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STATISTICAL THEORY OF GRAVITY AND CAPILLARY WAVES ON THE SURFACE OF A FINITE-DEPTH FLUID

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1. Introduction

In many physical situations, the oscillations of the free surface of a fluid is are a random process in space and time. This is equally correct for ripples in a tea cup as well as for large ocean waves. In both cases the situation must be described by the averaged equations imposed on a certain set of correlation functions. The derivation of such equations is not a simple problem even on a "physical" level of rigor. It is especially important to determine correctly the conditions of applicability for a given statistical description. For some physical reasons they might happen to be narrow. In this article we discuss the statistical description of potential surface waves on the surface of an ideal fluid of finite depth. We will show that this problem becomes nontrivial in the limit of long waves, i.e. in the case of "shallow water".

The most common tool for the statistical description of nonlinear waves is a kinetic equation for squared wave amplitudes. We will call it the "wave kinetic equation". Sometimes it is called "Boltzmann's equation". This is not exactly accurate. In fact, a wave kinetic equation and Boltzmann's equation are the opposite limiting cases of a more general kinetic equation for particles obeying Bose-Einstein statistics like photons in stellar atmospheres or phonons in liquid helium. It was derived by Peierls in 1929 and can be found now in any textbook on the physics of condensed matter. Both Boltzmann's equation and the wave kinetic equation can be simply derived from the quantum kinetic equation. In spite of this fact, the wave kinetic equation was derived independently and almost simultaneously by Patric, Petchek and others (see Kadomtsev, 1965) in plasma physics and by K. Hasselmann (1962) for surface waves on deep water. It was done in the early sixties. Recall that Boltzmann derived his equation in the last century. Some authors call this equation after Hasselmann. We will use a more general term – "kinetic wave equation".

The pioneers starting from Boltzmann did not care about rigorously justifying the kinetic equation and finding the exact limits of its applicability. This work was done later. Boltzmann's equation was derived in a systematic and self-consisted way by Bogoliubov in 1949. The quantum kinetic equation was studied systematically by the use of diagram technique in fifties.

The wave kinetic equation can be derived and justified in a similar way. It is a lengthy procedure, thus in this short article we will give the final results of the diagram procedure - the kinetic equation and the limits of its validity. We will see that in the case of shallow water the limits are very restrictive.

2. Hamiltonian formalism

We will study weakly-nonlinear waves on the surface of an ideal fluid in an infinite basin of constant depth h. The vertical coordinate is

$$-h < z < \eta(\vec{r}), \qquad \vec{r} = (x, y). \tag{2.1}$$

The fluid is incompressible,

$$\operatorname{div} V = 0 \tag{2.2}$$

and the velocity V is a potential field,

$$V = \nabla \Phi, \tag{2.3}$$

where the potential Φ satisfies the Laplace equation

$$\Delta \Phi = 0 \tag{2.4}$$

under the boundary conditions

$$\Phi|_{z=\eta} = \Psi(\vec{r}, t), \qquad \Phi_z|_{z=-h} = 0. \tag{2.5}$$

Let us assume that the total energy of the fluid, $H = T + U$, has the following expressions for kinetic and potential energies:

$$T = \frac{1}{2} \int dr \int_{-h}^{\eta} (\nabla \Phi)^2 dz, \tag{2.6}$$

$$U = \frac{1}{2} g \int \eta^2 \, dr + \sigma \int (\sqrt{1 + (\nabla \eta)^2} - 1) \, dr. \tag{2.7}$$

Here g is the acceleration of gravity, and σ is the surface tension coefficient.

The Dirichlet-Neumann boundary problem (2.4)–(2.5) is uniquely solvable, thus the flow is defined by fixing η and Ψ. This pair of variables is canonical, so the equation of motion for η and Ψ takes the form (Zakharov, 1968):

$$\frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \Psi}, \quad \frac{\partial \Psi}{\partial t} = -\frac{\delta H}{\delta \eta}. \tag{2.8}$$

Taking their Fourier transform yields

$$\frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \Psi(\vec{k})^*}, \quad \frac{\partial \Psi(\vec{k})}{\partial t} = -\frac{\delta H}{\delta \eta(\vec{k})^*}. \tag{2.9}$$

Here $\Psi(\vec{k})$ is the Fourier transform of $\Psi(\vec{r})$:

$$\Psi(\vec{k}) = \frac{1}{2\pi} \int \Psi(\vec{r}) e^{-i\vec{k}\cdot\vec{r}} dr. \tag{2.10}$$

The Hamiltonian H can be expanded in Taylor series in powers of η:

$$H = H_0 + H_1 + H_2 + \cdots \tag{2.11}$$

Omitting the procedure of calculating H_i we present the final expressions for the first three terms in this expansion:

$$H_0 = \frac{1}{2}\int \{A_k|\Psi_k|^2 + B_k|\eta_k|^2\}dk, \qquad A_k = k\,\tanh(kh),\;\; B_k = g + \sigma\, k^2 \tag{2.12}$$

$$H_1 = \frac{1}{2(2\pi)}\int L^{(1)}(\vec{k}_1,\vec{k}_2)\,\Psi_{k_1}\,\Psi_{k_2}\,\eta_{k_3}\,\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3)dk_1\,dk_2\,dk_3, \tag{2.13}$$

$$\begin{aligned} H_2 &= \frac{1}{2(2\pi)^2}\int L^{(2)}(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)\,\Psi_{k_1}\,\Psi_{k_2}\eta_{k_3}\,\eta_{k_4}\,\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)\,dk_1\,dk_2\,dk_3\,dk_4 \\ &\quad -\frac{\sigma^2}{8(2\pi)^2}\int(\vec{k}_1\vec{k}_2)(\vec{k}_3\vec{k}_4)\,\eta_{k_1}\,\eta_{k_2}\,\eta_{k_3}\,\eta_{k_4}\,\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)\,dk_1\,dk_2\,dk_3\,dk_4\,. \end{aligned} \tag{2.14}$$

The formulas for $L^{(1)}$ and $L^{(2)}$ were found in 1970 by Zakharov and Kharitonov (see also Craig and Sulem 1992, Zakharov 1998). Here are their expressions:

$$L^{(1)}(\vec{k}_1,\vec{k}_2) = -(\vec{k}_1\vec{k}_2) - |k_1||k_2|\,\tanh k_1 h\;\tanh k_2 h, \tag{2.15}$$

and

$$\begin{aligned} L^{(2)}(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4) &= \frac{1}{4}|k_1||k_2|\,\tanh k_1 h\;\tanh k_2 h \\ &\times \Big\{-\frac{2|k_1|}{\tanh k_1 h} - \frac{2|k_2|}{\tanh k_2 h} + |\vec{k}_1+\vec{k}_3|\tanh|\vec{k}_1+\vec{k}_3|h \\ &+ |\vec{k}_2+\vec{k}_3|\,\tanh|\vec{k}_2+\vec{k}_3|h + |\vec{k}_1+\vec{k}_4|\tanh|\vec{k}_1+\vec{k}_4|h + |\vec{k}_2+\vec{k}_4|\,\tanh|\vec{k}_2+\vec{k}_4|h\Big\} \\ &= \frac{1}{4}A_1A_2\left\{-\frac{2k_1^2}{A_1} - \frac{2k_2^2}{A_2} + A_{1+3} + A_{2+3} + A_{1+4} + A_{2+4}\right\} \end{aligned} \tag{2.16}$$

