Freely Decaying Weak Turbulence for Sea Surface Gravity Waves

M. Onorato, A. R. Osborne, and M. Serio
Dipartimento di Fisica Generale, Università di Torino, Via Pietro Giuria 1, 10125 Torino, Italy

D. Resio
Coastal and Hydraulics Laboratory, U.S. Army Engineer Research and Development Center, Halls Ferry Road, Vicksburg, Mississippi 39180

A. Pushkarev
Waves and Solitons LLC, 918 West Windsong Drive, Phoenix, Arizona 85045

V. E. Zakharov
Landau Institute for Theoretical Physics, Moscow, 117334, Russia
Department of Mathematics, University of Arizona, Tucson, Arizona 85721

C. Brandini
La.M.M.A., Regione Toscana, Via A. Einstein 35/b, 50013 Campi Bisenzio, Firenze, Italy
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We study the long-time evolution of deep-water ocean surface waves in order to better understand the behavior of the nonlinear interaction processes that need to be accurately predicted in numerical models of wind-generated ocean surface waves. Of particular interest are those nonlinear interactions which are predicted by weak turbulence theory to result in a wave energy spectrum of the form of $|\mathbf{k}|^{-2.5}$. We numerically implement the primitive Euler equations for surface waves and demonstrate agreement between weak turbulence theory and the numerical results.

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After the pioneering work by Kolmogorov [1] on the equilibrium range in the power spectrum of an homogeneous and isotropic hydrodynamic turbulent flow, there have been a number of studies on cascade processes in many other fields of physics such as plasma physics, ocean waves, nonlinear optics, etc. In some of these situations, the description of turbulence turns out to be more accessible than in the case of hydrodynamics because nonlinear terms, although still crucial, are smaller than the linear terms. In this framework, a systematic approach based on averaging the dynamical equations leads to an integro-differential equation known as a *wave kinetic equation* that describes the evolution of the spectral density function for the turbulent field [2]. The wave kinetic equation is of the same form as the standard kinetic equation for bosons, traditionally used in statistical physics since the 1920s. The turbulence that can be described by such an equation is known as *weak* or *wave turbulence* [3].

One remarkable property of wave turbulence is that, in contrast to hydrodynamic turbulence, power law spectra (also known as *Kolmogorov spectra*) have been found as exact solutions of the wave kinetic equation [4]. Since this discovery, weak turbulence has found applications in a wide variety of fields of physics including internal waves [5], waves on liquid hydrogen [6], Alfvén waves in a plasma [7], turbulence in nonlinear optics [8], Langmuir waves [9], etc. Predictions from this theory are very welcome in these various fields because theoretical results are normally hard to achieve and physical behavior is often obscured by little understood complex behaviors in numerical simulations.

In spite of the fact that weak turbulence theory is at a rather advanced state analytically, the actual verification of many of its predictions in comparison to the behavior of the full dynamical equations is often lacking. In this study of surface water waves, one begins with the classical Euler equations [10]. A long outstanding difficulty with wave turbulence theory has been actually verifying that the wave kinetic equation adequately describes the basic physics of the Euler equations. Here we present new results which suggest that the Kolmogorov spectra predicted by weak turbulence theory for surface gravity waves agree with numerical simulations of the Euler equations. This result is important for many fields in physics because it supports the idea that the formation of power laws in the power spectrum of ensembles of stochastic weakly nonlinearly interacting waves may generally be explained using weak turbulence theory [3].

The slope of the Kolmogorov spectrum for ocean surface gravity waves has been the subject of many controversies. The first seminal work was done by Phillips in 1958 [11]. Using dimensional arguments, he found that the frequency spectrum of the surface elevation in the inertial range is of the form $P(\omega) = \alpha g^2 \omega^{-5}$, where $P(\omega)$ is the energy density, g is gravity, and α is a dimensionless constant. This form of the spectrum was obtained by making the assumption that the only parameter that determines the slope of the spectrum was gravity.

Some years later, Zakharov and Filonenko [4] established from the wave kinetic equation for water waves in infinite depth that the nonlinear interactions should produce an energy spectrum of the surface elevation of the form $P(\omega) \sim \omega^{-4}$ (not $\sim\omega^{-5}$ as predicted by Phillips): the result was found as an exact solution of the wave kinetic equation [4,12].

