z=0. \tag{34}$$

Note that equations very close to (34) can be found in other papers: see Eq. (1b) in Ruvinsky et al. [8], Eq. (4.1) in Longuet-Higgins [1] and Eq. (6.26) in Joseph and Wang [5]. It is easy to check that the two boundary conditions (32) and (34) lead to the same dispersion relation as (28).

To summarize, the linearized water-wave problem in the framework of our new viscous potential theory is given by Eqs. (9), (32), (34) and (13).

3. Fully nonlinear equations with dissipation

In the previous section, we added viscous terms in the *linear* equations of the 2D potential flow of a fluid with a free surface. At this point a second natural question arises: *Can we generalize the analysis of the linear case to the fully nonlinear equations, or at least to weakly nonlinear equations?*

The situation is more complicated. As said above, the set of equations which is the closest to ours is that of Ruvinsky et al. [8]. The only difference is that they did not express the vertical component of the vortical part of fluid velocity $\partial A_y/\partial x|_{z=0}$ through η. However they provided the extension of their results to the fully nonlinear equations. Details are given in the appendix of Ruvinsky et al. [8]. Their linearized Bernoulli equation is the same as ours and their nonlinear Bernoulli equation (1b) reads, after adapting it to our notation,

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + g\eta = -2\nu\frac{\partial^2\phi}{\partial z^2} \quad \text{at } z=\eta. \tag{35}$$

The only question that remains is whether one can still express $\partial A_y/\partial x|_{z=\eta}$ through the second derivative of η. The fully nonlinear kinematic boundary condition of Ruvinsky et al. [8] reads

$$\frac{\partial\eta}{\partial t} + \frac{\partial\eta}{\partial x}\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial z} + \frac{\partial A_y}{\partial x} \quad \text{at } z=\eta, \tag{36}$$

together with

$$\frac{\partial^2 A_y}{\partial x\partial t} = 2\nu\frac{\partial^3\phi}{\partial^2 x\partial z} \quad \text{on the free surface } z=\eta.$$

At this stage, we conjecture that at low viscosity one can still express $\partial A_y/\partial x$ by $2\nu\partial^2\eta/\partial x^2$ and we provide what we believe to be the physically correct generalization of the kinematic boundary condition:

$$\frac{\partial \eta}{\partial t} + \nabla\eta \cdot \nabla\phi = \frac{\partial \phi}{\partial z} + 2\nu\Delta\eta \quad \text{at } z = \eta(x, y, t). \tag{37}$$

Note that we have also extended it to three-dimensional flows: in other words, $\phi = \phi(x, y, z, t)$ and $\eta = \eta(x, y, t)$. It is interesting to point out that Fuhrman et al. [14] in a paper devoted to Boussinesq models provide in their appendix a set of diffusive Boussinesq equations (A.1) and (A.2) which include a dissipative term $\Delta\eta$ in the equation for $\partial\eta/\partial t$ and a dissipative term Δu in the equation for $\partial u/\partial t$. (In (35), the term $-\partial^2\phi/\partial z^2$ can be replaced by $\Delta\phi$ if by analogy we express the horizontal Laplacian by Δ.)

Next we provide an additional argument in favor of our new set of nonlinear equations. It is well known that the modulations of weakly nonlinear (undamped) gravity waves in deep water, with basic wave number k_0 and frequency $\omega_0(k_0) = \sqrt{g|k_0|}$, can be described by the non-dissipative nonlinear Schrödinger (NLS) equation for the envelope A of a Stokes wavetrain (written here in the 2D case)

$$i\frac{\partial A}{\partial t} - \frac{\omega_0}{8k_0^2}\frac{\partial^2 A}{\partial x^2} - \frac{1}{2}\omega_0 k_0^2|A|^2 A = 0. \tag{38}$$

Recall that A is the envelope of the normal canonical variable a. The Fourier harmonics a satisfy the following relation:

$$a = \sqrt{\frac{\omega}{2|k|}}\eta_k + i\sqrt{\frac{|k|}{2\omega}}\psi_k, \tag{39}$$

where η_k is the Fourier transform of the free-surface elevation and ψ_k the Fourier transform of the velocity potential evaluated on the free surface, see Zakharov et al. [15]. What happens if one tries to derive a similar equation from the boundary conditions (35) and (37)? Dissipation then appears naturally in the NLS equation in the following way:

$$i\frac{\partial A}{\partial t} - \frac{\omega_0}{8k_0^2}\frac{\partial^2 A}{\partial x^2} - \frac{1}{2}\omega_0 k_0^2|A|^2 A = -2i\nu k_0^2 A. \tag{40}$$

It is important to recognize that Eq. (40) has often been used successfully by many authors but its physical origin was not really explained. Authors either simply refer to previous papers or argue that it is the simplest way to include dissipation. Here we made an attempt to provide a clear derivation of the damped NLS equation (40). It is also important to point out that if vorticity effects are not included in the kinematic boundary condition, then the resulting NLS equation is much more complicated.

4. Discussion

Let us first summarize the new set of equations we provide for the study of weakly damped three-dimensional free-surface flows:

$$\Delta\phi = 0,$$
$$\frac{\partial \eta}{\partial t} + \nabla\eta \cdot \nabla\phi = \frac{\partial \phi}{\partial z} + 2\nu\Delta\eta \quad \text{at } z = \eta(x, y, t),$$
$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + g\eta = -2\nu\frac{\partial^2 \phi}{\partial z^2} \quad \text{at } z = \eta(x, y, t),$$
$$|\nabla\phi| \to 0 \quad \text{at } z \to -\infty.$$

While the set of equations is fully justified for the linearized problem, it is still at the stage of a conjecture for the nonlinear problem.

In addition to Laplace's equation and the equation for the decay at infinity, Ruvinsky et al. [8] derived a system of three equations in the limit of small viscosity: one for the velocity potential, one for the surface elevation and one for the vertical component of the velocity. Longuet-Higgins [1] simplified these equations. Debnath [16] and Spivak et al. [17] applied these equations to particular problems.

Here we have shown that the vortical component of velocity can be excluded from the equations, thus leaving only two equations for a quasi-potential flow. A small amount of vorticity has been incorporated into the kinematic boundary condition. At this stage the generalization to nonlinear equations remains a conjecture but we provided at least two heuristic arguments in favor of the new formulation: the extension to nonlinear equations that Ruvinsky et al. [8] gave and the derivation of a damped NLS equation which is physically correct. However it is clear that these two arguments are valid only if $|\nabla\eta|$ remains small. If it is not the case, the additional mechanism of dissipation due to wave breaking plays a much more important role.

The set of dissipative equations we derived can be extended in several directions. First of all, as in Longuet-Higgins [1,18] and Ruvinsky et al. [8], one can include surface tension in the study of weakly damped waves. Surface tension is important for example to explain the generation of short capillary waves by long gravity waves of large amplitude. These capillary waves remove energy from the gravity waves. Energy is then dissipated faster since νk^2 is much larger for the capillary waves than it is for the gravity waves. Incidentally the above analysis applies to non-breaking and non-turbulent motions in which the kinematic viscosity ν represents the molecular viscosity. The set of dissipative equations can also be extended to interfacial waves and more generally to multi-layer configurations. Another extension is the generalization to finite depth. It was recently provided by Dutykh and Dias [19]. The zero velocity requirement at the bottom leads to a nonlocal viscous term in the bottom kinematic boundary condition.

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