One can introduce the normal variables a_k, a_k^*. They can be expressed as follows:

$$\begin{aligned} \eta_k &= \frac{1}{\sqrt{2}}\left(\frac{A_k}{B_k}\right)^{1/4}(a_k + a_{-k}^*) \\ \Psi_k &= \frac{i}{\sqrt{2}}\left(\frac{B_k}{A_k}\right)^{1/4}(a_k - a_{-k}^*) \\ a_k &= \frac{1}{\sqrt{2}}\left\{\left(\frac{B_k}{A_k}\right)^{1/4}\eta_k - i\left(\frac{A_k}{B_k}\right)^{1/4}\Psi_k\right\} \end{aligned} \tag{2.17}$$

The transformation $\Psi_k\,,\eta_k \to a_k$ is canonical. One can check that

$$\frac{\partial a_k}{\partial t} + i\frac{\delta H}{\delta a_k^*} = 0, \tag{2.18}$$

where the Hamiltonian H can be represented as the sum of two terms

$$H = H_0 + H_{int}. \tag{2.19}$$

For the first term we have

$$H_0 = \int \omega_k\, a_k\, a_k^*\, dk, \tag{2.20}$$

where $\omega_k > 0$ is defined by the formula

$$\omega_k = \sqrt{A_k B_k} = \sqrt{k\,\tanh(kh)\,(g + \sigma\,k^2)}. \tag{2.21}$$

The second term, H_{int}, is represented by the infinite series

$$H_{int} = \frac{1}{n!m!} \sum_{n+m\geq 3} \int V^{(n,m)}(\vec{k}_1, \dots, \vec{k}_n, \vec{k}_{n+1}, \dots, \vec{k}_{n+m}) a^*_{k_1} \cdots a^*_{k_n} a_{k_{n+1}} \cdots a_{k_{n+m}}$$

$$\times \delta(\vec{k}_1 + \dots + \vec{k}_n - \vec{k}_{n+1} - \dots - \vec{k}_{n+m})\, dk_1 \dots dk_{n+m} \tag{2.22}$$

In the case under consideration we have

$$V^{(n,m)}(P,Q) = V^{(m,n)}(Q,P), \tag{2.23}$$

where $P = (\vec{k}_1, \dots, \vec{k}_n)$ and $Q = (\vec{k}_{n+1}, \dots, \vec{k}_{n+m})$ are multi-indices.

For more general Hamiltonian systems (in the presence of wind, for instance), the coefficients $V^{(n,m)}(P,Q)$ are complex, and

$$V^{(n,m)}(P,Q) = V^{*(m,n)}(Q,P). \tag{2.24}$$

The condition (2.24) guarantees that the Hamiltonian H_{int} is real.

For surface waves the coefficients can be written as

$$\begin{aligned} V^{(1,2)}(\vec{k}, \vec{k}_1, \vec{k}_2) &= \frac{1}{4\pi\sqrt{2}} \left\{ \left(\frac{A_k\, B_{k_1}\, B_{k_2}}{B_k\, A_{k_1}\, A_{k_2}} \right)^{1/4} L^{(1)}(\vec{k}_1, \vec{k}_2) - \right. \\ &\quad \left. - \left(\frac{B_k\, A_{k_1}\, B_{k_2}}{A_k\, B_{k_1}\, A_{k_2}} \right)^{1/4} L^{(1)}(-\vec{k}, \vec{k}_1) - \left(\frac{B_k\, B_{k_1}\, A_{k_2}}{A_k\, A_{k_1}\, B_{k_2}} \right)^{1/4} L^{(1)}(-\vec{k}, \vec{k}_2) \right\} \end{aligned} \tag{2.25}$$

$$\begin{aligned} V^{(0,3)}(\vec{k}, \vec{k}_1, \vec{k}_2) &= \frac{1}{4\pi\sqrt{2}} \left\{ \left(\frac{A_k\, B_{k_1}\, B_{k_2}}{B_k\, A_{k_1}\, A_{k_2}} \right)^{1/4} L^{(1)}(\vec{k}_1, \vec{k}_2) + \right. \\ &\quad \left. + \left(\frac{B_k\, A_{k_1}\, B_{k_2}}{A_k\, B_{k_1}\, A_{k_2}} \right)^{1/4} L^{(1)}(\vec{k}, \vec{k}_1) + \left(\frac{B_k\, B_{k_1}\, A_{k_2}}{A_k\, A_{k_1}\, B_{k_2}} \right)^{1/4} L^{(1)}(\vec{k}, \vec{k}_2) \right\} \end{aligned} \tag{2.26}$$

In this paper we will use only one coefficient of fourth order $V^{(2,2)}(P,Q)$. After a simple calculation we can obtain the following expression for this coefficient:

$$\begin{aligned} V^{(2,2)}(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) &= -\frac{1}{2\pi^2} \Big\{ \tilde{L}^{(2)}(-\vec{k}_1, -\vec{k}_2, \vec{k}_3, \vec{k}_4) + \tilde{L}^{(2)}(\vec{k}_3, \vec{k}_4, -\vec{k}_1, -\vec{k}_2) - \tilde{L}^{(2)}(-\vec{k}_1, \vec{k}_3, -\vec{k}_2, \vec{k}_4) \\ &\quad - \tilde{L}^{(2)}(-\vec{k}_1, \vec{k}_4, -\vec{k}_2, \vec{k}_3) - \tilde{L}^{(2)}(-\vec{k}_2, \vec{k}_3, -\vec{k}_1, \vec{k}_4) - \tilde{L}^{(2)}(-\vec{k}_2, \vec{k}_4, -\vec{k}_1, \vec{k}_3) \Big\} \\ &\quad - \frac{\sigma^2}{16\pi^2} \Big\{ (\vec{k}_1, \vec{k}_2)(\vec{k}_3, \vec{k}_4) + (\vec{k}_1, \vec{k}_3)(\vec{k}_2, \vec{k}_4) + (\vec{k}_1, \vec{k}_4)(\vec{k}_2, \vec{k}_3) \Big\} \left(\frac{A_{k_1}\, A_{k_2}\, A_{k_3}\, A_{k_4}}{B_{k_1}\, B_{k_2}\, B_{k_3}\, B_{k_4}} \right)^{1/4}, \end{aligned} \tag{2.27}$$

where

$$\tilde{L}^{(2)}(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) = \frac{1}{4} \left(\frac{B_{k_1}\, B_{k_2}\, A_{k_3}\, A_{k_4}}{A_{k_1}\, A_{k_2}\, B_{k_3}\, B_{k_4}} \right)^{1/4} L^{(2)}(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4). \tag{2.28}$$