Experimental support of the theory of Zakharov and Filonenko was given by Toba [13] who was completely unaware of their paper. He reformulated the Phillips' equilibrium range taking into account in the dimensional analysis not only gravity, but also the friction velocity u_*. His prediction for the frequency spectrum in the inertial range, in agreement with Zakharov and Filonenko, was $P(\omega) = \beta g u_* \omega^{-4}$, with β a dimensionless constant. After the work by Toba, successive experimental observations of the ω^{-4} law have been made by a number of researchers [14–17]. (Note that, using the linear dispersion relation in infinite water $\omega^2 = g|\mathbf{k}|$, an ω^{-4} frequency spectrum corresponds to a $|\mathbf{k}|^{-2.5}$ wave-number spectrum.)

Even though there is a consensus on the experimental result by Toba [13], it must be stressed that so far the verification of the weak turbulence theory for surface gravity waves has never been established directly from the Euler equations, and moreover the mechanisms that lead to the power law ω^{-4} are not universally recognized: geometrical aspects related to wave breaking, without invoking the nonlinear wave-wave interaction mechanism, are still retained by many oceanographers as fundamental for generating an ω^{-4} power law. Numerical simulations of the wave kinetic equation with forcing and dissipation recently performed [18] show the formation of a power law very close to ω^{-4}. Nevertheless, it must be underlined that the kinetic equation is derived from the Euler equations under an averaging process (a closure is also needed in order to model higher-order moments that derive from the averaging process of the nonlinear terms [19]); therefore, it cannot be concluded *a priori* that power law solutions of the kinetic equation are also shared by the fully nonlinear Euler wave equations.

One way to verify weak turbulence theory is, therefore, to perform direct numerical simulations of the Euler equations. This is indeed the approach we have used in this Letter. The fluid is considered inviscid, irrotational, and incompressible (we also assume infinite depth). Under these conditions, the velocity potential $\phi(x, y, z, t)$ satisfies Laplace's equation everywhere in the fluid [10]. The boundary conditions are such that the vertical velocity at the bottom, $z = -\infty$, is zero and on the free surface, $z = \eta(x, y, t)$, the kinematic and dynamic boundary conditions are satisfied for the velocity potential $\psi(x, y, t) = \phi[x, y, \eta(x, y, t), t]$ (see [12]):

$$\psi_t + g\eta + \tfrac{1}{2}[\psi_x^2 + \psi_y^2 - (\phi_z|_\eta)^2(1 + \eta_x^2 + \eta_y^2)] = 0, \tag{1}$$

$$\eta_t + \psi_x\eta_x + \psi_y\eta_y - \phi_z|_\eta(1 + \eta_x^2 + \eta_y^2) = 0. \tag{2}$$

Numerical simulation of these equations is not an easy task; different numerical approaches can be found in the literature; see [20] for a review. In our numerical simulations, we have used a higher-order spectral (HOS) method, introduced independently by West *et al.* [21] and by Dommermuth *et al.* [22] (see also the recent work by Tanaka [23]). The method is based on the series expansion of the potential velocity around the surface. The small parameter in the expansion is the wave steepness. This expansion is used in the computation of the vertical velocity in (1) and (2). In our numerical computation, we have considered the expansion up to the third order so that four-wave interactions are included (see [23] for a discussion on the relation between the HOS method and the Zakharov equation). Numerical work for forced and dissipated water waves has been performed by Willemsen [24]. Very recently, three new methods have been proposed as very promising for simulating water waves [25–27]. Results using these new approaches on turbulent cascades are still to be completed.

