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# Kolmogorov Spectra in One-Dimensional Weak Turbulence[1]

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In this article, we report the results of our numerical simulation of a one-dimensional modified MMT model, which includes the processes of "one-to-three" wave interactions. We show that this model, with properly chosen parameters, behaves according to the weak-turbulence theory. In particular, it demonstrates the validity of the Kolmogorov spectrum over a wide range of wave numbers. *© 2001 MAIK "Nauka/Interperiodica".*

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**1.** The problem of Kolmogorov spectra is the core of the theory of weak wave turbulence. These spectra appear as exact solutions of stationary kinetic equation for mean squared wave amplitudes [1]. In our opinion, the Kolmogorov weak-turbulence spectra should be used for the theoretical explanation of power-law spectral distributions of energy in ensembles of stochastic nonlinearly interacting waves of any nature. Spectra of such type are observed systematically. The most conspicuous example of this sort is the spectrum $\epsilon_\omega \simeq gv/\omega^4$, which is routinely observed in the systems of wind-driven gravity sea waves.

However, this viewpoint is not shared by everybody. Moreover, the very applicability of the kinetic equation for waves to the real situation is under discussion (see, for instance, [2]). The derivation of the kinetic equation from the initial dynamic equation implies the validity of the assumption of phase randomness, which can be destroyed by the formation of some coherent structures, like solitons or wave collapses. Actually, this criticism has serious foundations. In real situations, the coherent structures are common, but there is no reason for complete abandonment of the weak-turbulence theory. The real life is multicolor, and in many particular situations the coherent structures coexist with weak turbulence, sharing the processes of transport and dissipation of energy and other integrals of motion.

Hence, there is a strong motivation to continue the study of weak turbulence and explore both the case where the coherent structures are important and the case when the influence of such structures is negligible.

**2.** One of the most promising approaches for the study of weak turbulence is a direct numerical simulation of nonlinear dynamic equations describing wave systems. In many cases, these equations can be effectively solved by the use of a spectral code. Of course, numerical simulation of the equation completely relevant to the real physical situation is most preferable. However, a considerable amount of interesting information can be extracted from the solution of simplified, more or less artificial models, which account for the basic features of the real physical equations. Since the weak turbulence is a very general theory, its main statements—applicability of the kinetic equation, existence of the Kolmogorov-type spectra and structures of high-order correlation functions, etc.—can be properly tested with these simple models, for which computer simulation can be carried out more easily.

One-dimensional models are the most attractive ones from this point of view. Even a modest modern computer makes it possible to perform numerical simulation of a system of nonlinear waves including a thousand modes and three decades of scaling. Historically, the first such simulation was accomplished in 1997 by the authors of [2]. They used a model which is called now, after their names, the MMT model. The MMT model is the generalized nonlinear Schrödinger equation conserving not only energy and momentum but also the wave action (number of particles).

Further simulation with the MMT model was performed by the same authors in [3]. Later on, numerical experiments with the MMT model were performed in [4, 5]. The results of both groups basically coincide. The MMT model demonstrates a complicated many-variant behavior that cannot be considered as a certain confirmation of the weak-turbulence theory. In our opinion, this is so because of an interference of the coherent structures, which are present in all versions of the MMT model.

In this article, we report results of our numerical simulation with a modified MMT model, which includes processes "one-to-three" wave interactions not conserving wave action. We will show that this model, with properly chosen parameters, behaves

[1] This article was submitted by the authors in English.

according to the weak-turbulence theory. In particular, it demonstrates the validity of the Kolmogorov spectrum in the range of more than two decades.

**3.** We study the following model:

$$i\left(\frac{\partial\psi}{\partial t} + \gamma_k\psi_k\right) = k^\alpha\psi + a\left\{\int(|k||k_1||k_2||k_3|)^{\beta/4}\right.$$
$$\times\psi^*_{k_1}\psi_{k_2}\psi_{k_3}\delta_{k+k_1-k_2-k_3}dk_1dk_2dk_3$$
$$+g\int(|k||k_1||k_2||k_3|)^{\beta/4}(\psi_{k_1}\psi_{k_2}\psi_{k_3}\delta_{k-k_1-k_2-k_3} \tag{1}$$
$$\left.+3\psi_{k_1}\psi^*_{k_2}\psi^*_{k_3}\delta_{k-k_1+k_2+k_3})dk_1dk_2dk_3\right\}.$$

If $g = 0$, this model becomes the MMT model.

Model (1) can be written as

$$i\left(\frac{\partial\psi}{\partial t} + \gamma_k\psi_k\right) = \frac{\delta H}{\delta\psi^*_k}, \tag{2}$$

where

$$H = H_0 + H_{int}, \quad H_0 = \int|k|^\alpha\psi_k\psi^*_k dk, \tag{3}$$

$$H_{int} = a\left\{\frac{1}{2}\int(kk_1k_2k_3)^{\beta/4}\psi^*_k\psi^*_{k_1}\psi_{k_2}\psi_{k_3}\delta_{k+k_1-k_2-k_3}\right.$$
$$+g\int(kk_1k_2k_3)^{\beta/4}[\psi^*_k\psi^*_{k_1}\psi^*_{k_2}\psi_{k_3} + \psi_k\psi_{k_1}\psi_{k_2}\psi^*_{k_3}] \tag{4}$$
$$\left.\times\delta_{k+k_1+k_2-k_3}dk_1dk_2dk_3\right\}.$$

Hamiltonian $H$ describes the following four-wave processes:

(a) scattering obeying the resonant conditions

$$\omega_k + \omega_{k_1} = \omega_{k_2} + \omega_{k_3}, \quad k + k_1 = k_2 + k_3; \tag{5}$$

(b) “one wave-to-three” decay and the reverse process of gluing three waves to one wave, obeying the resonant conditions

$$\omega_k = \omega_{k_1} + \omega_{k_2} + \omega_{k_3}, \quad k = k_1 + k_2 + k_3. \tag{6}$$

Here, $\omega_k = |k|^\alpha$.