We will not discuss the five-wave processes systematically. This makes it possible to use the following approximation for the Hamiltonian:

$$\begin{aligned} H &= \int \omega_k |a_k|^2 dk + \frac{1}{2}\int V^{(1,2)}(\vec{k},\vec{k}_1,\vec{k}_2)(a_k\, a^*_{k_1}\, a^*_{k_2} + a^*_k\, a_{k_1}\, a_{k_2})\,\delta(\vec{k}-\vec{k}_1-\vec{k}_2)\,dk\,dk_1\,dk_2 + \\ &+\frac{1}{6}\int V^{(0,3)}(\vec{k},\vec{k}_1,\vec{k}_2)(a_k\, a_{k_1}\, a_{k_2} + a^*_k\, a^*_{k_1}\, a^*_{k_2})\,\delta(\vec{k}+\vec{k}_1+\vec{k}_2)\,dk\,dk_1\,dk_2 + \\ &+\frac{1}{4}\int V^{(2,2)}(\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3)\,a^*_k\, a^*_{k_1}\, a_{k_2}\, a_{k_3}\,\delta(\vec{k}+\vec{k}_1-\vec{k}_2-\vec{k}_3)\,dk\,dk_1\,dk_2\,dk_3 \end{aligned} \tag{2.29}$$

3. Canonical Transformation

In this chapter we will study only gravity waves and put $\sigma = 0$, so that

$$\omega_k = \sqrt{gk\,\tanh(kh)}\,. \tag{3.1}$$

The dispersion relation (3.1) is of the "non-decay type" and the equations

$$\omega_k = \omega_{k_1} + \omega_{k_2} \qquad \vec{k} = \vec{k}_1 + \vec{k}_2 \tag{3.2}$$

have no real solution. This means that in the limit of small nonlinearity, the cubic terms in the Hamiltonian (2.11) can be excluded by a proper canonical transformation. The transformation

$$a(k,t) \to b(k,t) \tag{3.3}$$

must transform equation (2.18) into the same equation:

$$\frac{\partial b_k}{\partial t} + i\frac{\delta H}{\delta b^*_k} = 0. \tag{3.4}$$

This requirement imposes the following conditions on Poisson's brackets between a_k and b_k:

$$\{a_k, a_{k'}\} = \int \left\{ \frac{\delta a_k}{\delta b_{k''}}\frac{\delta a_{k'}}{\delta b^*_{k''}} - \frac{\delta a_k}{\delta b^*_{k''}}\frac{\delta a_{k'}}{\delta b_{k''}} \right\} dk'' = 0 \tag{3.5}$$

$$\{a_k, a^*_{k'}\} = \int \left\{ \frac{\delta a_k}{\delta b_{k''}}\frac{\delta a^*_{k'}}{\delta b^*_{k''}} - \frac{\delta a_k}{\delta b^*_{k''}}\frac{\delta a^*_{k'}}{\delta b_{k''}} \right\} dk'' = \delta(k-k') \tag{3.6}$$

$$\{b_k, b_{k'}\} = \int \left\{ \frac{\delta b_k}{\delta a_{k''}}\frac{\delta b_{k''}}{\delta a^*_{k'}} - \frac{\delta b_k}{\delta a^*_{k''}}\frac{\delta b_{k''}}{\delta a_{k'}} \right\} dk'' = 0 \tag{3.7}$$

$$\{b_k, b^*_{k'}\} = \int \left\{ \frac{\delta b_k}{\delta a_{k''}}\frac{\delta b^*_{k''}}{\delta a^*_{k'}} - \frac{\delta b_k}{\delta a^*_{k''}}\frac{\delta b^*_{k''}}{\delta a_{k'}} \right\} dk'' = \delta(k-k') \tag{3.8}$$

The canonical transformation excluding cubic terms is given by the infinite series:

$$a_k = a^{(0)}_k + a^{(1)}_k + a^{(2)}_k + \cdots \tag{3.9}$$

where

$$\begin{aligned}
a_k^{(0)} &= b_k \\
a_k^{(1)} &= \int \Gamma^{(1)}(\vec{k},\vec{k}_1,\vec{k}_2)\, b_{k_1}\, b_{k_2}\, \delta(\vec{k}-\vec{k}_1-\vec{k}_2)\, dk_1\, dk_2 - 2\int \Gamma^{(1)}(\vec{k}_2,\vec{k},\vec{k}_1)\, b^*_{k_1}\, b_{k_2}\, \delta(\vec{k}+\vec{k}_1-\vec{k}_2)\, dk_1\, dk_2 \\
&+ \int \Gamma^{(2)}(\vec{k},\vec{k}_1,\vec{k}_2)\, b^*_{k_1}\, b^*_{k_2}\, \delta(\vec{k}+\vec{k}_1+\vec{k}_2)\, dk_1\, dk_2 \\
a_k^{(2)} &= \int B(\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3)\, b^*_{k_1}\, b_{k_2}\, b_{k_3}\, \delta(\vec{k}+\vec{k}_1-\vec{k}_2-\vec{k}_3)\, dk_1\, dk_2\, dk_3 + \cdots
\end{aligned} \tag{3.10}$$

Plugging (3.9) into (3.5)–(3.8), we obtain infinite series in powers of b, b^*, which must identically cancel out at all orders except zero.

Let us assume that

$$\Gamma^{(2)}(\vec{k},\vec{k}_1,\vec{k}_2) = \Gamma^{(2)}(\vec{k}_1,\vec{k},\vec{k}_2) = \Gamma^{(2)}(\vec{k}_2,\vec{k},\vec{k}_1). \tag{3.11}$$

This condition guarantees that (3.11), (3.5)–(3.8) are satisfied at first order in b, b^*. Substituting (3.9) into H we observe that the cubic terms cancel out:

$$\Gamma^{(1)}(\vec{k},\vec{k}_1,\vec{k}_2) = -\frac{1}{2}\frac{V^{(1,2)}(\vec{k},\vec{k}_1,\vec{k}_2)}{(\omega_k-\omega_{k_1}-\omega_{k_2})} \tag{3.12}$$

$$\Gamma^{(2)}(\vec{k},\vec{k}_1,\vec{k}_2) = -\frac{1}{2}\frac{V^{(0,3)}(\vec{k},\vec{k}_1,\vec{k}_2)}{(\omega_k+\omega_{k_1}+\omega_{k_2})} \tag{3.13}$$

A simple method for the recurrent calculation of $B(\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3)$ and higher terms in the expansion (3.9) was found by the author in the article (Zakharov, 1998). By the use of this method one can find

$$\begin{aligned}
B(\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3) &= \Gamma^{(1)}(\vec{k}_1,\vec{k}_2,\vec{k}_1-\vec{k}_2)\,\Gamma^{(1)}(\vec{k}_3,\vec{k},\vec{k}_3-\vec{k}) + \Gamma^{(1)}(\vec{k}_1,\vec{k}_3,\vec{k}_1-\vec{k}_3)\,\Gamma^{(1)}(\vec{k}_2,\vec{k},\vec{k}_2-\vec{k}) \\
&-\Gamma^{(1)}(\vec{k},\vec{k}_2,\vec{k}-\vec{k}_2)\,\Gamma^{(1)}(\vec{k}_3,\vec{k}_1,\vec{k}_3-\vec{k}_1) - \Gamma^{(1)}(\vec{k}_1,\vec{k}_3,\vec{k}_1-\vec{k}_3)\,\Gamma^{(1)}(\vec{k}_2,\vec{k}_1,\vec{k}_2-\vec{k}_1) \\
&-\Gamma^{(1)}(\vec{k}+\vec{k}_1,\vec{k},\vec{k}_1)\,\Gamma^{(1)}(\vec{k}_2+\vec{k}_3,\vec{k}_2,\vec{k}_3) + \Gamma^{(2)}(-\vec{k}-\vec{k}_1,\vec{k},\vec{k}_1)\,\Gamma^{(2)}(-\vec{k}_2-\vec{k}_3,\vec{k}_2,\vec{k}_3)
\end{aligned} \tag{3.14}$$

