Physics Letters A 372 (2008) 4230–4233

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

The role of the generalized Phillips' spectrum in wave turbulence

A.C. Newell *, V.E. Zakharov

Department of Mathematics, The University of Arizona, 617 N. Santa Rita Ave., PO Box 210089, Tucson, AZ 85721-0089, USA

ARTICLE INFO

Article history:
Received 5 January 2008
Received in revised form 18 March 2008
Accepted 28 March 2008
Available online 3 April 2008
Communicated by A.P. Fordy

PACS:
47.27.Ek
47.35.Jk

ABSTRACT

We suggest the generalized Phillips' spectrum, which we define as that spectrum for which the statistical properties of wave turbulence inherit the symmetries of the original governing equations, is, in many circumstances, the spectrum which obtains in those regions of wavenumber space in which the Kolmogorov–Zakharov (KZ) spectra are no longer valid. This spectrum has many very special properties. We discuss its connection with the singularities which are associated with the whitecap events observed in windblown seas.

Wave turbulence, the study of the long time statistical behavior of a sea of weakly nonlinear dispersive waves, usually in the presence of sources and sinks, tends to be considered in the category of solved problems. After all, one has a natural asymptotic closure, manifested as a kinetic equation for the spectral number density n_k and a nonlinear frequency renormalization depending only on n_k. Moreover, the kinetic equation has stationary solutions corresponding both to equipartition (Rayleigh–Jeans) and finite flux or Kolmogorov–Zakharov (KZ) which allow for constant transport of some conserved density (such as energy or wave-action/particle number) between sources and sinks in wavenumber space [1]. A complete theory? Nothing could be further from the truth.

The main reason that the theory is incomplete is that it is almost never valid over all length scales [2,3]. In one dimension, in fact, there are circumstances in which it is not valid at any length scale, as the resonance transfer mechanism is replaced by a transfer process based on recurring phase slips leading to a gas of coherent and narrowing solitary pulses and the so-called CMMT spectra [4]. Even in situations where resonances are important, finite size effects can play an inhibitory role. Because the spectrum is discrete, very few of the resonances lie on the wavevector grid. Even though the resonant band broadens as the wave slope increases, the transfer tends to be intermittent [5,6]. But even in situations where the dimension d is two or more and the spectrum is continuous, the story is far from over. Again, the theory is incomplete because it almost always fails either at very high or at very low wavenumbers. It is only when the KZ spectra inherit the symmetries of the governing unaveraged equations for the field variables that the theory is uniformly valid [2]. This, of course, is the exception. In general, all the markers of wave turbulence, the dominance of the linear over nonlinear terms, the separation of the linear and nonlinear time scales essential for the asymptotics, the proximity of the long time state to one of joint-Gaussian statistics, all fail at very high or very low wavenumbers. If the failure region is masked by viscous dissipation, or by the nonuniversal input region, or some other physical process such as capillary wave action in the sea, then such processes serve to regularize the breakdown. But, in many cases, for sufficiently large values of the energy or wave-action fluxes, the breakdown region is not masked. Then one is faced with the challenge of constructing a solution for the breakdown region which will coexist with that of the region in which the KZ solution is valid. It is not at all clear, a priori, that such a coexistence is possible. After all, the problem is nonlinear and this means that length scales from both regions are dynamically connected.

There are two notable examples. The first we have discussed before [7,8] and arises in optical turbulence and in situations involving condensate formation. Because it has lessons useful for the topic of this Letter, we briefly recount the main points. Here the wave turbulence description of the inverse cascade of optical power (equivalent to wave-action or wavenumber density) by four wave resonant interactions breaks down at very long scales or low frequencies. Without any regularization (removal of the power by some dissipative process at long scales), one must deal with either (a), in the case of defocussing medium, a fully developed condensate along with the dynamical structures (waves with a dispersion relation which can transfer energy via three wave resonances, fully nonlinear vortices) it can support, or (b), in the case of a focussing medium, an unstable condensate and collapsing filaments. It is the latter case we describe here. An initial inverse cascade rate Q_0 leads to the nucleating of collapses (via the modulational insta-

* Corresponding author.
E-mail addresses: anewell@math.arizona.edu (A.C. Newell), zakharov@math.arizona.edu (V.E. Zakharov).

