How Classical Physics Helps Mathematics

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1 Introduction

The history of the relations between Physics and Mathematics is a long and romantic story. It begins in the time of Archimedes, and up to the seventeen and eighteen centuries the relations were quite cordial. Mathematics supplied the tools for the solution of physical problems, and in its turn, the necessity to develop proper tools was a very strong factor in stimulating progress in mathematics itself. The problem of the brachistochrone, which was a starting point in the creation of variational calculus, is a classic example. In those times most outstanding mathematicians were also physicists.

In the nineteen century the relations were still close, but some tendency to alienation and separation had become visible. Riemann was both mathematician and physicist, while Weierstrass was a pure mathematician and Faraday was a pure physicist. Until the middle of the last century physics was not divided into theoretical and experimental branches. In the second half of the century the efforts of giants like Maxwell and Boltzmann gained for theoretical physics the status of an independent power. What they created was classical theoretical physics.

The profession of the theoretical physicist was new for that time. Like mathematicians, theoretical physicists use only paper and pen. However, they did not identify themselves with mathematicians. They were sure that what they study is not a world of abstract mathematical concepts, but real nature. On the other hand, pure mathematicians righteously considered results obtained by theoretical physicists as not rigorously justified. Only a few outstanding mathematicians, like Poincare, were at the same time qualified theoretical physicists.

Alienation between physics and mathematics increased after the First World War, which took the lives of a lot of young talents educated by old masters. The creation of quantum mechanics boosted tremendously this process. The logic and intuition of quantum physics was so dramatically different from the "classical" intuition that those who studied the quantum world usually lost interest in other parts of science and preferred to stay in this field forever. An explosive progress of experimental atomic and nuclear physics stimulated enormously the development of theoretical physics. The subject was so hot that physicists just had no time to follow the progress in contemporary mathematics.

At the same time mathematicians after the First World War were too busy to think about physics. At hand were the problems of the "axiomatic revolution", the time of rethinking and reassessment of the very foundations of mathematics. During this time mathematics took its contemporary shape. Set theory, mathematical logic, abstract algebra, topology, and functional analysis were created at that time. Thus mathematicians had a good excuse for paying very limited attention to physicists and their theoretical activity.

I think that the maximum of alienation was in the middle of the fifties. At that time physicists had in their possession a substantial mathematical apparatus, including special functions, complex analysis, representation of finite groups and Lie groups. They knew how to perform sophisticated asymptotic expansion. If they felt a lack of mathematical tools, they invented new ones and used them boldly, not caring too much about their rigor. A δ-function, offered by Dirac, is the most impressive invention of that sort. However, the use of sophisticated machinery was not necessary in many cases. As Laurent Schwarts said bitterly in fifties, the development of the perturbation technique diminished the volume of mathematics used by physicists to elementary algebra and the knowledge of Greek and Latin alphabets.

I started my scientific career in the beginning of the sixties and was one of the few students who were equally keen on physics and mathematics. During my whole life I couldn't make a real choice between these two sister branches of science. Since my youth I had close friends in both scientific communities, and I testify that in the beginning of the sixties these two communities were almost completely divided. In fact, by that time the maximum of separation was already over.

Some important steps to if not reunification but convergence of physics and mathematics were done by mathematicians. In the middle of the century the great axiomatic revolution was over, and mathematicians changed the focus of their interest to more "pragmatic" objects, like Partial Differential

Equations (what are they good for?), or infinite representations of Lie groups (can it be found useful by anybody?). Finally, some mathematicians started to express an interest in the relentless activity of physicists, who every day performed unjustified, risky but unquestionably efficient operations, making possible to obtain quite reasonable results.

The Dirac's δ-function is an especially intriguing object. Its mathematical nature was understood by Laurent Schwarts in 1950 []. This was an event of tremendous importance. It led to explosive development of the theory of generalized functions, the theory of linear topological spaces, to an real breakthrough in the theory of PDE. The accurate use of the theory of generalized functions was important for theoretical physics as well. It made possible to develop a consistent theory of renormalization in quantum electrodynamics.

Discovery of generalized functions was the first move to the renewal of the romance between physics and mathematics. I think that the δ-function, born inside the world of quantum physics, was the most valuable gift presented by physicists to mathematicians in twentieth century. Since that time quantum physics presented to mathematicians several such gifts. Quantum groups and topological quantum field theory are the recent ones.

However, in this article I would like to discuss a quite different subject. Quantum physics dominated in the physical world until the middle of sixties, but then classical physics came out of the shadow and started to grow steadily and persistently. I dare to say that today it has a status equal with quantum physics.

The rise of classical physics in the last four decades was a direct sequence of general technical progress in these years. The invention and fast development of lasers led to the creation of nonlinear optics. Massive use of satellites for monitoring oceans and the atmosphere stimulated the development of physical oceanography and geophysical hydrodynamics. Extensive efforts towards the realization of controlled thermonuclear fusion together with progress in observational astronomy caused an explosive growth of plasma physics and magnetohydrodynamics. All these disciplines became parts of renewed, mostly nonlinear, classical physics, which covers also an essential part of the theory of superfluidity and magnetism. Enormous progress of computers made possible the numerical solving of certain vital problems of classical physics, giving another boost for its progress.

One can say that the "old" classical physics gave to mathematics the linear partial differential equations. All three basic classes of them - elliptic (Laplace equation), parabolic (heat transport equation), and hyperbolic (wave equation) - were born in the classical physics of eighteenth century. Needless to say, how important a role they played in the progress of even the most pure of mathematics. The "new" classical physics opened for mathematicians the magic world of the nonlinear PDE. In a sense, history repeats itself. Among a whole variety of linear PDE, only a few basic ones (Laplace, heat transport and wave equation) play a really fundamental role. If linear PDE form a sea, nonlinear PDE form an immense ocean. But again, only a few selected systems, like Korteveg-de Vries, Nonlinear Schrödinger, and Sine-Gordon equations are really interesting both from a physical and mathematical point of view. It would be very difficult to pull these equations out of the ocean without understanding their fundamental role in classical physics.

One can say that classical physics made mathematics. Two of the most valuable presents - solitons and fractal sets, appearing in the theory of turbulence. In fact, these objects appeared before in pure mathematics, but their fundamental role was not properly estimated. It is natural to add to this list the discovery of nonlinear integrable Hamiltonian systems with infinite number of degrees of freedom, but this subject is closely connected with mathematical theory of solitons. Any relatively complete review of these subjects will take at least two full-scale monographs.