The series (3.10) should be at least asymptotic. Hence we require

$$|a_k^{(1)}| \ll |b_k| \tag{3.15}$$

Let us consider the limit of shallow water as $kh \to 0$. We will assume also that the wave packet is narrow in angle: $k_y \ll k_x$. In this limit

$$\omega_k \to s|k|\left(1-\frac{1}{3}(kh)^2+\cdots\right), \qquad s=\sqrt{gh}, \tag{3.16}$$

and

$$L^{(1)}(\vec{k}_1,\vec{k}_2) \simeq -k_1k_2\,, \quad A_k \simeq h|k|^2\,, \quad B_k \simeq g\,, \quad V^{(1,2)}(\vec{k},\vec{k}_1,\vec{k}_2) \simeq \frac{3}{4\pi\sqrt{2}}(kk_1k_2)^{1/2}\left(\frac{g}{h}\right)^{1/4}.$$

Denoting $k_y = q$, $k_x = p$ and $|p| \gg |q|$, one obtains:

$$\omega(p,q) \simeq s\left(p+\frac{1}{2}\frac{q^2}{p}-\frac{1}{3}h^2p^3\right), \quad V^{(1,2)} \simeq \frac{3}{4\pi\sqrt{2}}\left(\frac{g}{h}\right)^{1/4}(pp_1p_2)^{1/2}. \tag{3.17}$$

We will study two opposite cases - wave packets narrow in angle and broad in angle. In both cases we will look only for the leading order terms in $1/kh$. For a packet which is very narrow in angle:

$$\begin{aligned} a^{(0)}(p,q) &\simeq b(p)\,\delta(q), \quad a^{(1)}(p,q) = b^{(1)}(p)\,\delta(q) \\ b^{(1)}(p) &\simeq \frac{3}{8\pi\sqrt{2}}\left(\frac{g}{h}\right)^{1/4}\frac{1}{sh^2}\left\{\int_0^p \frac{b(p_1)b(p-p_1)}{pp_1(p-p_1))^{1/2}}\,dp_1 + 2\int_0^\infty \frac{b^*(p_1)b(p+p_1)}{pp_1(p+p_1))^{1/2}}\,dp_1\right\} + \cdots \end{aligned} \tag{3.18}$$

The condition (3.15) now reads now

$$\frac{|b|}{p^{1/2}}\left(\frac{g}{h}\right)^{1/4}\frac{1}{sh^2} \ll 1 \tag{3.19}$$

Let a be a characteristic elevation of the free surface, $\mu = (ka)^2$, $\delta = kh$. The condition (3.19) is equivalent to

$$\mu \ll \delta^6\,, \quad \text{or} \quad N = \frac{\mu^{1/2}}{\delta^3} \ll 1\,. \tag{3.20}$$

N is known as "Stokes number".

For wave packets which are broad in angle the condition (3.15) is less restrictive. In this case the denominator of $\Gamma^{(1)}(\vec{k},\vec{k}_1,\vec{k}_2)$ is small if all three vectors $\vec{k},\vec{k}_1,\vec{k}_2$ are parallel. Let us put $\vec{k} = (p,q)$, $\vec{k}_1 = (p_1,q_1)$, $\vec{k}_2 = (p_2,-q_2)$. Then $\Gamma^{(1)}(p,p_1,p_2,q)$ has a sharp maximum at $q=0$. Performing integration over q yields

$$\begin{aligned} b^{(1)}(p,0) = \frac{3}{8\sqrt{2}}\left(\frac{g}{h}\right)^{1/4}\frac{1}{shp^{1/2}}&\left\{\int_0^p p_1^{1/2}(p-p_1)^{1/2}\,b(p_1,0)\,b(p-p_1,0)dp_1\right. \\ &\left. + 2\int_0^\infty p_1^{1/2}\,(p+p_1)^{1/2}\,b^*(p_1,0)\,b(p+p_1,0)dp_1\right\} + \cdots \end{aligned} \tag{3.21}$$

The condition

$$|b^{(1)}\,(p,0)| \ll |b^{(0)}\,(p,0)| \tag{3.22}$$

now reads

$$\mu \ll \delta^4. \tag{3.23}$$

4. **Effective Hamiltonian**

After performing the canonical transformation the cubic terms in the Hamiltonian cancel out. In new variables b_k we have

$$H = H_0 + H_2 + H_3 + \cdots\,, \tag{4.1}$$

$$H_0 = \int \omega_k\,|b_k|^2\,dk\,, \tag{4.2}$$

$$H_2 = \frac{1}{4}\int T(\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3)\,b_k^*\,b_{k_1}^*\,b_{k_2}\,b_{k_3}\,\delta(\vec{k}+\vec{k}_1-\vec{k}_2-\vec{k}_3)\,dk\,dk_1\,dk_2\,dk_3\,, \tag{4.3}$$

$$H_3 = \ldots$$

where

$$
\begin{aligned}
T(\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3) &= \frac{1}{4}\Big(\tilde{T}(\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3)+\tilde{T}(\vec{k}_1,\vec{k},\vec{k}_2,\vec{k}_3)+\tilde{T}(\vec{k}_2,\vec{k}_3,\vec{k},\vec{k}_1)+\tilde{T}(\vec{k}_3,\vec{k}_2,\vec{k},\vec{k}_1)\Big)\\
\tilde{T}(\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3) &= V^{(2,2)}(\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3)+R^{(1)}(\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3)+R^{(2)}(\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3)
\end{aligned}
\tag{4.4}
$$

and

$$
R^{(1)}(\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3) = -\frac{V^{(0,3)}(-\vec{k}-\vec{k}_1,\vec{k},\vec{k}_1)\,V^{(0,3)}(-\vec{k}_2-\vec{k}_3,\vec{k}_2,\vec{k}_3)}{\omega(-k-k_1)+\omega(k)+\omega(k_2)}, \tag{4.5}
$$

$$
\begin{aligned}
R^{(2)}(\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3) = &-\frac{V^{(1,2)}(\vec{k}+\vec{k}_1,\vec{k},\vec{k}_1)\,V^{(1,2)}(\vec{k}_2+\vec{k}_3,\vec{k}_2,\vec{k}_3)}{\omega_{k+k_1}-\omega_k-\omega_{k_1}} -\\
&-\frac{V^{(1,2)}(\vec{k},\vec{k}_2,\vec{k}-\vec{k}_2)\,V^{(1,2)}(\vec{k}_3,\vec{k}_3-\vec{k}_1,\vec{k}_1)}{\omega_{k-k_2}-\omega_k+\omega_{k_2}}-\frac{V^{(1,2)}(\vec{k},\vec{k}_3,\vec{k}-\vec{k}_3)\,V^{(1,2)}(\vec{k}_2,\vec{k}_2-\vec{k}_1,\vec{k}_1)}{\omega_{k-k_3}-\omega_k+\omega_{k_3}} -\\
&-\frac{V^{(1,2)}(\vec{k}_2,\vec{k},\vec{k}_2-\vec{k})\,V^{(1,2)}(\vec{k}_1,\vec{k}_1-\vec{k}_3,\vec{k}_3)}{\omega_{k_2-k}+\omega_k-\omega_{k_2}}-\frac{V^{(1,2)}(\vec{k}_3,\vec{k},\vec{k}_3-\vec{k})\,V^{(1,2)}(\vec{k}_2,\vec{k}_2-\vec{k}_1,\vec{k}_1)}{\omega_{k_3-k}+\omega_k-\omega_{k_3}}
\end{aligned}
\tag{4.6}
$$

In the presence of capillarity, the expression (4.6) makes sense everywhere except in the vicinity of the zeros of the denominators. The width of these vicinities depends on the level of nonlinearity.