0375-9601/$ – see front matter
doi:10.1016/j.physleta.2008.03.043

bility) and, in two space dimensions, each of these carries a fixed amount of power (N_0) back to small scales where dissipation sinks, such as phonon coupling, are present. The burnout of each collapse is incomplete and only a fraction fN_0, $f \simeq 1/5$, of the power in the collapsing filament is actually lost. The remainder is returned to the wave field as both power and the conserved density (not easily identified physically) corresponding to the Hamiltonian. The former increases the inverse cascade of power until it eventually reaches an amount Q_0/f. Only then will sufficient collapses be nucleated to bring the injected power to the dissipative scales and the system to a statistically stationary state. The time dependent shape of the coherent structure remains the same but the density increases to bring about the required dissipation. Current work [9] seeks to verify by numerical simulation our predictions, including the distribution of collapse events (Poisson), its parametric dependence on the inverse power flux as it evolves from Q_0 to Q_0/f and the fact that the long time behavior of the system can be described by two very different coexisting states.

The principal focus of this Letter, however, concerns the breakdown which, for gravity waves, occurs at high wavenumbers and frequencies. The Kolmogorov–Zakharov (KZ) spectrum corresponding to a constant flux of energy P from low to high wavenumbers no longer satisfies the premises on which the theory is based at wavenumbers k_1 such that $P^{\frac{2}{3}}\frac{k_1}{g}$, g gravity, is of order unity. For $P > P_0 = (\sigma g)^{\frac{3}{4}}$, the breakdown wavenumber k_1 is less than that wavenumber $k_0 = (g/\sigma)^{\frac{1}{2}}$, $\sigma = S/\rho$, S surface tension, at which capillary effects come into play. The questions then are: What happens for wavenumbers $k_1 < k < k_0$? Is there another spectrum valid for $k > k_1$, which can coexist with the KZ spectrum? What equation, what physical balances, does this new spectrum satisfy? How does the flux of energy P behave for $k > k_1$? Does energy dissipate? If so, how?

It is our suggestion that the coexisting spectrum is the spectrum introduced by Phillips in a pioneering paper [10] in 1958. This spectrum is associated with the formation of wedge-like structures, crests of length L with derivative discontinuities in the surface elevation in a direction perpendicular to the wave crests. A naive argument would suggest that for such a crest $k\hat{\eta}_k \sim k^{-1}$ where $\hat{\eta}_k$ is the Fourier transform of the surface elevation $\eta(\vec{x}(x,y),t)$ and that therefore the average density $\langle\hat{\eta}_k^2\rangle$ should scale as k^{-4}. But Kuznetsov et al. [11] have shown, if one averages over all angles between the crest and the wave, then, for $kL \gg 1$, one obtains an additional factor of k^{-1} leading to an energy density of k^{-5}. At the end of this Letter, we show this extra factor is offset by averaging over wedge scales. For the present, however, we choose to introduce and define the Phillips' spectrum in a totally different and very general way. The generalized Phillips' spectrum has several very important and intriguing properties, the first of which is a nontrivial result and will serve as our definition: (1) It is the spectrum for which the statistical properties of the system inherit the symmetries of the original unaveraged equations. It is not at all obvious a priori that there should be such a spectrum. (2) It is the only spectrum for which wave turbulence theory is uniformly valid for all wavenumbers k. (3) Most important, it is the only spectrum which is independent of the flux P and the breakdown wavenumber k_1, but for which the energy P flowing though k_1 each second is entirely absorbed in the region $k > k_1$. (4) Its entropy production divided by the frequency ω associated with the wavevector $\vec{k}$ is constant. The results are independent of the order of the resonance (three wave, four wave).

We begin with the equations for field variables written in Fourier coordinates $A^s_{\vec{k}}$ where a $\vec{k}$ dependent linear combination of $A^+_{\vec{k}}$ and $A^-_{\vec{k}}$ is the Fourier transform of a physically measurable field such as the surface elevation. The equation is

$$\frac{\mathrm{d}A^s_{\vec{k}}}{\mathrm{d}t} - \mathrm{i}s\omega_k A^s_{\vec{k}} = \sum_{r=2}^{+\infty}\sum_{s_1\ldots s_r}\int L^{ss_1\ldots s_r}_{kk_1\ldots k_r} A^{s_1}_{\vec{k}_1}\cdots \times A^{s_r}_{\vec{k}_r}\,\delta(\vec{k}_1+\cdots+\vec{k}_r-\vec{k})\,\mathrm{d}\vec{k}_1\cdots\mathrm{d}\vec{k}_r. \tag{1}$$