In this article we discuss an application of ideas of classical physics to several important problems of pure mathematics. One idea runs through our examples: classical physics can help mathematicians to handle with Nonlinear Partial Differential Equations.

2 n-wave equations and n-orthogonal coordinate systems

Classical physics is a rich source of "good" nonlinear PDE, but mathematics has its own source - in Differential Geometry. What is remarkable, the "best" equations generated by these two quite different sources sometimes become close related or even identical. Of course, any comment on this phenomenon belongs to metaphysics, and is beyond the scope of this purely scientific article.

A following situation is typical for different physical applications: three wave trains, possibly of a different physical nature, propagate in a weakly nonlinear conservative medium. Their leading wave vectors $\vec{k_1}$, $\vec{k_2}$, $\vec{k_3}$ and corresponding frequencies ω_1, ω_2, ω_3 satisfy the resonant conditions

$$\begin{aligned} \omega_1 &= \omega_2 + \omega_3, \\ \vec{k_1} &= \vec{k_2} + \vec{k_3}. \end{aligned} \tag{2.1}$$

The wave trains are described by complex-valued functions $\psi_i(\vec{x},\, t)$,$\vec{x} = (x_1,\, x_2,\, x_3)$, obeying Hamiltonian equations

$$\frac{\partial \psi_k}{\partial t} = i \frac{\delta H}{\delta \psi_k^*}, \tag{2.2}$$

$$H = \sum_{k=1}^{3} \int Im\, \psi_k (\vec{v_k} \nabla)\, \psi_k^*\, d\vec{x} + \lambda \int (\psi_1^*\, \psi_2\, \psi_3 + \psi_1\, \psi_2^*\, \psi_3^*) d\vec{x}. \tag{2.3}$$

Here $\vec{v_k}$ are group velocities and λ is an interaction coefficient. One can put $\lambda = 1$, then the equations read

$$\begin{aligned} \frac{\partial \psi_1}{\partial t} + (v_1\, \nabla)\psi_1 &= i\, \psi_2\, \psi_3, \\ \frac{\partial \psi_2}{\partial t} + (v_2\, \nabla)\psi_2 &= i\, \psi_1\, \psi_3^*, \\ \frac{\partial \psi_3}{\partial t} + (v_3\, \nabla)\psi_3 &= i\, \psi_1\, \psi_2^*. \end{aligned} \tag{2.4}$$

From the physical view-point system (2.4) is fundamental. It describes an important phenomena - stimulated Raman scattering as well as a three-wave resonant interaction of wave packets. System (2.4) is usually called the "three wave system". To make this system looking more "mathematical" one should introduce new variables

$$\frac{\partial}{\partial u_i} = \frac{\partial}{\partial t} + (v_i\, \Delta), \tag{2.5}$$

and put

$$\begin{aligned} \psi_1 &= -i\, Q_{23} = i\, Q_{32}^*, \\ \psi_2 &= -i\, Q_{13} = i\, Q_{31}^*, \\ \psi_3 &= -i\, Q_{12} = i\, Q_{21}^*. \end{aligned} \tag{2.6}$$

Then system (2.4) takes a form

$$\frac{\partial Q_{ij}}{\partial x_k} = Q_{ik}\, Q_{ki}, \quad i \neq j \neq k, \quad i = 1, 2, 3. \tag{2.7}$$

One can generalize system (2.7) to an n-dimensional case just by putting in (2.7) $i, j, k = 1, \ldots, n$. For $n > 3$ system (2.7) is overdetermined. A further generalization can be done as follows.

Let A be an associate algebra (for instance, algebra of $N \times N$ matrix $N > n$) and I_k $(k = 1, \ldots, n)$ is a set of commuting projectors

$$I_i\, I_k = I_k\, \delta_{ik}. \tag{2.8}$$

A generalization of (2.7) reads

$$I_i\, \frac{\partial Q}{\partial x_k}\, I_j = I_i\, Q\, I_k\, Q\, I_j. \tag{2.9}$$

System (2.9) can be called a "general n-wave system".

In the physical case (2.4) matrix Q is Hermitian, $Q^+ = Q$. This is an example of "reductions" - additional restrictions imposed on Q and compatible with system (2.9). This is an example of a more general reduction

$$Q^+ = J\, Q\, J, \quad [J,\, I_k] = 0, \quad J^2 = 1. \tag{2.10}$$

For $n = 3$ a choice $J = diag(-1, 1, 1)$ leads to a so-called "explosive three-wave system".

It is remarkable that systems (2.7), (2.9) can be applied for solution of some important problems in Differential Geometry. The most famous one is the problem of n-orthogonal coordinate systems.

Suppose S is a domain in R^n. How to find all orthogonal curvilinear coordinate systems in S? Let $x = (x_1, \ldots, x_n)$ be such coordinates. In this coordinate system the matrix tensor is diagonal

$$ds^2 = \sum H_i^2\, dx_i^2. \tag{2.11}$$

Hamiltonian $H_i = H_i(x)$ and the Lamé coefficients are subjects for determination. They satisfy a heavily overdetermined system of nonlinear PDE, the Gauss-Lamé equations. These equations read:

$$\frac{\partial Q_{ij}}{\partial x_k} = Q_{ik}\, Q_{kj}, \;\; i \neq j \neq k, \tag{2.12}$$

$$E_{ij} = \frac{\partial Q_{ij}}{\partial x^j} + \frac{\partial Q_{jk}}{\partial x^i} + \sum_{n \neq i,j} Q_{ik}\, Q_{jk} = 0, \ \ i \neq j. \tag{2.13}$$