The equation of motion (3.4) in new variables takes the form

$$
\frac{\partial b_k}{\partial t}+i\,\omega_k\,b_k = -\frac{i}{2}\int T(\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3)\,b^*_{k_1}\,b_{k_2}\,b_{k_3}\,\delta(\vec{k}+\vec{k_1}-\vec{k_2}-\vec{k_3})\,dk_1\,dk_2\,dk_3 \tag{4.7}
$$

The term $T(\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3)$ is defined on the resonance manifold

$$
\omega_k+\omega_{k_1}=\omega_{k_2}+\omega_{k_3},\quad \vec{k}+\vec{k_1}=\vec{k_2}+\vec{k_3}\,. \tag{4.8}
$$

Further we will omit the wave numbers k and keep only their labels. After a series of transformations the four-wave interaction coefficient T can be simplified into the form

$$
\begin{aligned}
T_{1234} &= \frac{1}{2}(\tilde{T}_{1234}+\tilde{T}_{2134}),\\
\tilde{T}_{1234} &= -\frac{1}{16\pi^2}\left(\frac{A_1A_2A_3A_4}{B_1B_2B_3B_4}\right)^{1/4}\\
&\times\left[k_1^2\,B_1+k_2^2\,B_2+k_3^2\,B_3+k_4^2\,B_4-(\omega_1-\omega_3)^2\,A_{1-3}-(\omega_1-\omega_4)^2\,A_{1-4}-(\omega_1+\omega_2)^2\,A_{1+2}\right]\\
&-\frac{1}{16\pi^2}\left(\frac{B_1B_2B_3B_4}{A_1A_2A_3A_4}\right)^{1/4}\times\left\{\frac{1}{B_{1+2}}\left[L_{1,2}\,L_{3,4}+\frac{u_{1,2}\,u_{3,4}}{\omega_{1+2}^2-(\omega_1+\omega_2)^2}\right]\right.\\
&\left.+\frac{1}{B_{1-3}}\left[L_{-1,3}\,L_{-2,4}+\frac{u_{-1,3}\,u_{-2,4}}{\omega_{1-3}^2-(\omega_1-\omega_3)^2}\right]+\frac{1}{B_{1-4}}\left[L_{-1,4}\,L_{-2,3}+\frac{u_{-1,4}\,u_{-2,3}}{\omega_{1-4}^2-(\omega_1-\omega_4)^2}\right]\right\}
\end{aligned}
\tag{4.9}
$$

Here

$$
A_k=k\tanh kh\,,\quad B_k=g+\sigma k^2\,,\quad L_{1,2}=-(\vec{k_1}\cdot\vec{k_2})-A_1A_2\,,\quad \omega_k=\sqrt{A_kB_k}. \tag{4.10}
$$

The expression for $u_{1,2}$ is

$$\begin{aligned}
u_{1,2} &= (\vec{k}_1\cdot\vec{k}_2)\left[\omega_1\Big(1+\frac{B_{1+2}}{B_1}\Big)+\omega_2\Big(1+\frac{B_{1+2}}{B_2}\Big)\right] \\
&+\frac{B_{1+2}}{B_2}\,\omega_2\,k_1^2+\frac{B_{1+2}}{B_1}\,\omega_1\,k_2^2+\Big(\frac{A_1A_2}{B_1B_2}\Big)^{1/2}(\omega_1\,\omega_2-\omega_{1+2}^2)(\omega_1+\omega_2) \\
u_{-1,3} &= -(\vec{k}_1\cdot\vec{k}_3)\left[\omega_1\Big(1+\frac{B_{1-3}}{B_1}\Big)-\omega_3\Big(1+\frac{B_{1-3}}{B_3}\Big)\right] \\
&-\frac{B_{1-3}}{B_3}\,\omega_3^2\,k_1^2+\frac{B_{1-3}}{B_1}\,\omega_1k_3^2+(\omega_1-\omega_3)(\omega_1\omega_3+\omega_{1-3}^2)\Big(\frac{A_1A_3}{B_1B_3}\Big)^{1/2}
\end{aligned} \tag{4.11}$$

The above expression is the most general form of four-wave interaction coefficient and is applicable for gravity as well as for capillary waves on an arbitrary depth. It can be simplified in different limiting cases.

In the absence of capillarity $\sigma=0$, $B_k=g$ and

$$\begin{aligned}
u_{1,2} &= (\omega_1+\omega_2)\left\{2(\vec{k}_1\cdot\vec{k}_2)+\frac{1}{g}\omega_1\omega_2(\omega_1^2+\omega_2^2-\omega_{1+2}^2)\right\} \\
u_{-1,3} &= (\omega_1-\omega_3)\left\{-2(\vec{k}_1\cdot\vec{k}_3)+\frac{1}{g}\omega_1\omega_3(\omega_{1-3}^2-\omega_1^2+\omega_3^2)\right\}
\end{aligned} \tag{4.12}$$

5. Deep water limit

The coefficient of four-wave interaction for pure gravity waves on deep water was calculated by many authors since Hasselmann (1962). We present here a relatively compact expression for this coefficient.