In regions of wavenumber where only one physical process dominates, ω_k and $L^{ss_1\ldots s_r}_{kk_1\ldots k_r}$ will be homogeneous functions of $\vec{k}$; i.e. $\omega_{\lambda\vec{k}} = \lambda^\alpha\omega_k$, $L^{ss_1\ldots s_r}_{\lambda k\lambda k_1\ldots\lambda k_r} = \lambda^{\gamma_r}L^{ss_1\ldots s_r}_{kk_1\ldots k_r}$. For gravity (capillary) waves $\alpha = \frac{1}{2}$, $\gamma_2 = \frac{7}{4}$, $\gamma_3 = 3$ ($\alpha = \frac{3}{2}$, $\gamma_2 = \frac{9}{4}$, $\gamma_3 = 3$). The dimension d of $\vec{k}$ is 2. Implied in (1) is a small parameter ε, $0 < \varepsilon \ll 1$, a dimensionless wave amplitude which for water waves is the root mean squared wave slope. In what follows, we absorb this parameter into the energy flux P.

The first important result [2] is that, in the region of homogeneity, the governing equation (1) is invariant under the transformations $\vec{k} \to \vec{K} = \lambda\vec{k}$, $t \to T = \lambda^{-\alpha}t$, $A^s_k(t) \to \lambda^\nu B^s_K(T)$ if $\nu = d + \frac{\gamma_r-\alpha}{r-1}$. One sees this by a direct substitution of the transformation into (1) using homogeneity properties of the dispersion $\omega(\vec{k})$ and the nonlinear coupling coefficients $L^{ss_1\ldots s_r}_{kk_1\ldots k_r}$. The result implies there exists a relation between the homogeneity indices which indeed holds; namely, $\frac{\gamma_r-\alpha}{r-1}$ is independent of r. One can check this for both pure gravity waves where g is the only additional parameter and pure capillary waves where σ is the only parameter. In both cases, $\gamma_3 + \alpha = 2\gamma_2$. For gravity waves, $\nu = 13/4$; for capillary waves, $\nu = 11/4$. We then ask: How does the symmetry carry over to the statistics of wave turbulence? All the important long time statistics depend principally on the number density n_k defined by $\langle A^s_k A^{-s}_{k'}\rangle = n_k\delta(\vec{k}+\vec{k}')$. First, it satisfies the closed kinetic equation. Second, all the long time behaviors of the higher order cumulants, called the asymptotic survivors, which are regenerated by nonlinear interactions, depend only on n_k. Third, the frequency renormalization depends only on n_k. It is therefore natural to for what value αx does the spectrum $n_k = Ck^{-\alpha x}$ and all the system's long time statistics inherit the symmetry of the original equation (1), namely when $\langle A^s_k A^{-s}_{k'}\rangle = \delta(\vec{k}+\vec{k}')k^{-\alpha x}$ implies $\langle B^s_K B^{-s}_{K'}\rangle = \delta(\vec{K}+\vec{K}')K^{-\alpha x}$. This only happens when $\alpha x = \alpha x_P = d + 2\frac{\gamma_r-\alpha}{r-1}$. We define this to be the generalized Phillips' spectrum. The generalized Phillips' spectrum is the only spectrum for which all the long time statistics of the mean field inherit the symmetries of the original field.

We now turn to the second important property. For sufficiently small and suitably nondimensionalized constant c, $n_k = cDk^{-\alpha x}$, D a dimensional constant, wave turbulence theory is uniformly valid at all scales only when $\alpha x = \alpha x_P$. Using standard methods [1,2], the equations for the long time behavior of n_k and the frequency renormalization are

$$\frac{dn_k}{dt} = T_2[n_k] + T_4[n_k] + T_6[n_k] + \cdots, \tag{2}$$

$$s\omega_k \to s\omega_k + \Omega^s_2[n_k] + \Omega^s_4[n_k] + \cdots. \tag{3}$$