Here

$$Q_{ij} = \frac{1}{H_j} \frac{\partial H_i}{\partial x^j}. \tag{2.14}$$

One can see that the first group in the Gauss-Lamé equations (2.12) exactly coincides with the n-wave equations (2.7), (2.9). The second group of equations (2.13) can be treated as a reduction. As far as Q_{ik} is real, one more reduction is imposed

$$Q_{ij} = \bar{Q}_{ij}. \tag{2.15}$$

To understand the origin of the Gauss-Lamé equations, one can consider that a domain is not in R^n but in some Riemann space, admitting introduction of "diagonal" coordinate (2.11), and calculate the Riemann curvature tensor R_{ijkl}. One finds that by the virtue of (2.11)

$$\begin{aligned} R_{ij,kl} &= 0, \quad i \neq j \neq k \neq l, \\ R_{ik,jk} &= -H_i\, H_j \, (\frac{\partial Q_{ij}}{\partial x^k} - Q_{ik}\, Q_{kj}), \\ R_{ij,ij} &= -H_i\, H_j\, E_{ij}. \end{aligned} \tag{2.16}$$

The curvature tensor is a symmetric matrix in the space of bivectors in TS. If only (2.12) is satisfied, this matrix is diagonal. A corresponding Riemann space S can be called a space of "diagonal curvature". Riemann spaces of diagonal curvature are a very interesting class of objects. They include, for instance, homogeneous spaces as well as conformal flat spaces. If the second system (2.13) is satisfied, $E_{ij} = 0$, the space is flat and S is a domain in R^n.

We see that the Gauss-Lamé equations differ from the n-wave system only by a choice of reduction. All these systems are completely integrable and can be efficiently solved by the use of the method of Inverse Scattering Transform, elaborated in the theory of solitons. We will present here the most advanced version of this method known as a "Dressing Method". It makes possible to construct solitonic, multisolitonic, and more general solutions of integrable equations locally in X-space. We will do this in a maximally general form assuming that all unknown functions belong to some associative algebra A over the complex field.

We introduce again a set of projectors I_k satisfying the condition (2.8) and construct an element $\Phi \in A$:

$$\Phi = \sum_{i=1}^{n} x_i \, I_i. \tag{2.17}$$

Let λ is a point on the complex plane, $\chi = \chi(\lambda, \bar{\lambda})$ is an A-valued quasianalityc function on C. Suppose that $\chi(\lambda, \bar{\lambda}, x)$ is a solution of the following non-local $\bar{\delta}$-problem:

$$\frac{\partial \chi}{\partial \bar{\lambda}} = \chi \times R = \int \chi(\nu, \, \bar{\nu}, x) \, R(\nu, \, \bar{\nu}, \, \lambda, \, \bar{\lambda}, \, x) \, d\lambda \, d\bar{\lambda}, \tag{2.18}$$

normalized by the condition

$$\chi \to 1 \quad at \lambda \to \infty. \tag{2.19}$$

In (2.18)

$$R(\nu, \bar{\nu}, \lambda, \bar{\lambda}) = e^{\Phi \, \nu} \, T \, e^{-\Phi \, \lambda}, \tag{2.20}$$

where $T = T(\nu, \, \bar{\nu}, \, \lambda, \, \bar{\lambda})$ does not depend on x. Function T is a "free parameter" of the theory. It should be chosen by such a way that (2.18) has a unique solution for all $x \in S$. At $\lambda \to \infty$ this solution has an asymptotic expansion

$$\chi = 1 + \frac{Q}{\lambda} + \frac{P}{\lambda^2} + \cdots. \tag{2.21}$$

According to the Freidholm's alternative, any solution $\tilde{\chi}$ of $\bar{\delta}$-problem vanishing at infinity is identically zero

$$\tilde{\chi} \equiv 0, \;\; if \;\; \tilde{\chi} \to 0 \;\; at \;\; \lambda \to \infty. \tag{2.22}$$

The solution of $\bar{\delta}$-problem (2.21) is called "dressing", while a free kernel T is called a "dressing function".

The following statement is a cornerstone of the theory:

Theorem 1 *For $x \in S$ the term Q in (2.21) is a solution of n-wave system (2.9).*

Proof:

Let us construct a set of differential operators L_{ij} acting on χ as follows

$$L_{ij}\,\chi = I_i\,(\frac{\partial\chi}{\partial\chi^j} + \lambda\,\chi\,I_j) - I_i\,Q\,I_j\,\chi. \qquad (2.23)$$

A straightforward calculation shows that $L_{ij}\,\chi$ satisfies the same $\bar{\delta}$-problem

$$\frac{\partial}{\partial\bar{\lambda}}\,L_{ij}\,\chi = L_{ij}\,\chi \times R. \qquad (2.24)$$

Substitution of asymptotic (2.21) into $L_{ij}\,\chi$ shows that $L_{ij}\,\chi \to v(\frac{1}{\lambda})$ at $\lambda \to \infty$. Hence

$$L_{ij}\,\chi = 0, \qquad (2.25)$$

and

$$L_{ij}\,\chi\,I_k = 0, \quad i \neq j \neq k. \qquad (2.26)$$

Substituting asymptotic expansion (2.21) into (2.24) and taking into account only the leading nonvanishing terms of order $1/\lambda$, one can see that Q satisfies the equation (2.9).

The solution of (2.9) can be found in closed algebraic form if the kernel T is degenerated:

$$T(\nu,\,\bar{\nu},\,\lambda,\,\bar{\lambda}) = \sum_{k=1}^{N} A_k\,(\nu,\bar{\nu})\,B_k(\lambda,\,\bar{\lambda}). \qquad (2.27)$$

In the most simple case

$$T = A(\nu,\,\bar{\nu})\,B(\lambda,\,\bar{\lambda})$$

the solution has a compact form

$$Q = < A > (1 - (B|A))^{-1} < B > . \qquad (2.28)$$

Here

$$\begin{aligned} < A > &= \int e^{\lambda\Phi}\,A(\lambda,\,\bar{\lambda})\,d\lambda\,d\bar{\lambda}, \\ < B > &= \int B(\lambda,\,\bar{\lambda})\,e^{-\lambda\,\Phi}\,d\lambda\,d\bar{\lambda}, \\ < B|A > &= \frac{1}{\pi}\int \frac{B(\nu,\,\bar{\nu})\,e^{(\lambda-\mu)\,\Phi}A(\lambda,\,\bar{\lambda})}{\nu - \lambda}\,d\nu\,d\bar{\nu}\,d\lambda\,d\bar{\lambda}. \end{aligned} \qquad (2.29)$$

The equation (2.27) is a general solitonic solution of the n-wave system. It is quite nontrivial solution describing a set of interesting physical phenomena. In a general case the equation (2.27) leads to N-solitonic solution. The role of solitonic solution in Differential Geometry has not yet been studied in a proper degree.