$$\begin{aligned}
T_{1234} &= -\frac{1}{16\pi^2}\frac{1}{(k_1k_2k_3k_4)^{1/4}}\Bigg\{-12k_1k_2k_3k_4- \\
&-2(\omega_1+\omega_2)^2\Big[\omega_3\omega_4\Big((\vec{k}_1\cdot\vec{k}_2)-k_1k_2\Big)+\omega_1\omega_2\Big((\vec{k}_3\cdot\vec{k}_4)-k_3k_4\Big)\Big]\frac{1}{g^2} \\
&-2(\omega_1-\omega_3)^2\Big[\omega_2\omega_4\Big((\vec{k}_1\cdot\vec{k}_3)+k_1k_3\Big)+\omega_1\omega_3\Big((\vec{k}_2\cdot\vec{k}_4)+k_2k_4\Big)\Big]\frac{1}{g^2} \\
&-2(\omega_1-\omega_4)^2\Big[\omega_2\omega_3\Big((\vec{k}_1\cdot\vec{k}_4)+k_1k_4\Big)+\omega_1\omega_4\Big((\vec{k}_2\cdot\vec{k}_3)+k_2k_3\Big)\Big]\frac{1}{g^2} \\
&+[(\vec{k}_1\cdot\vec{k}_2)+k_1k_2][(\vec{k}_3\cdot\vec{k}_4)+k_3k_4]+[-(\vec{k}_1\cdot\vec{k}_3)+k_1k_3][-(\vec{k}_2\cdot\vec{k}_4)+k_2k_4] \\
&+[-(\vec{k}_1\cdot\vec{k}_4)+k_1k_4][-(\vec{k}_2\cdot\vec{k}_3)+k_2k_3] \\
&+4(\omega_1+\omega_2)^2\frac{[(\vec{k}_1\cdot\vec{k}_2)-k_1k_2][-(\vec{k}_2\cdot\vec{k}_4)+k_2k_4]}{\omega_{1+2}^2-(\omega_1+\omega_2)^2}+4(\omega_1-\omega_3)^2\frac{[(\vec{k}_1\cdot\vec{k}_3)+k_1k_3][(\vec{k}_2\cdot\vec{k}_4)+k_2k_4]}{\omega_{1-3}^2-(\omega_1-\omega_3)^2} \\
&+4(\omega_1-\omega_4)^2\frac{[(\vec{k}_1\cdot\vec{k}_4)+k_1k_4][(\vec{k}_2\cdot\vec{k}_3)+k_2k_3]}{\omega_{1-4}^2-(\omega_1-\omega_4)^2}\Bigg\}
\end{aligned} \tag{5.1}$$

Here $\omega_i=\sqrt{g\,|k_i|}$.

In spite of its complexity the expression (5.1) has an inner symmetry and beauty. It was mentioned that in the one dimensional case the coefficient T_{1234} cancels out (Dyachenko and Zakharov, 1994). This result was obtained earlier by computer. We will obtain it below "by hand". Another compact expression for T_{1234} was found by Webb (1978). Both expressions coincide on the resonant surface (5.2), but a proof of cancellation of T_{1234} in a one dimensional geometry is more difficult with the Webb formula.

In the one-dimensional case the resonant conditions

$$\begin{aligned} \omega_2+\omega_2 &= \omega_3+\omega_3 \\ k_1+k_2 &= k_3+k_4 \end{aligned} \tag{5.2}$$

have trivial solutions $k_3=k_1, k_4=k_2, k_3=k_2, k_4=k_1$ describing wave scattering without momentum exchange, and nontrivial solutions providing the momentum exchange. For these solutions the sign of one of the wave vectors is opposite to others. For instance, we can put

$$k_1>0,\ \ k_2<0,\ \ k_3>0,\ \ k_4>0.$$

In the one-dimensional case most of the terms in (5.1) cancel out, and the expression is simplified down to the form

$$T_{1234}=-\frac{1}{8\pi^2}\,\omega_1(\omega_1\omega_2\omega_3\omega_4)^{1/2}\Big\{-3\omega_2\omega_3\omega_4+\omega_2(\omega_1+\omega_2)^2-\omega_3(\omega_1-\omega_3)^2-\omega_4(\omega_1-\omega_4)^2\Big\} \tag{5.3}$$

The resonant conditions (5.2) can be solved by the parametrization

$$\begin{aligned} \omega_1=A(1+\xi+\xi^2),\ \omega_2=A\xi,\ \omega_3=A(1+\xi),\ \omega_4=A\xi(1+\xi) \\ k_1=A^2(1+\xi+\xi^2)^2,\ k_2=-A^2\xi^2,\ k_3=A^2(1+\xi)^2,\ k_4=A^2\xi^2(1+\xi)^2 \end{aligned} \tag{5.4}$$

By plugging the parametrization (5.4) into (5.3) we get

$$T_{1234}=-\frac{1}{4\pi^2}\omega_1(\omega_1\omega_2\omega_3\omega_4)^2\,A^3\xi(1+\xi)\left(-3\xi(1+\xi)+(1+\xi)^3-1-\xi^3\right)\equiv 0 \tag{5.5}$$

6. Shallow water limit

The shallow water limit takes place if $kh\to 0$. In this limit

$$\begin{aligned} A_k\to k^2h\quad \omega_k\to sk\,,\quad s^2=gh\,,\quad L_{12}\to-(\vec{k}_1\cdot\vec{k}_2)\,,\quad u_{1,2}\to s(k_1+k_2)(\vec{k}_1\cdot\vec{k}_2)\,, \\ u_{-1,3}\to -s(k_1-k_3)(\vec{k}_1\cdot\vec{k}_3). \end{aligned} \tag{6.1}$$

The coefficient (4.9) can be simplified into the form

$$\begin{aligned} T_{1234} = &-\frac{1}{16\pi^2 h}\frac{1}{(k_1k_2k_3k_4)^{1/2}}\Bigg\{(\vec{k}_1\cdot\vec{k}_2)(\vec{k}_3\cdot\vec{k}_4)+(\vec{k}_1\cdot\vec{k}_3)(\vec{k}_2\cdot\vec{k}_4)+(\vec{k}_1\cdot\vec{k}_4)(\vec{k}_2\cdot\vec{k}_3) \\ &+2\left[\frac{(\vec{k}_1\cdot\vec{k}_2)(\vec{k}_3\cdot\vec{k}_4)(k_1-k_2)^2}{(\vec{k}_1\cdot\vec{k}_2)-k_1k_2}-\frac{(\vec{k}_1\cdot\vec{k}_3)(\vec{k}_2\cdot\vec{k}_4)(k_1-k_3)^2}{(\vec{k}_1\cdot\vec{k}_3)-k_1k_3}-\frac{(\vec{k}_1\cdot\vec{k}_4)(\vec{k}_2\cdot\vec{k}_3)(k_1-k_4)^2}{(\vec{k}_1\cdot\vec{k}_4)-k_1k_4}\right]\Bigg\} \end{aligned} \tag{6.2}$$

The three terms in (6.2) are singular if the vectors k_i are parallel. But there is a remarkable fact: these singularities cancel and the whole expression (6.2) is a regular continuous function. The cancellation of singularities is a quite nontrivial circumstance. It could be checked by a straightforward calculation.

The singular part of $\tilde{T}_{1234}$ can be written as follows:

$$\tilde{T}_{1234}=-\frac{1}{4\pi^2 h}\frac{k_2k_3k_4}{(k_1k_2k_3k_4)^{1/2}}\left[\frac{(k_1+k_2)^2}{k_2(\cos\phi_2-1)}-\frac{(k_1-k_3)^2}{k_3(\cos\phi_3-1)}-\frac{(k_1-k_4)^2}{k_4(\cos\phi_4-1)}\right] \tag{6.3}$$

Here $\cos\phi_i = (\vec{k}_1 \cdot \vec{k}_i)/k_1 k_i$.