For the theory to be valid, Eqs. (2) and (3) and the ratios of linear to nonlinear timescales $\frac{t_L}{t_{NL}} = \frac{1}{n_k\omega_k}\frac{dn_k}{dt}$ must be uniformly valid asymptotic expansions and small respectively when considered as functions of $\vec{k}$ on some solution n_k of the truncated equations; e.g. $T_2[n_k] = 0$. In (2), $T_2[n_k]$ contains three wave resonant processes, $T_4[n_k]$ contains four wave resonant processes and convolutions (both resonances and modal interactions) of three wave processes. For example, for gravity waves $T_2[n_k] \equiv 0$ and

$$T_4[n_k] = 12\pi\int\left|\hat{L}^{++--}_{kk_1k_2k_3}\right|^2 nn_1n_2n_3\left(\frac{1}{n}+\frac{1}{n_1}-\frac{1}{n_2}-\frac{1}{n_3}\right) \times\delta(\vec{k}+\vec{k}_1-\vec{k}_2-\vec{k}_3)\delta(\omega+\omega_1-\omega_2-\omega_3)\,d\vec{k}_1\,d\vec{k}_2\,d\vec{k}_3, \tag{4}$$

where $\omega_k = \sqrt{gk}$ and where $\hat{L}^{++--}_{kk_1k_2k_3}$ is built from combinations of $L^{++--}_{kk_1k_2k_3}$ from which quadratic products of combinations of $L^{ss_1s_2}_{kk_1k_2}$ have been subtracted (see [2, p. 540]). Its degree of homogeneity is γ_3. The effects of forcing and damping, not known exactly, are often included phenomenologically by adding $-\gamma(k)$ to (2) where $\gamma(k)$ may depend on n_k and is positive (negative) in the dissipation (forcing) regions. From solutions of (2), we can calculate the asymptotic survivors of the higher Fourier space cumulants (they only depend on n_k) and the corresponding physical space cumulants and moments. In particular, we can check [3] when the statistics are close to joint Gaussian by looking at the structure functions $S_N(\vec{r}) = \langle(\eta(\vec{x}+\vec{r},t) - \eta(\vec{x},t))^N\rangle$ and estimating ratios such as $\tilde{S}_4 = \frac{(S_4-3S_2^2)}{S_2^2}$ as functions of separation $r = |\vec{r}|$. Our second important result is that, on the spectrum $n_k \propto k^{-\alpha x}$, the smallness of $\frac{t_L}{t_{NL}} = \frac{1}{\omega n_k}\frac{dn_k}{dt}$, $\tilde{S}_4$ and the uniform validity of asymptotic relations (2), (3) namely ratios such as $\frac{T_6[n_k]}{T_4[n_k]}$, $\frac{\Omega_2^s[n_k]}{\omega_k}$, are all measured by $Ck^{\gamma_3+d-\alpha-\alpha x}$ (for the case where three wave resonances are absent; $Ck^{2\gamma_2+d-2\alpha-\alpha x}$ when they dominate) where C is a calculable dimension dependent constant. These estimates depend on the convergence of integrals appearing in C and must be checked a posteriori [3]. The convergence of $T_4[n_k]$ on the spectrum $n_k \propto k^{-\alpha x}$ requires αx to lie in a certain window. For gravity waves, this window is $\frac{5}{2} < \alpha x < 5$ in which both the KZ ($\alpha x_0 = 4$) and Phillips' ($\alpha x_P = \frac{9}{2}$) spectra lie. The smallness of $\tilde{S}_4$ is measured by $C' r^{-\gamma_3-d+\alpha+\alpha x}$ where $r = |\vec{r}|$. These ratios are only k and r independent on the generalized Phillips' spectrum. The constant energy flux KZ solutions, $n_k = C_3 P^{\frac{1}{3}} k^{-\frac{2\gamma_3}{3}-d}$ for four wave resonances, $n_k = C_2 P^{\frac{1}{2}} k^{-\gamma_2-d}$ for three wave resonances, are only uniformly valid when the KZ spectra are the same as the generalized Phillips' spectra, namely when $\gamma_3 = 3\alpha$ or $\gamma_2 = 2\alpha$. For $\gamma_3 > 3\alpha$, $\alpha x_P > \alpha x_0 = \frac{2\gamma_3}{3} + d$, the KZ value, breakdown occurs at high wavenumbers and the Phillips' spectrum is steeper than the KZ spectrum. Inserting the proper dimensional factors, the equivalent expressions for gravity and surface tension waves are $C_3 P^{\frac{1}{3}} \frac{k^{\frac{1}{2}}}{g^{\frac{1}{2}}}$ and $C_2 P^{\frac{1}{2}}/((\sigma k)^{\frac{3}{4}})$ respectively. The KZ solutions break down at wavenumbers k_1 and k_2 such that $P^{\frac{1}{3}} k_1^{\frac{\gamma_3}{3}-\alpha}$ and $P^{\frac{1}{2}} k_2^{\gamma_2-2\alpha}$ are order unity. For gravity and capillary waves these wavenumbers are $k_1 = \frac{g}{P^{\frac{2}{3}}}$ and $k_2 = \frac{P^{\frac{2}{3}}}{\sigma}$ respectively. The breakdown region is $k_1 < k < k_2$ if $P > P_0 = (\sigma g)^{\frac{3}{4}}$, which also means that $k_1 < k_0 < k_2$ where $k_0 = \sqrt{\frac{g}{\sigma}}$. The criterion $P > (\sigma g)^{\frac{3}{4}}$, since $P^{\frac{1}{3}} \propto (\rho_a/\rho)^{\frac{1}{2}} U$, ρ_a air density, U windspeed, $U > (\sigma g)^{\frac{1}{4}} (\rho_\alpha/\rho)^{\frac{1}{2}}$ gives a critical wind speed of about 6 m/s at which the ocean wave breaking phenomenon, commonly referred to as whitecapping, is generally observed to occur. This process involves the breaking of the crest of a wedge-like surface into an emulsion of air and water droplets and is clearly dissipative. The parameter $P^{\frac{1}{3}}\sqrt{k/g}$ (or $P^{\frac{1}{3}}(\sigma k)^{-\frac{1}{2}}$ for capillary waves) is also the ratio of the fluid velocity at the surface to phase speed and can be considered a local (in wavenumber) measure of the wave slope.