So far we did not impose on the solution of the system (2.13) any additionally restrictions (reductions). They can be imposed by imposing of some additional constrains on the "dressing function" $T(\nu,\, \bar{\nu},\, \lambda,\, \bar{\lambda})$. Imposing of condition

$$\bar{T}(\bar{\nu},\, \nu,\, \bar{\lambda},\, \lambda) = T(\nu,\, \bar{\nu},\, \lambda,\, \bar{\lambda}) \tag{2.30}$$

makes Q real:

$$\bar{Q} = Q.$$

Condition

$$T^{+}(\bar{\nu},\, \nu,\, \bar{\lambda},\, \lambda) = J\, T(\nu,\, \bar{\nu},\, \lambda,\, \bar{\lambda}) J, \quad J^2 = 1, \quad [J,\, \Phi] = 0 \tag{2.31}$$

leads to the reduction

$$Q^{+} = J\, Q\, J, \tag{2.32}$$

the most important from a physical view-point. Finally, condition

$$T^{tr}(-\mu, -\bar{\mu}, -\lambda, -\bar{\lambda}) = \frac{\mu}{\lambda} T(\lambda, \bar{\lambda}, \mu, \bar{\mu}) \tag{2.33}$$

provides satisfaction of the last set of equations (2.13).

Formula (2.14) shows that the n^2 elements of matrix Q are expressed through n Lamé coefficients $H^i(x)$. It might make an impression that we are looking for some special solution of the equation (2.12). This is not actually true. Any solution of this system can be presented in a form (2.14) by many different ways.

Indeed, one can introduce a new function,

$$\psi = \chi\, e^{\lambda \Phi}, \tag{2.34}$$

which satisfies the equation

$$I_i \frac{\partial \psi}{\partial x^j} - I_i\, Q\, I_j\, \psi = 0. \tag{2.35}$$

Let $A_l(\lambda, \bar{\lambda}), l = 1, \ldots, n$ is an arbitrary set of functions of variables λ, $\bar{\lambda}$, and

$$H_i = \int \sum_{l=1}^{n} \psi_{il}(\lambda, \bar{\lambda}, x)\, A_l(\lambda, \bar{\lambda}) d\lambda\, d\bar{\lambda}. \tag{2.36}$$

one can see that

$$\frac{\partial H_i}{\partial x^j} = Q_{ij}\, H_j. \tag{2.37}$$

A different choice of $A_l(\lambda, \bar{\lambda})$ leads to a different set of H_i. All these sets are called Combescure equivalent. One can see that a classical problem of classification of all Combescure equivalent arrays of n-orthogonal coordinate systems is solved efficiently in this "solitonic" formalism.

3 Theory of surfaces as a chapter of theory of solitons

In previous chapter we saw how easily the method of "mathematical theory of solitons", elaborated in the classical theory of integrable systems, makes it possible to solve a classical problem of differential geometry. In this chapter we will develop this success and find a way to solve another important problem in differential geometry - the classification of surfaces in R^3.

Let Γ be a surface in R^3. One can introduce on Γ coordinates x_1, x_2 such that both the first and the second quadratic forms are diagonal:

$$\begin{aligned} \Omega_1 &= p^2\, dx_1^2 + q^2\, dx_2^2\,, \\ \Omega_2 &= pA\, dx_1^2 + qB\, dx_2^2\,. \end{aligned} \tag{3.1}$$

Coordinates x_1, x_2 are defined up to trivial transformations $x_1 = x_1(u_1), x_2 = x_2(u_2)$. Coefficients of the two quadratic forms Ω_1, Ω_2 cannot be chosen independently. Four functions p, q, A, B are connected by three nonlinear PDE known as Gauss-Codazzi equations (GCE). We will now show that these equations are simply a very degenerate case of a classical three-wave system. They can be integrated by a minor modification of the dressing method used for the integration of n-orthogonal coordinate system.

Let us imbed the surface Γ in a special three-orthogonal coordinate system in R^3 in vicinity of F

$$ds^2 = H_1^2\, dx_1^2 + H_2^2\, dx_2^2 + dx_3^2, \tag{3.2}$$

where

$$H_1 = p + A\,x_3, \quad H_2 = q + B\,x_3, \;\; H_3 = 1. \tag{3.3}$$

Obviously,

$$Q_{31} = Q_{32} = 0,$$

and other coefficients of matrix Q_{ij} do not depend on x_3. Indeed,

$$Q_{13} = \frac{1}{H_3}\,\frac{\partial}{\partial x_3}\,H_1 = A, \;\; Q_{23} = \frac{1}{H_3}\,\frac{\partial}{\partial x_3}\,H_2 = B, \tag{3.4}$$

so

$$\frac{\partial}{\partial x_2}(p + A\,x_3) = Q_{12}(q + B\,x_3),$$
$$\frac{\partial}{\partial x_1}(q + B\,x_3) = Q_{21}(p + A\,x_3).$$

Hence

$$Q_{12} = \frac{1}{B}\,\frac{\partial A}{\partial x_2} = \frac{1}{q}\,\frac{\partial p}{\partial x_2},$$
$$Q_{21} = \frac{1}{A}\,\frac{\partial B}{\partial x_1} = \frac{1}{p}\,\frac{\partial q}{\partial x_1}.$$

In this case only two equations survive in the system (2.12):

$$\frac{\partial Q_{13}}{\partial x_2} = Q_{12}\,Q_{23}, \;\; \frac{\partial Q_{23}}{\partial x_1} = Q_{21}\,Q_{13}. \tag{3.5}$$

The reduction condition (2.13) leads to one more equation,

$$\frac{\partial Q_{12}}{\partial x_2} + \frac{\partial Q_{21}}{\partial x_1} + Q_{13}\,Q_{23} = 0. \tag{3.6}$$

Let us denote $Q_{12} = \alpha, \;\; Q_{21} = \beta$. The Gauss-Codazzi equations read

$$\frac{\partial \alpha}{\partial x_2} + \frac{\partial \beta}{\partial x_1} + A\,B = 0,$$
$$\frac{\partial A}{\partial x_2} = \alpha\,B, \;\; \frac{\partial B}{\partial x_1} = \beta\,A. \tag{3.7}$$

The system (3.7) should be accompanied by equations for elements of first quadratic form, p and q:

$$\frac{\partial p}{\partial x_2} = \alpha\,q, \;\; \frac{\partial q}{\partial x_1} = \beta\,p. \tag{3.8}$$

Comparing (3.7), (3.8) one can realize that A, B and p, q are Combescure equivalent pairs of Lamé coefficients. Physicists know the equation (3.8) as the two-dimensional Dirac system.