We have studied only the case $\alpha > 1$. In this case, resonant conditions (5) have only a trivial solution

$$k_2 = k, \quad k_3 = k_1 \text{ or } k_2 = k_1, \quad k_3 = k, \tag{7}$$

while resonant conditions (6) describe $2 - D$ manifold in space ($k$, $k_1$, $k_2$, $k_3$). If $a > 0$ and $g$ is small, then Hamiltonian (4) is positively defined. This makes it possible to get rid of any kinds of localized structures.

Under these assumptions, system (1) is described by the kinetic equation

$$\partial n/\partial t + 2\gamma_k n_k + st(n, n, n), \tag{8}$$

where

$$st(n,n,n) = 4\pi a^2g^2\left\{\int(kk_1k_2k_3)^{\beta/2}\right.$$
$$\times(n_{k_1}n_{k_2}n_{k_3} - n_kn_{k_1}n_{k_2} - n_kn_{k_1}n_{k_3} - n_kn_{k_2}n_{k_3})$$
$$\times\delta(k-k_1-k_2-k_3)\delta(\omega_k-\omega_{k_1}-\omega_{k_2}-\omega_{k_3})dk_1dk_2dk_3$$
$$+3\int(kk_1k_2k_3)^{\beta/2}(n_{k_1}n_{k_2}n_{k_3} + n_kn_{k_1}n_{k_2} \tag{9}$$
$$+n_kn_{k_1}n_{k_3} - n_kn_{k_2}n_{k_3})\delta(k-k_1+k_2+k_3)$$
$$\left.\times\delta(\omega_k-\omega_{k_1}+\omega_{k_2}+\omega_{k_3})dk_1dk_2dk_3\right\}.$$

By definition,

$$\langle\psi_k\psi^*_{k'}\rangle = n_k\delta_{k-k'}. \tag{10}$$

A stationary equation

$$st(n, n, n) = 0 \tag{11}$$

has, for the proper values of $\alpha$ and $\beta$, a power-low solution

$$n_k = \alpha p^{1/3}/k^\lambda, \tag{12}$$

$$\lambda = \frac{2}{3}\beta + 1. \tag{13}$$

This is the Kolmogorov spectrum carrying a constant flux of energy $p$ to the large-$k$ region, and $\alpha$ is the dimensionless Kolmogorov constant.

**4.** We have performed the numerical simulation of Eq. (1) by the use of the standard spectral code. We set $\alpha = 3/2$, $\beta = 9/4$, $a = 1$ for different values of the dimensionless parameter $g = 0, 0.05, 0.1, 0.15, 0.2$.

Our spectral array included 2048 modes, $-1024 < k < 1023$. The system was pumped at low wave numbers, $\gamma_k = -0.005$ at $5 \le |k| \le 10$. The energy sink at large wave numbers was provided by damping, $\gamma_k = 400(k/512 - 0.5)^2$ at $|k| > 512$.

In all variants of our computations, we observed a growth and stabilization of the total energy $H$ of the wave system. According to the weak-turbulence theory, the stabilization level depends drastically on the $g$ parameter.

Figure 1 clearly demonstrates that $1 \longleftrightarrow 3$ processes play the main role in establishing equilibrium.

Figure 2 displays typical stationary spectra at $g = 0.15$. One can see that, in the range $30 < k < 300$, they can be well approximated by the Kolmogorov exponent $\lambda = 5/2$. A typical value of nonlinearity

$$\epsilon = H_{int}/H$$

is $\epsilon \simeq 0.15$.

In conclusion, we would like to claim that our result is the first clear confirmation of the validity of the

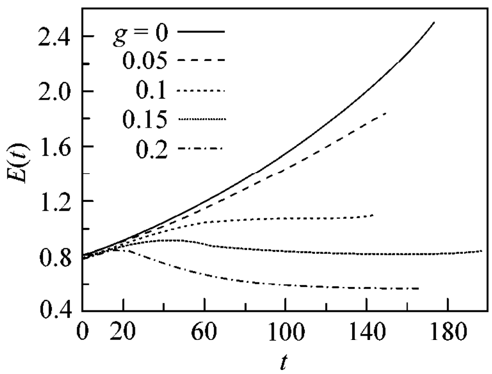


**Fig. 1.** Total energy of the pumped system versus time for different coefficients of "three-to-one" process.

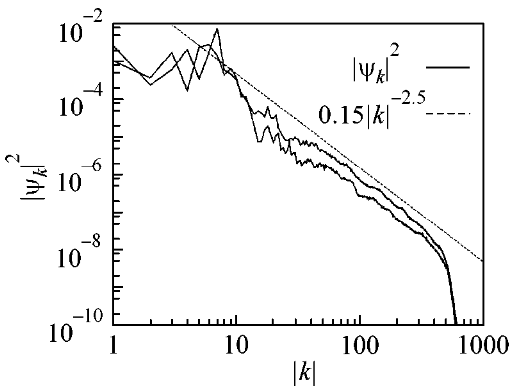


**Fig. 2.** $|\psi_k|^2$ averaged over time 100. Spectra for positive and negative $k$ are shown.

weak-turbulence theory for the $1 - D$ case. A similar confirmation for the $2 - D$ case was done in work [6]. However, in the present $1 - D$ case, the range of scales where the Kolmogorov spectrum is observed is substantially larger.

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