To summarize: The KZ solutions, valid for $k < k_1$ and $k > k_2$ are exact solutions of $T_4[n_k] = 0$ and $T_2[n_k] = 0$ but they are no longer valid in the range $k_1 < k < k_2$ whenever $\gamma_3 > 3\alpha$ and $\gamma_2 < 2\alpha$. In other words, the KZ spectrum fails at high wavenumbers in the gravity waves range and at low wavenumbers in the capillary wave range. On the other hand, the generalized Phillips' spectrum with $n_k = Ck^{-(\gamma_3+d-\alpha)}$ for four wave processes and with $n_k = Ck^{-(2\gamma_2+d-2\alpha)}$ for capillary wave processes are uniformly valid at all scales. But these spectra do not satisfy either $T_4[n_k] = 0$ or $T_2[n_k] = 0$. The questions are: What equations do the generalized Phillips spectra satisfy? To which physical processes are they connected?

Our answer to this question, for which there is yet no irrefutable evidence, has a simple elegance. We assert that the generalized Phillips' spectrum obtains as an integral balance between nonlinear transfer terms and intermittent dissipation. We can represent the flux of energy $E_k = 2\pi e_k k^{d-1}$, $e_k = \omega_k n_k$, in an isotropic sea as

$$\frac{\partial E_k}{\partial t} = -\frac{\partial p(k)}{\partial k} - \gamma_k, \quad k > k_1, \tag{5}$$

where $\frac{\partial p(k)}{\partial k} = 2\pi k^{d-1}(T_4[n_k] + \text{higher order terms})$ is the nonlinear transfer, $p(k)$ is the flux at wavenumber k, and $k_1 (P^{\frac{1}{3}} k_1^{\frac{\gamma_3}{3}-\alpha} = O(1))$ the breakdown wavenumber. For the moment, we do not try to say what γ_k is. It may not even have a local expression in k. What we do know, however, is that, in the absence of capillary wave action, the cumulative effect of the whitecaps events represented by γ_k is to absorb all the energy flux passing through $k = k_1$ from the KZ solution. We stipulate therefore that (5) holds in the integral time averaged sense so that

$$\int_{k_1}^{\infty} -\frac{\partial p}{\partial k} dk = \int_{k_1}^{\infty} \gamma_k \, dk = P. \tag{6}$$