System (3.7) consists of three equations imposed on four unknown functions α, β, A, B. Hence, its general solution should be parametrized by some functional parameters. To perform the solution one should remember that system (3.5), (3.6) is a special case of Gauss-Lamé equations. Thus, one can use the standard scheme described in the previous chapter. In another words, one can solve the $\bar{\delta}$-problem

$$\begin{aligned} \frac{\partial \chi}{\partial \lambda} &= \chi \times R, \\ R_{ij}(\lambda, \bar{\lambda}, \mu, \bar{\mu}) &= e^{\lambda \chi_i - \mu \chi_j} T_{ij}(\lambda, \bar{\lambda}, \mu, \bar{\mu}). \end{aligned} \tag{3.9}$$

The dressing function T should satisfy the condition (2.33) and condition of reality,

$$\bar{T}(\bar{\lambda}, \lambda, \bar{\mu}, \mu) = T(\lambda, \bar{\lambda}, \mu, \bar{\mu}), \tag{3.10}$$

and must satisfy one more condition,

$$\frac{\partial R_{ij}}{\partial x_3} \equiv 0. \tag{3.11}$$

One can assume also that

$$R_{11} = R_{22} = R_{33} = 0, \;\; T_{11} = T_{22} = T_{33} = 0.$$

Under these assumptions the matrix function is defined uniquely.

Let

$$\begin{aligned} & f_1(\lambda, \bar{\lambda}) = \bar{f}_1(\bar{\lambda},\, \lambda), \\ & f_2(\lambda, \bar{\lambda}) = \bar{f}_2(\bar{\lambda},\, \lambda), \\ & R(\mu,\, \bar{\mu}, \lambda, \bar{\lambda}) = \bar{R}(\bar{\mu},\, \mu,\, \bar{\lambda},\, \lambda), \end{aligned} \tag{3.12}$$

be arbitrary functions. Then all non-zero elements of the dressing matrix $T_{ij}(\lambda, \bar{\lambda}, \mu, \bar{\mu})$ are the following:

$$\begin{aligned} & T_{12} = \mu\, R(\mu, \bar{\mu}, \lambda, \bar{\lambda}), \\ & T_{21} = -\mu\, R(-\lambda, -\bar{\lambda}, -\mu, -\bar{\mu}), \end{aligned}$$

$$\begin{aligned}
T_{13} &= -\mu\, f_1(-\mu,-\bar{\mu})\delta(\lambda)\,\delta(\bar{\lambda}),\\
T_{23} &= -\mu\, f_2(-\mu,-\bar{\mu})\,\delta(\lambda)\,\delta(\bar{\lambda}),\\
T_{31} &= \mu\,\delta(\mu)\,\delta(\bar{\mu})\, f_1(\lambda,\bar{\lambda}),\\
T_{32} &= \mu\,\delta(\mu)\,\delta(\bar{\mu})\, f_2(\lambda,\bar{\lambda}).
\end{aligned}\tag{3.13}$$

The $\bar{\delta}$-problem is equivalent to the integral equation

$$\chi_{ij}(\lambda,\bar{\lambda},x) = \delta_{ij} + \frac{1}{\pi}\int \frac{\chi_{ij}(\mu,\bar{\mu},x)\,R_{kl}(\mu,\bar{\mu},\xi,\bar{\xi},x)}{\lambda-\xi} d\mu\, d\bar{\mu}\, d\xi\, d\bar{\xi}. \tag{3.14}$$

According to (3.13) the kernel R is partly degenerated. One can obtain from (3.14) the following relations:

$$\begin{aligned}
&\chi_{31} = \chi_{32} = 0,\quad \chi_{33} = 1,\\
&\chi_{13} = \frac{1}{\lambda}A,\quad \chi_{23} = \frac{1}{\lambda}B.
\end{aligned}\tag{3.15}$$

$$\begin{aligned}
A &= -\frac{1}{\pi}\int \mu \sum_{k=1}^{\infty} \chi 1k(\mu,\bar{\mu},x)\, f_k(-\mu,-\bar{\mu}) e^{-\mu x_k} d\mu\, d\bar{\mu},\\
B &= -\frac{1}{\pi}\int \mu \sum_{k=1}^{\infty} \chi 2k(\mu,\bar{\mu},x)\, f_k(-\mu,-\bar{\mu}) e^{-\mu x_k} d\mu\, d\bar{\mu}.
\end{aligned}\tag{3.16}$$

From (3.15), (3.16) one observe that it is possible to construct a closed system of integral equations for the left upper block of the matrix χ_{ij}. Omitting intermediate calculations we present only the result:

Theorem 2 *The solution of the Gauss-Codazzi equation is given by the solution of the integral equation imposed on a* 2×2 *complex matrix* $Q_{ij}(\lambda,\bar{\lambda},x)$*:*

$$Q_{ij}(\lambda,\bar{\lambda},x) = \delta_{ij} + \frac{1}{\pi}\int \frac{\mu\, Q_{ik}(\mu,\bar{\mu},x)\, S_{ij}(\mu,\bar{\mu},\xi,\bar{\xi})\, e^{\mu x_k - \xi x_l}}{\lambda-\xi} d\mu\, d\bar{\mu}\, d\xi\, d\bar{\xi}. \tag{3.17}$$

Here

$$\begin{aligned}
S_{ij} &= S_{ij}^{(1)} + S_{ij}^{(2)},\\
S_{ij}^{(1)}(\mu,\bar{\mu},\lambda,\bar{\lambda}) &= \begin{bmatrix} 0 & R(\lambda,\bar{\lambda},\mu,\bar{\mu}) \\ -R(-\mu,-\bar{\mu},-\lambda,-\bar{\lambda}) & 0 \end{bmatrix},\\
S_{ij}^{(2)}(\mu,\bar{\mu},\lambda,\bar{\lambda}) &= -\frac{1}{\pi}\, f_i(-\mu,-\bar{\mu})\, f_j(\lambda,\bar{\lambda}).
\end{aligned}\tag{3.18}$$

If this equation uniquely resolves, then

$$Q_{12} \to \frac{\alpha}{\lambda}, \;\; Q_{21} \to \frac{\beta}{\lambda} \;\; at \;\; \lambda \to \infty. \tag{3.19}$$

Formulae (3.16), (3.19) generate a solution of the Gauss-Codazzi system.