In this Letter, we establish numerically, integrating Eqs. (1) and (2), that nonlinear interactions in surface gravity waves are sufficient for generating a cascade power law in wave spectra. Numerical simulations are organized as follows: we start with a generic wave field (see below) and let that evolve in time numerically, looking for the formation of a power law in the power spectrum of the surface elevation. In order to avoid the effects of external forcing, we considered the case of a freely decaying wave field. If the simulations, as we will see, show the formation of a power law, then the conjecture that this power law is caused by forcing and wave breaking must be excluded, since forcing is absent and wave breaking cannot be taken into account properly using the numerical method considered. From a physical point of view, the freely decaying case corresponds to the evolution of a swell wave field. Since numerical computations are limited by the dimension of the grid considered, and hence limited by the Nyquist wave number, an artificial dissipation is needed at high wave numbers in order to absorb the energy cascading from low wave numbers (see also [18]). We have considered the dissipation phenomenon of the wave field to be similar to the one that takes place in a turbulent flow, i.e., that is mathematically expressed by a Laplacian that operates on the velocity. This methodology of "filtering" high wave numbers is not new in the context of inviscid water waves [18,22]. As is usually done in direct numerical simulations of box turbulent flows, in order to increase the inertial range, we have used a higher-order diffusive term. More explicitly, on the right-hand side of Eqs. (1) and (2), we have added, respectively, two extra terms: $-\nu(-\nabla^2)^n\psi$ and $-\mu(-\nabla^2)^m\eta$, where ν and μ represent an artificial viscosity coefficient and ∇^2 is the horizontal Laplacian. If n and m are greater than 1, the viscosity is known as "hyperviscosity." It should be noted that a very high power of the Laplacian unfortunately could bring about the "bottleneck effect" [28], i.e., an accumulation of energy at high wave numbers that could distort the power law expected [29]. In our numerical simulations, we use $\nu = \mu = 3 \times 10^4$ and $n = m = 8$. These values have been selected after some trial during the development of the numerical code. In our numerical simulations we did not impose any dissipation at low wave numbers.

Concerning the initial conditions for numerical simulations, it has to be underlined that wind waves are not isotropic. For a realistic construction of the initial conditions, a directional frequency power spectrum $S(\omega, \theta)$ is considered [$S(\omega, \theta)$ is an energy density as a function of wave direction and frequency]. As is usual in ocean waves, the angular dependence is factorized as follows: $S(\omega, \theta) = P(\omega)G(\theta)$ with $\int_{-\pi}^{\pi} G(\theta)d\theta = 1$ [16,23]. Without loss of generality here we have used a cosine-squared function for $G(\theta)$ in which only the first lobe (relative to the dominant wave direction) is considered [16,30] [the shape of the spreading function $G(\theta)$ is the same used recently by Tanaka [23]; see also [31]]. It has to be underlined that, even though our initial condition is not isotropic, weak turbulence theory predicts *Kolmogorov spectra* with some very small correction due to anisotropy [2,18]. $P(\omega)$ is chosen to be any localized spectrum at low frequencies. We have performed numerical simulations with $P(\omega)$ taken to be a Gaussian function or a "chopped JONSWAP" (Joint Oceanographic North Sea Wave Analysis Program) spectrum [30] (a JONSWAP spectrum with amplitudes equal to zero for frequencies greater than 1.5 times the peak frequency). Both Gaussian and JONSWAP spectra led to the same results in terms of the turbulent cascade. For the case of the Gaussian function, wave numbers lower than a selected threshold have been set to zero in order to avoid extremely long and large waves. From the directional frequency spectrum, $S(\omega, \theta)$, the two-dimensional surface $\eta(x, y, t = 0)$ is computed using first the linear dispersion relation to move from (ω, θ) to wave number (k_x, k_y) coordinates, and then the inverse Fourier transform with the random phase approximation. The velocity potential $\psi(x, y, t = 0)$ is then obtained from the surface using linear theory.

In our numerical simulations, the spectrum at time $t = 0$ was centered at $\omega_0 = 0.628$ rad/s; i.e., we are considering 10 s waves. Using the linear dispersion relation in infinite water depth, this corresponds to a wavelength $\lambda = 2\pi/k_0 = 156$ m. The wave steepness, $\varepsilon = k_0 H_s/2$, the standard nonlinear parameter for deep-water waves,

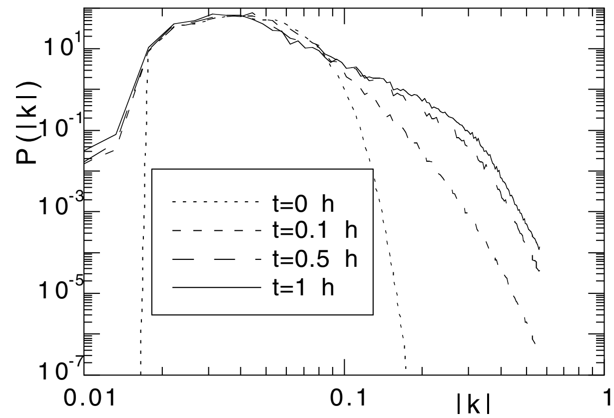

FIG. 1. Wave spectra at different times.