Eq. (6) is

$$p(k_1) = P. \tag{7}$$

Next we can compute $p(k)$ on a spectrum

$$n_k = CD_1 k^{-\alpha x}, \tag{8}$$

where C and D_1 are respectively dimensionless and dimensional constants. If we assume for the moment that in (2) the leading order term, $T_4[n_k]$, dominates, (7) becomes

$$p(k_1) = C^3 I(x) D_2 k_1^{3\alpha(x_0-x)} = P \tag{9}$$

where D_2 is dimensional and $I(x)$ is a smooth function of x with a zero at $x = x_0 - \frac{1}{3}$ in a band of convergence that includes both the KZ and Phillips values of x. Now let us demand further that x be chosen so that the constant c is universal, independent both of the energy flux P arriving through $k = k_1$ from the KZ spectrum and the breakdown wavenumber k_1 itself. This is entirely reasonable for why should c change as the flux or windspeed change? There is only one possible choice of x. Knowing that the relation between P and k_1 is $Pk_1^{\gamma_3-3\alpha}$ is of order unity, we must have that $3\alpha(x_0 - x) = -\gamma_3 + 3\alpha$ or $\alpha x = \gamma_3 + d - \alpha$, the generalized Phillips' spectrum. For this spectrum we are on the margin of applicability of wave turbulence theory. For the three wave case, $\alpha x = 2\gamma_2 + d - 2\alpha$, the generalized Phillips' spectrum in that situation.

Further, from the second property of the generalized Phillips' spectrum, namely that the ratios $\frac{T_{2N}[n_k]}{T_4[n_k]}$, $N > 2$, are independent of k, we can add the correction to (9) arising from the higher order terms in (2). We find that the left-hand side of (9) is multiplied by $f(C)$, where $f(0) = 1$ and $f(C)$ is a power series in C^2. We have not proved, of course, that $f(C)$ converges but the available evidence suggests that, for gravity waves where the Phillips' spectrum is $n_k = Cg^{\frac{1}{2}} k^{-\frac{9}{2}}$, C is about $\frac{1}{5}$. This value is obtained from the expression of Donelan et al. [12] for the KZ spectrum (including constants). Assuming continuity between that and the Phillips' spectrum at values of k where a clear break from the KZ spectrum is observed in the experimental data determines C. Note that the wavenumber at which the KZ ($C_3 P^{\frac{1}{3}} k^{-4}$) and Phillips' spectra ($Cg^{\frac{1}{2}} k^{-\frac{9}{2}}$) intersect is again k_1. We stress that we define the Phillips' spectrum in terms of wavenumber. The naive translation of this result to frequency space via the linear dispersion relation may not necessarily hold for the breakdown structures we describe below.

This result is much stronger than, but of course consistent with, the result obtained from pure dimensional analysis. Demanding that n_k only depend on g and k and be independent of P leads to the gravity wave Phillips' spectrum. But what we have shown is that this spectrum is also the unique spectrum which can absorb the KZ flux P between the breakdown wavenumber $k = k_1$ and $k = \infty$.

We emphasize the similarities with the situation of optical turbulence. In that context, in order to dissipate the power cascading to low wavenumbers, the system employed randomly occurring, fully nonlinear collapses to carry the power from small to large wavenumbers. The time dependent shape of each carrier is the same but the average frequency of events increases to absorb whatever power is fed to the low wavenumbers. We suggest that the same situation obtains for gravity waves and with the generalized Phillips' spectrum. Although each event is time dependent, we argue that on average, the set of such events consist of a distribution of wedge like shapes $\eta(x, y)$ with a fixed derivative discontinuity across each wave crest, $\eta(x, y) \sim \frac{L}{2s} e^{-s|x|} e^{-y^2/L^2}$ where L, the length scale along the crest, is larger than the largest length scale (which we associate with λ_1, the breakdown wavelength) across the crest. The resulting power spectrum $\langle \hat{\eta}_k^2 \rangle$ is proportional to $(s^2 + k^2 \cos^2\varphi)^{-2} \exp(-\frac{k^2 L^2 \sin^2\varphi}{2})$ where k is the wavenumber modulus and φ is the angle the wavenumber $\vec{k}$ makes with crest normal. Integrating (averaging) over φ (assuming uniform distribution) introduces an additional k in the denominator so that $\langle \hat{\eta}_k^2 \rangle \sim k^{-5}$. This is Kuznetsov's argument. But we will first integrate over a spectrum of uniformly distributed scales $s > k_1$ from which calculation we obtain $\frac{L}{k^3} \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\cos^3\varphi} \mathrm{e}^{\frac{-k^2 L^2 \sin^2\varphi}{2}} (\frac{\pi}{4} - \frac{1}{2} \tan^{-1} \frac{k_1}{k\cos\varphi} - \frac{1}{2} \frac{k k_1 \cos\varphi}{k_1^2 + k^2 \cos^2\varphi})$ (we have written $\int_0^{2\pi} = 4\int_0^{\frac{\pi}{2}}$). For $kL \gg 1$, the principal contribution comes from near $\varphi \simeq 0$. Near $\varphi = \frac{\pi}{2}$, the third order zero of $\cos^3\varphi$ is cancelled by the terms in angular brackets. We obtain that, for $kL \gg 1$, $\langle \hat{\eta}_k^2 \rangle \sim \frac{1}{k^4}$ which is the Phillips' spectrum. Of course, there are many unknowns which are yet far from understood, the nature of the evolving breaker, the typical crest length, the way (perhaps analogous to the Rayleigh–Taylor instability) in which the dynamics close to the wedge breaks the surface, the amount of dissipation per event. But what we can say is that this simplified calculation yields a result which is compatible with the integral power balance reflected in (9).