One should also mention that a solution of the Gauss-Codazzi equation does not define the surface uniquely. It defines a whole class of surfaces with different elements of the first quadratic form p^2, q^2. One can find them using the formulae

$$\begin{aligned} p &= \int \Big[Q_{11}(\lambda, \bar{\lambda}, x) e^{-\lambda x_1} u(\lambda, \bar{\lambda}) + Q_{12}(\lambda, \bar{\lambda}, x) e^{-\lambda x_2} v(\lambda, \bar{\lambda}) \Big] d\lambda\, d\bar{\lambda}, \\ q &= \int \Big[Q_{21}(\lambda, \bar{\lambda}, x) e^{-\lambda x_1} u(\lambda, \bar{\lambda}) + Q_{22}(\lambda, \bar{\lambda}, x) e^{-\lambda x_2} v(\lambda, \bar{\lambda}) \Big] d\lambda\, d\bar{\lambda}. \end{aligned} \tag{3.20}$$

Here

$$u(\lambda,\, \bar{\lambda}) = \bar{u}(\bar{\lambda},\, \lambda), \quad v(\lambda,\, \bar{\lambda}) = \bar{v}(\bar{\lambda},\, \lambda), \tag{3.21}$$

are arbitrary functions. In particular, one can choose the case

$$\begin{aligned} u(\lambda,\, \bar{\lambda}) &= -\frac{1}{\pi}\, \lambda\, f_1(-\lambda,\, -\bar{\lambda}), \\ v(\lambda,\, \bar{\lambda}) &= -\frac{1}{\pi}\, \lambda\, f_2(-\lambda,\, -\bar{\lambda}), \end{aligned} \tag{3.22}$$

where $p = A$, $q = B$, and the surface has a constant curvature.

The theory of surfaces in R^3 is a classical chapter in differential geometry, still actively developing. We can offer a "solitonic" program to the systematic study and classification of surfaces. It includes the following two steps.

1. Classification of solutions of the Gauss-Codazzi system. Each solution define the whole class of Combescure-equivalent surfaces.

2. Study of surfaces in the framework of a given class.

3. Embedding the surface into R^3. As far as we know the first and second quadratic terms, according to the Bonnet theorem this embedding is unique up to Eucledean motion. The presented method for solution of the Gauss-Codazzi equation makes possible to perform embedding efficiently. In this article we just announce the result, the details will be published separately.

The solution of the integral equation (3.17) can be done explicitly in a closed algebraic form if the kernel R in $\bar{\delta}$-problem (3.9) is degenerated

$$R(\mu, \bar{\mu}, \lambda, \bar{\lambda}) = \sum_{k=1}^{n} A_k(\mu, \bar{\mu})\, B_k(\lambda, \bar{\lambda}). \tag{3.23}$$

The class of surfaces arising from the solution of the $\bar{\delta}$-problem can be called $n+1$ solitonic surfaces. In the simplest case $R = 0$, and the surface is one-solitonic. In this case the integral equation ( ) can be easily solved. The results are:

$$\begin{aligned}
\alpha &= -\frac{2h_1^1(x_1)\, h_2(x_2)}{1 + h_1^2(x_1) + h_2^2(x_2)},\\
\beta &= = -\frac{2h_2^1(x_2)\, h_1(x_1)}{1 + h_1^2(x_1) + h_2^2(x_2)},\\
A &= -\frac{2h_1^1(x_1)}{1 + h_1^2(x_1) + h_2^2(x_2)},\\
B &= -\frac{2h_2^1(x_2)}{1 + h_1^2(x_1) + h_2^2(x_2)},
\end{aligned} \tag{3.24}$$

where

$$h_1(x_1) = \sqrt{2\pi} \int f_1(\mu, \bar{\mu})\, e^{-\mu x_1}\, d\mu\, d\bar{\mu},$$

$$h_2(x_2) = \sqrt{2\pi} \int f_2(\mu, \bar{\mu})\, e^{-\mu x_2}\, d\mu\, d\bar{\mu}.$$

One can see that the choice of f_1, f_2 is just a redefinition of variables x_1, x_2.

Even this simplest class of "solitonic" surfaces is not studied properly. The simplest representative of this class arises if one puts $p = A$, $q = B$. This is a sphere in stereographic projection.

4 Long-time asymptotics in the Hamiltonian PDE equation

In the previous chapter we formulated more or less rigorous mathematical statements. Now we will speak the language of theoretical physics. This language has its own logic, convincing for physicists, who I hope, will agree with final conclusions of our consideration.

Mathematicians quite rightfully will consider these statements as not proved rigorously enough. We would be quite satisfied, if they will treat them as plausible conjectures, stimulating their curiosity to examine and finally either to prove or to refuse them. One has just to remember that the way from the "physical" and the "mathematical" versia of scientific truth might be long and difficult.

In this chapter we will discuss long-time asymptotics of certain nonlinear PDE equations. As far as nonlinear PDE is the most common tool for the study of very different mathematical phenomena from black holes to dynamic of population nobody can believe that a kind of general theory can be anticipated here. Thus we restrict our consideration only by evolutionary equation of Hamiltonian type. These equations preserve energy and, possibly, other constants of motion. We will ask ourselves the following question. Suppose we impose to our equation or system of equations an initial data, having a certain level of smoothness. Will this smoothness improve or determinate in time?

In dissipative systems, like heat transport equations, rough initial data have tendency to become as smooth as it is compatible with the boundary condition. In Hamiltonian systems the situation is quite opposite. One should expect that their solutions will lose their smoothness and become in process of evolution more and more rough. Only in exceptional cases, for linear and for integrable systems (also for systems asymptotically linear or integrable) the smoothness will tend in time to a certain finite limit.

This very general statement is just a consequence of the second law of thermodynamics. Conservative PDE describe Hamiltonian systems with infinite number of degrees of freedom. As all Hamiltonian systems in the world in process of their evolution they tend to thermodynamic equilibrium.