was chosen to be around 0.15 (H_s is the significant wave height and has been computed as 4 times the standard deviation of the surface elevation). This value is typical for wind waves. The wave field was contained in a square domain (the resolution is 256×256) of length $L = 1417.6$ m. The time step considered was 1/50 the dominant frequency; i.e., $\Delta t = 0.2$ s. In Fig. 1 we show the evolution of the wave-number spectrum for different real times ($t = 0$, 0.1, 0.5, 1 h). We see that, as expected, the tail of the spectrum begins to grow. This process seems to be quite rapid: as shown in the figure, after a few dominant wave periods considerable energy is already injected into high wave numbers. The dynamical process in which the spectrum approaches the "correct" power law then slows, especially for low wave numbers. This may be due to the frozen turbulent phenomenon [32], i.e., a condition in which the energy fluxes towards high wave numbers are reduced because of the discreteness of the spectrum. Note that there is also a downshifting of the peak of the spectrum towards lower wave numbers; as a consequence, the wave steepness subsequently decreases over time. The time scale of the nonlinear energy transfer becomes larger and larger. In Fig. 2 we show the spectrum of the surface elevation after 4 h real time (the steepness of the

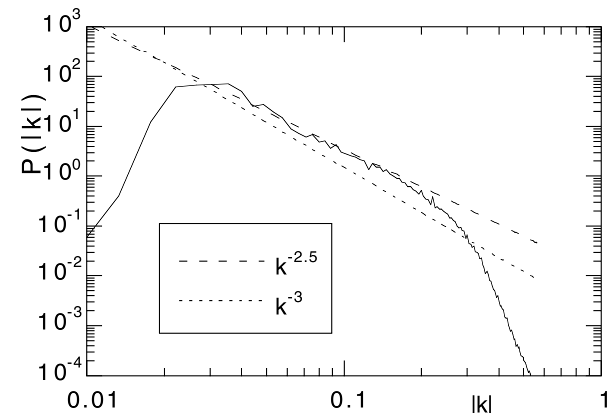

FIG. 2. Wave spectrum at $t = 4$ h. A $k^{-2.5}$ (dashed line) and a k^{-3} (dotted line) power law are also plotted.

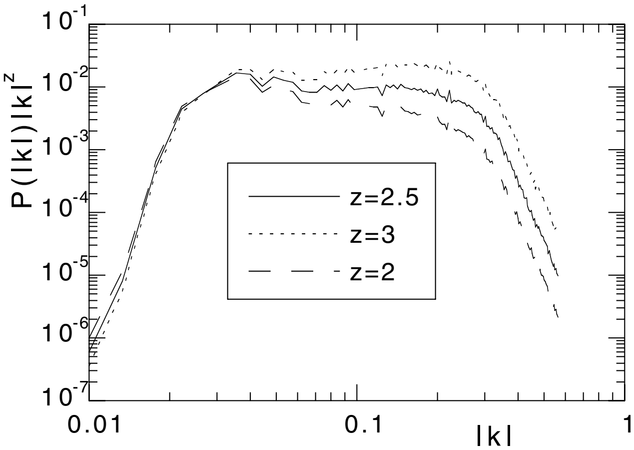

FIG. 3. Compensated wave power spectra for different values of the compensation power: $z = 2$ (dashed line), $z = 2.5$ (solid line), and $z = 3$ (dotted line).

wave field is $\varepsilon \simeq 0.07$). In the same plot we show two power laws $\sim|\mathbf{k}|^{-2.5}$ and $\sim|\mathbf{k}|^{-3}$: the first one seems to better fit the data. A fit in the range from $|\mathbf{k}| = 0.026\ \mathrm{m}^{-1}$ to $|\mathbf{k}| = 0.16\ \mathrm{m}^{-1}$ gives an exponent 2.58 ± 0.10. Additional simulations that we have performed have shown that the same power law behavior persists for longer times.

We also show in Fig. 3 compensated spectra with different compensation powers. There seems to be ample evidence that the power law detected numerically is in good agreement with the value predicted from weak turbulence theory.

After the pioneering work by Zakharov and Filonenko, the kinetic wave theory has developed further, making available quantitative predictions for other physical observables such as energy fluxes, downshifting of the peak, energy dissipation, etc. All these quantities will be examined and results will be reported in future papers. Other questions naturally arise from our results: in our HOS simulations the order of the computation was set to four-wave interactions; do higher-order interactions influence the cascade process? Our computation has been performed in a freely decaying case; could external forcing (especially if anisotropic) influence the power law? And more, what would be the influence of the water depth? These are all questions to be answered in the near future.

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