There are two other intriguing properties enjoyed by the Phillips' spectrum. First, for both gravity and capillary waves, the quantity $\langle \hat{\eta}_k^2 \rangle \sim \frac{1}{k^4}$ (for capillary waves, $n_k \propto k^{-\frac{5}{2}}, e_k = \sigma k^2 \langle \hat{\eta}_k^2 \rangle = \omega_k n_k \propto k^{-2}$). Moreover, one can show that if one had an additional process associated with the potential energy $k^s \langle \hat{\eta}_k^2 \rangle$ (for capillary waves, $s = 2$), one obtains again that $\langle \hat{\eta}_k^2 \rangle \sim \frac{1}{k^4}$. Second, one can associate an entropy production $\dot{S_k} = \frac{1}{n_k} \frac{dn_k}{dt}$ with wave turbulence. For Phillips, $\frac{\dot{S_k}}{\omega}$ is constant. This is consistent with property 2, where we showed that the ratio of linear to nonlinear scales $\frac{t_L}{t_{NL}}$ is independent of k on the generalized Phillips' spectrum. But $\frac{t_L}{t_{NL}} = \frac{1}{\omega} \dot{S_k}$.

We next ask what happens to the argument given in (9) if capillary waves are present. Since $p(k)$ will increase on the Phillips' spectrum ($\alpha x_P < \alpha x_{KZ}$ for capillary waves), a physically impossible situation (where would the additional energy come from?), the Phillips' spectrum must end once three wave capillary processes begin. We therefore conjecture that the Phillips' spectrum window will end at k_0 at the point where the flux P has been reduced to that amount P_0 which the capillary wave flux can accommodate. It must be remembered, however, that near k_0, the simple expressions for $p(k)$ obtained when only one physical process was operative, no longer apply.

Acknowledgements

This was supported by NSF grant DMS 0404577.

References

[1] V.E. Zakharov, V.S. L'vov, G. Falkovich, Kolmogorov Spectra of Turbulence, Springer, Berlin, 1992.
[2] A.C. Newell, L. Biven, S.V. Nazarenko, Physica D 152/153 (2001) 520.
[3] L. Biven, C. Connaughton, A.C. Newell, Physica D 184 (2003) 98.
[4] D. Cai, A. Majda, D. McLaughlin, E. Tabak, Physica D 152–153 (2001) 551.
[5] S. Nazarenko, J. Stat. Mech. (2006) L02002.
[6] V.Z. Zakharov, A.O. Korotkevich, A.N. Pushkarev, A.I. Dyachenko, JETP Lett. 82 (2005) 487.
[7] S. Dyachenko, A.C. Newell, A. Pushkarev, V.E. Zakharov, Physica D 57 (1992) 96.
[8] A.C. Newell, V.E. Zakharov, Optical turbulence, in: P. Tabeling, O. Cardoso (Eds.), Turbulence: A Tentative Dictionary, Plenum Press, New York, 1995.
[9] A. Korotkevich, A.C. Newell, V.E. Zakharov, in preparation.
[10] O.M. Phillips, J. Fluid Mech. 4 (1958) 426.
[11] E.A. Kuznetsov, V. Naulin, A.H. Neilsen, J. Rasmussen, Phys. Fluids 19 (2007) 105110.
[12] M.A. Donelan, J. Hamilton, W.H. Hui, Philos. Trans. R. Soc. London A 315 (1985) 509.