However, thermodynamic equilibrium for classical continuous systems means equipartition of energy between all degrees of freedom and, consequently, excitation of all possible spatial harmonics. This is exactly the lack of smoothness. Any initial data tending to thermodynamic equilibrium becomes more and more rough. Such phenomena as formation of the "islands of stability" or KANM tori can slow this process but cannot completely stop it.

In other words, the right question is about the rate - how soon a system loses its smoothness and relaxes to the thermodynamic equilibrium. Theoretical physicists offer several ways to answer these important and interesting questions.

First of them is using a statistical description of the initial system. One can make a conjecture that the system being still far from thermodynamic equilibrium, nevertheless displays a chaotic behaviour which should be described in terms of correlating functions. One can try to find closed equations for these functions and find their solution asymptotically in time. It might be much easier than to follow the lack of smoothness directly in the initial dynamic system.

We illustrate this idea on one basic example. We will study "defocusing" nonlinear Scrödinger equation in infinite three-dimensional space

$$i\Psi_t + \Delta\,\Psi - |\Psi|^2\,\Psi = 0, \quad x \in R^3. \tag{4.1}$$

We will assume that no boundary condition is imposed and that the initial data

$$\Psi|_{t=0} = \Psi_0(2)$$

is an infinitely smooth function. More exactly we will assume for any finite domain $\Omega in R^3$

$$W_2^l(\Omega) < A_2^l\, V_\Omega. \tag{4.2}$$

Here V_Ω is the volume of Ω, A_2^l is a set of positive constants. We ask now the question: how fast does A_2^l grow in time?

Equation (4.1) is a Hamiltonian system and can be written in a form

$$i\Psi_t = \frac{\partial H}{\partial \Psi^k} \tag{4.3}$$

$$H = \int \left\{ |\nabla\,\Psi|^2 + \frac{1}{2}|\Psi|^4 \right\} d\vec{r} \tag{4.4}$$

Hamiltonian H is a constant of motion, another constant of motion is "number of particles" $N = \int |\Psi|^2\, d\vec{r}$.

One can assume that the initial data $\Psi_0(r)$ is homogeneous in space stochastic process described by a pair of correlation functions in Fourier space

$$< \Psi_{0k}\,\Psi^*_{0k'} >= n_0(k)\,\delta_{k-k'}. \tag{4.5}$$

Now estimate (4.2) reads

$$\int k^{2l}\, n_0(k)\, d < A_2^l, \tag{4.6}$$

and $n_0(k)$ decays at $k \to \infty$ faster than any powerlike function.

In Fourier space Hamiltonian is

$$\begin{aligned} H &= H_0 + H_{int}, \\ H_0 &= \int \omega_k |\Psi_k|^2 \, dk, \\ H_{int} &= \frac{1}{2} \int \Psi_{k_1}^* \, \Psi_{k_2}^* \, \Psi_{k_3} \, \Psi_{k_4} \, \delta_{k_1+k_2-k_3-k_4} \, dk_1 \, dk_2 \, dk_3 \, dk_4, \end{aligned} \tag{4.7}$$

where $\omega_k = k^2$ is a symbol of linear part of equation (4.1), known in physics as the "dispersion law". Equation (4.4) is a nonlinear wave equation describing interaction of spatial harmonics with different wave vectors $\vec{k}$.

A thermodynamic equilibrium in a system of harmonics can be reached on the Rayligh-Jeans spectrum in presence of condensate

$$n_k^T = A_0 \, \delta(k) + \frac{T}{\omega_k}, \tag{4.8}$$

where T is temperature and A - intensity of condensate.

Conservation of particle number N means that L_2-norm (A_2^0) is a constant of motion. Meanwhile for spectrum (3.8) $A_2^0 = \infty$ at any finite T. This is a completely general fact.

Thermodynamic equilibria in classical wave system can be reached only if constants of motion (wave number, energy) are infinite. Hence any initial data with finite integrals can evolve only into the state with $T = 0$. The second law of thermodynamics leads to following plausible

Statement 1.

Let initial data for equation (4.1) be a stationary homogeneous field described by spectral density $n_0(k)$ and

$$\int n_0(k) \, dk = A_2^0. \tag{4.9}$$

Then in the Hilbert space L_2

$$n_0(k) \to A_2^0 \, \delta(k), \tag{4.10}$$

in other words, if you wait long enough time "almost all" particles from the initial distribution will be concentrated in the condensate.

From the mathematical point of view "almost all particles" mean that convergence (4.10) takes place only in L_2, not in higher Sobolev spaces. To find plausible estimates for behavior of higher norms A_2^l one should use a kinetic equation for n_k. It reads

$$\frac{\partial n_k}{\partial t} = 4\pi \int |T_{kk_1k_2k_3}|^2 \, \delta(\vec{k} + \vec{k_1} - \vec{k_2} - \vec{k_3}) \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \times (n_{k_1} n_{k_2} n_{k_3} + n_k n_{k_2} n_{k_3} - n_k n_{k_1} n_{k_2} - n_k n_{k_1} n_{k_3}) \, dk_1 \, dk_2 \, dk_3, \quad (4.11)$$

and is written for a more general Hamiltonian system, when

$$H_{int} = \frac{1}{2} \int T_{kk_1k_2k_3} \, \Psi_{k_1}^* \, \Psi_{k_2}^* \, \Psi_{k_2} \, \Psi_{k_3} \, \delta_{k+k_1-k_2-k_3} \, dk \, dk_1 \, dk_2 \, dk_3. \quad (4.12)$$