The resonant conditions are:

$$\begin{aligned} k_1 + k_2 &= k_3 + k_4, \\ k_1 + k_2\cos\phi_2 &= k_3\cos\phi_3 + k_4\cos\phi_4, \\ k_2\sin\phi_2 &= k_3\sin\phi_3 + k_4\sin\phi_4. \end{aligned} \tag{6.4}$$

For small angles $|\phi_i| \ll 1$, we can put approximately

$$\cos\phi_i - 1 \simeq -\frac{\phi_i^2}{2}, \quad \sin\phi_i \simeq \phi_i.$$

The resonant conditions now become now

$$k_2\phi_2^2 = k_3\phi_3^2 + k_4\phi_4^2, \quad k_2\phi_2 = k_3\phi_3 + k_4\phi_4 \tag{6.5}$$

The most singular part of $\tilde{T}_{1234}$ is

$$\tilde{T}_{sing} \simeq -\frac{1}{2\pi^2 h}\frac{(k_3k_3k_4)^{1/2}}{k_1^{1/2}}\left\{-\frac{(k_1+k_2)^2}{k_2\phi_2^2} + \frac{(k_1-k_3)^2}{k_3\phi_3^3} + \frac{(k_1-k_4)^2}{k_4\phi_4^2}\right\} \tag{6.6}$$

But one can check by a direct calculation that

$$\frac{(k_1+k_2)^2}{k_2\phi_2^2} - \frac{(k_1-k_3)^2}{k_3\phi_3^2} - \frac{(k_1-k_4)^2}{k_4\phi_4^2} \equiv 0 \tag{6.7}$$

in virtue of (6.5). Hence the singularities cancel and (6.2) is a regular function.

We can calculate $\tilde{T}_{1234}$ more accurately by putting

$$\tilde{T}_{1234} = -\frac{1}{16\pi^2 h}\frac{1}{(k_1k_2k_3k_4)^{1/2}}\left\{(\vec{k}_1\vec{k}_2)(\vec{k}_3\vec{k}_4) + (\vec{k}_1\vec{k}_3)(\vec{k}_2\vec{k}_4) + (\vec{k}_1\vec{k}_4)(\vec{k}_2\vec{k}_3)\right.$$
$$\left. + 4s^2\left[\frac{(\vec{k}_1\vec{k}_2)(\vec{k}_3\vec{k}_4)(k_1+k_2)^2}{\omega_{1+2}^2 - (\omega_1+\omega_2)^2} - \frac{(\vec{k}_1\vec{k}_3)(\vec{k}_2\vec{k}_4)(k_1-k_3)^2}{\omega_{1-3}^2 - (\omega_1-\omega_3)^2} - \frac{(\vec{k}_1\vec{k}_4)(\vec{k}_2\vec{k}_4)(k_1-k_4)^2}{\omega_{1-4}^2 - (\omega_1-\omega_4)^2}\right]\right\} \tag{6.8}$$

Here we put

$$\omega(k) = sk\left(1 - \frac{1}{3}(kh)^2\right) \tag{6.9}$$

Now denominators in (6.8) cannot reach zero, but for almost parallel k_i they are of order $(kh)^2$ and small if $kh \to 0$. As a result, some terms in (6.8) are large, of order $1/h^3$, but in fact they cancel each other. The major terms in (6.8) are

$$\tilde{T}_{sing} \simeq \frac{1}{2\pi^2 h}\frac{(k_2k_3k_4)^{1/2}}{k_1^{1/2}}\left\{\frac{(k_1+k_2)^2}{k_2[\phi_2^2 + h^2(k_1+k_2)^2]} - \frac{(k_1-k_3)^2}{k_3[\phi_3^2 + h^2(k_1-k_3)^2]} - \frac{(k_1-k_4)^2}{k_4[\phi_4^2 + h^2(k_1-k_4)^2]}\right\} = 0 \tag{6.10}$$

The expression (6.10) is identically zero in virtue of (6.5). As $h \to 0$ (6.10) goes to (6.7).

Cancellations (6.7), (6.10) have a very deep hidden reason - they are consequencies of the integrability of the KP-2 equations (see Zakharov 1998).

7. Statistical description

The statistical description of nonlinear wave fields is realized by the correlation function

$$< a^*_{k_1} \cdots a^*_{k_n} a_{k_{n+1}} \cdots a_{k_{n+m}} >= J^{n,m}(\vec{k}_1 \cdots \vec{k}_n, \vec{k}_{n+1} \cdots \vec{k}_{n+m}) \delta(\vec{k}_1 + \cdots + \vec{k}_n - \vec{k}_{n+1} \cdots - \vec{k}_{n+m}) \tag{7.1}$$

The presence of δ-functions in (7.1) is a result of spatial uniformity of the wave field.

In the same way we can introduce correlation functions for the transformed variables b_k:

$$< b^*_{k_1} \cdots b^*_{k_n} b_{k_{n+1}} \cdots b_{k_{n+m}} >= I^{n,m}(\vec{k}_1 \cdots \vec{k}_n, \vec{k}_{n+1} \cdots \vec{k}_{n+m}) \delta(\vec{k}_1 + \cdots + \vec{k}_n - \vec{k}_{n+1} \cdots - \vec{k}_{n+m}) \tag{7.2}$$

To find the connection between $J^{n,m}$ and $I^{n,m}$ one has to substitute (3.9) into (7.1) and perform the averaging. The following pair of correlation functions is the most important:

$$\begin{aligned} < a_k a^*_{k'} > &= n_k \, \delta(k - k') \\ < b_k b^*_{k'} > &= N_k \, \delta(k - k') \end{aligned} \tag{7.3}$$

Here n_k and N_k are different functions. n_k is a measurable quantity, connected directly with observable correlation functions. For instance, from (2.17) we get

$$I_k =< |\eta_k|^2 >= \frac{1}{2} \left(\frac{A_k}{B_k} \right)^{1/2} (n_k + n_{-k}) = \frac{1}{2} \frac{\omega_k}{B_k} (n_k + n_{-k}) \tag{7.4}$$

The function N_k cannot be measured directly. It is an important auxiliary tool used in analytical constructions. In most articles on physical oceanography the authors make no difference between n_k and N_k. This is a source of persistent and systematic mistakes. We will see that the difference between n_k and N_k is especially important on shallow water.

Plugging (3.9) into (7.3) we get:

$$n_k = N_k + < a_k^{(0)} a_k^{(1)*} > + < a_k^{(0)*} a_k^{(1)} > + < a_k^{(1)} a_k^{(1)*} > + < a_k^{(0)} a_k^{(2)*} > + < a_k^{(0)*} a_k^{(2)} > + \cdots \tag{7.5}$$

Terms $< a_k^{(0)} a_k^{(1)*} >$, $< a_k^{(0)*} a_k^{(1)} >$ are expressed through triple correlation functions $< b^* bb >$ and $< bbb >$. As far as the cubic terms in the effective Hamiltonian are cancelled, triple correlation is defined by the fifth-order correlation functions and is small and can be neglected. In fact, $I^{(1,2)} \simeq n^5$.