In our case $T \equiv 1$, and in the presence of condensate equation (4.11) is valid in the limit of high wave numbers

$$k^2 \gg A_0 \quad (4.13)$$

Equation (4.11) has the following constants of motion

$$N = \int n_k \, d\vec{k}, \quad (4.14)$$

$$\vec{p} = \int \vec{k} \, n_k \, dk, \quad (4.15)$$

$$E = \int \omega_k \, n_k \, dk, \quad (4.16)$$

which can be interpreted as densities of number of particles, momentum and energy. It has, for $T = 1, \omega_k = k^2$, a family of self-similar solutions

$$n = \frac{1}{t^{\frac{4a+1}{2}}} \, f\Big(\frac{\vec{k}}{t^a}\Big), \quad (4.17)$$

where a is an still unknown constant. To find a, one should use constants of motion (4.14-4.16). It is reasonable to assume that $n(\vec{k})$ is spherically symmetric, hence $\vec{p} = 0$. Almost all number of particles concentrate in condensate, hence conservation law (4.16) should be disregarded. Conservation of energy reads

$$E = \int k^2 \, n_k \, d\vec{k} = const, \quad (4.18)$$

from which we obtain $a = 1/6$, and solution (4.17) takes a form

$$n = \frac{1}{t^{5/6}} f\Big(\frac{\vec{k}}{t^{1/6}}\Big). \tag{4.19}$$

For higher Sobolev norms one can obtain

$$A_2^l = \frac{1}{t^{\frac{4a+1}{2}}} \int k^{2l} f\Big(\frac{k}{t^a}\Big)\, dk \simeq t^{(2l+1)a-1/2}, \tag{4.20}$$

assuming $a = 1/6$ one obtains

$$A_2^l \simeq t^{1/3(l-1)}. \tag{4.21}$$

Formula (4.20) is the central result of this consideration. According to our assumption the first Sobolev's norm A_2^1 is constant and all higher norms grow. L_2-norm, which is a number of particles outside the condensate decreases in time, $L_2 \simeq t^{-1/3}$. The whole picture in nontrivial and even tricky. Decreasing the number of particles outside the condensate provides increasing roughness of the solution.

One should note that infinity of the domain and statistical homogeneity of $\Psi_0(x)$ are very essential. If equation (4.1) would be put in a finite domain, for instance in the box $0 < x_i < 2\pi$ with zero or periodic boundary conditions, a situation would be completely different. The equation (3.1) in this case can be treated as a discrete system of infinite number of oscillators, and kinetic equation (3.11) cannot be applicable. One can expect that in finite domain the rate of relaxation to thermodynamic equilibrium and growing of roughness will be in a discrete case much more slow than in continuous.

The branch of theoretical physics studying statistical properties of nonlinear waves is known as "weak turbulence". It is actively developing field having important applications in physical oceanology, meteorology and astrophysics. On our opinion, methods of weak turbulence could be very helpful for solution of pure mathematical questions on nonlinear partial differential equations.

5 Briefly on collapses

Another subject equally interesting both from mathematical and physical point of view is formation of finite-time singularities in the nonlinear PDE.

A classical example of the system having collapsing solution is the focusing nonlinear Schrödinger equation in R^3

$$i\,\Psi_t + \Delta\,\Psi + |\Psi|^2\,\Psi = 0, \;\; \Psi|_{t=0} = \Psi_0(r). \tag{5.1}$$

The Cauchy problem for (5.1) explodes in a finite time of

$$H = \int \Big\{ |\Delta\,\Psi_0|^2 - \frac{1}{2}|\Psi_0|^4 \Big\} d\vec{r} < 0, \tag{5.2}$$

and this collapse leads to formation of integrable singularities $|\Psi|^2 \to c/r^2$. All Sobolev's norms in the moment of singularity become infinite.

Existence of singularities in (5.1) is a rigorous mathematical fact. Existence of finite-time singularities in another fundamental system, the Navier-Stokes equation,

$$\frac{\partial v}{\partial t} + v\,\nabla\,v + \nabla p = v\Delta v, \;\; div\,v = 0, \tag{5.3}$$

is an open question. Moreover, as much as one million dollars will presented to a lucky mathematician, who will manage to prove the existence or absence of finite-time singularity in (5.2). Note that the question about singularities is open only for Euler equation arising from (5.2) if $v = 0$. Today both Euler and Navier-Stokes systems are very hot business.

Here we will show that in the limit of "almost two-dimensional" hydrodynamics The Euler equations have solutions collapsing in finite time. Let us study a system of almost parallel vorticies, crossing the perpendicular plane in point

$$\omega = x + iy, \;\; \omega = \omega(\vec{s}),$$

where $\vec{s}$ is two-dimensional marker of the vortex line. Let z is a coordinate along the vortex, $\Gamma(\vec{s})$ is distribution of vorticity which can be treated as a measure of s-plane. If bending of a vortex line is small, one can describe the system of vortecies by the following Nonlinear Schrödinger equation [5],

$$-i\,\frac{\partial w}{\partial t} + \Gamma(s)\,\frac{\partial^2 w}{\partial z^2} + \int \frac{\Gamma(\vec{s'})\,d\vec{s'}}{\bar{w}(s) - \bar{w}(s')} = 0. \tag{5.4}$$

Equation (5.4) has following self-similar solution

$$w = (t_0 - t)^{1/2+i\epsilon}\,F\Big(\frac{z, \vec{s}}{(t_0 - t)^{1/2}}\Big), \tag{5.5}$$

where $F(\xi, \vec{s})$ satisfies the equation

$$-\Big(i\frac{1}{2}+\epsilon\Big)\,F+\frac{i}{2}\xi\,F_\xi+\Gamma(s)\,F_{\xi\xi}+\int\frac{\Gamma(s')}{\bar{F}(s)-\bar{F}(s')}\,ds'=0 \tag{5.6}$$

Here ϵ is an arbitrary real number and $\Gamma(s)$ is an arbitrary, not necessary positive measure. Let us take

$$\epsilon=0,\;\;\Gamma(s)=\Gamma\,\delta(s+1)-\Gamma\delta(s-1),F(1)=A+iB,\;\;F(-1)=A-iB,$$

with A, B satisfying the system of equations

$$\begin{aligned}\frac{1}{2}(-A+\xi\,A') &= -B^{"}+\frac{1}{2B}\\ \frac{1}{2}(-B+\xi\,B') &= A^{"}\end{aligned} \tag{5.7}$$

Asymptotically,

$$\begin{aligned}&A\to\alpha z,\;\;B\to+\beta z\;\;at\;z\to\infty\\ &A\to-\alpha z,\;\;B\to\beta z\;\;at\;x\to-\infty.\end{aligned} \tag{5.8}$$

This solution describes a collapse of two antiparallel vortex tubes. There is some hope that similar collapsing solution still exists in presence of a very small viscosity.

In conclusion the author express a deep gratitude to Tel-Aviv University and to organizers of the conference "Vision in Mathematics in 2000" for this extremely interesting and intellectually brilliant event.

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