The next terms in (7.5) are expressed through quartic correlation. Only one quartic correlation function is really important

$$< b_k^* \, b_{k_1}^* \, b_{k_2} \, b_{k_3} >= I^{(2,2)}(\vec{k}, \vec{k}_1, \vec{k}_2, \vec{k}_3) \, \delta(\vec{k} + \vec{k}_1 - \vec{k}_2 - \vec{k}_3) \tag{7.6}$$

We study only weakly nonlinear waves and can assume that the stochastic process of surface oscillations is close to Gaussian. Thus we can put approximately

$$I^{(2,2)}(\vec{k}, \vec{k}_1, \vec{k}_2, \vec{k}_3) = N_k N_{k_2} \, \delta(\vec{k} - \vec{k}_3) + N_k N_{k_3} \, \delta(\vec{k} - \vec{k}_2) \tag{7.7}$$

By the use of (7.7) we obtain the following expression:

$$\begin{aligned} n_k \;=\;& N_k + 2\int |\Gamma^{(1)}(\vec{k},\vec{k}_1,\vec{k}_2)|^2\, N_{k_1}\, N_{k_2}\, \delta(\vec{k}-\vec{k}_1-\vec{k}_2) dk_1 dk_2 \\ &+2\int |\Gamma^{(1)}(\vec{k}_2,\vec{k},\vec{k}_1)|^2\, N_{k_1}\, N_{k_2}\, \delta(\vec{k}+\vec{k}_1-\vec{k}_2)\, dk_1 dk_2 + \\ &+2\int |\Gamma^{(1)}(\vec{k}_1,\vec{k},\vec{k}_2)|^2\, N_{k_1}\, N_{k_2}\, \delta(\vec{k}-\vec{k}_1+\vec{k}_2)\, dk_1 dk_2 + \\ &+2\int |\Gamma^{(2)}(\vec{k},\vec{k}_1,\vec{k}_2)|^2\, N_{k_1}\, N_{k_2}\, \delta(\vec{k}+\vec{k}_1+\vec{k}_2)\, dk_1 dk_2 - 4N_k \int B(\vec{k},\vec{k}_1,\vec{k},\vec{k}_1)\, N_{k_1}\, dk_1 \end{aligned} \tag{7.8}$$

Using the expression (3.14) for B and formulae (3.12), (3.13) we get the final result:

$$\begin{aligned} n_k \;=\;& N_k + \frac{1}{2}\int \frac{|V^{(1,2)}(\vec{k},\vec{k}_1,\vec{k}_2)|^2}{(\omega_k-\omega_{k_1}-\omega_{k_2})^2}\,(N_{k_1}N_{k_2} - N_k N_{k_1} - N_k N_{k_2})\,\delta(\vec{k}-\vec{k}_1-\vec{k}_2)\, dk_1 dk_2 + \\ &+\frac{1}{2}\int \frac{|V^{(1,2)}(\vec{k},\vec{k}_1,\vec{k}_2)|^2}{(\omega_{k_1}-\omega_k-\omega_{k_2})^2}\,(N_{k_1}N_{k_2} + N_k N_{k_1} - N_k N_{k_2})\,\delta(\vec{k}_1-\vec{k}-\vec{k}_2)\, dk_1 dk_2 + \\ &+\frac{1}{2}\int \frac{|V^{(1,2)}(\vec{k}_2,\vec{k},\vec{k}_1)|^2}{(\omega_{k_2}-\omega_k-\omega_{k_1})^2}\,(N_{k_1}N_{k_2} + N_k N_{k_2} - N_k N_{k_1})\,\delta(\vec{k}_2-\vec{k}-\vec{k}_1)\, dk_1 dk_2 + \\ &+\frac{1}{2}\int \frac{|V^{(0,3)}(\vec{k},\vec{k}_1,\vec{k}_2)|^2}{(\omega_k+\omega_{k_1}+\omega_{k_2})^2}\,(N_{k_1}N_{k_2} + N_k N_{k_1} + N_k N_{k_2})\,\delta(\vec{k}+\vec{k}_1+\vec{k}_2)\, dk_1 dk_2 \end{aligned} \tag{7.9}$$

On deep water all the terms in (7.9) are of the same order, and the difference between n_k and N_k is small:

$$\frac{n_k - N_k}{n_k} \simeq \mu \tag{7.10}$$

However, in shallow water, denominators in (7.9) are small, and this difference can be dangerously big. The integration in (7.9) for a wave distribution which is broad in angle in the perpendicular direction can be performed explicitly. The last, nonresonant, term in (7.9) must be neglected. It is suitable to present the result in polar coordinates in the k-plane. The final formula is astonishingly simple:

$$n(k,\theta) = N(k,\theta) + \frac{9}{64}\left(\frac{h}{g}\right)^{1/2}\frac{1}{h^5 k}\left\{\int_0^k N(k_1,\theta)\, N(k-k_1,\theta) dk_1 + 2\int_0^\infty N(k_1,\theta)\, N(k+k_1,\theta) dk_1\right\} \tag{7.11}$$

Comparing the leading term with the next terms in (7.11) we obtain

$$\frac{n_k - N_k}{n_k} \simeq \mu/\delta^5 \quad \delta \sim (kh) \tag{7.12}$$

Then the condition of applicability for a weakly-nonlinear statistical theory of waves on shallow water becomes

$$\mu \ll \delta^5 \tag{7.13}$$

For a very shallow water, $kh \simeq 0.1$, this condition can practically never be satisfied. But for a moderately shallow water, $kh \simeq 0.3$, it could be satisfied for small amplitude waves, $\mu \simeq 10^{-4}$. In many real situations the corrections in (7.11) are important and cannot be neglected. Generally speaking, the weakly-nonlinear theory has narrow frames of applicability in shallow water.

8. Kinetic equation

The function n_k is usually named "wave action distribution". There is no standard name for the function N_k so far. We will call it "renormalized wave action". It is very important that the kinetic equation is imposed, not on the wave action n_k but on the renormalized wave action N_k.

To derive this equation we can begin from the equation (4.7). It imposes an infinite set of relations on correlation functions. The statistical description means a loss of time reversibility and needs an introduction of negligibly small damping. It can be done by replacing in (4.7)

$$\omega_k \to \omega_k + i\gamma_k$$

Directly from (4.7) we obtain

$$\frac{\partial N_k}{\partial t} + 2\gamma_k N_k = \int T(\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3)\, J_m\, I(\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3)\,\delta(\vec{k}+\vec{k}_1-\vec{k}_2-\vec{k}_3)dk_1 dk_2 dk_3 \tag{8.1}$$

We will shorten the notation further.

$$\frac{\partial}{\partial t} I_{1234} + (i\Delta + \Gamma)\, I_{1234} = -\frac{i}{2}\int \Big\{T_{1567}\,\delta_{1+5-6-7}\, I_{267345} +$$
$$+T_{2567}\,\delta_{2+5-6-7}\, I_{167345} - T_{3567}\, I_{125467}\,\delta_{3+5-6-7} - T_{4567}\, I_{125367}\,\delta_{4+5-6-7}\Big\} dk_5 dk_6 dk_7 \tag{8.2}$$

Here

$$\begin{aligned}\Delta = \Delta_{1234} &= -\omega_1 - \omega_2 + \omega_3 + \omega_4\\ \Gamma &= \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4\end{aligned} \tag{8.3}$$

To make a closure in the system we perform the canonical expansion of the correlation function

$$I_{1234} = N_1 N_2(\delta_{13} + \delta_{14}) + \tilde{I}_{1234} \tag{8.4}$$

into

$$\begin{aligned}I_{123456} = N_1N_2N_3(\delta_{14}\delta_{25} + \delta_{14}\delta_{26} + \delta_{15}\delta_{24} + \delta_{15}\delta_{26} + \delta_{16}\delta_{24} + \delta_{16}\delta_{25}) +\\ +N_4((\tilde{I}_{2356}\delta_{14} + \tilde{I}_{1356}\delta_{24} + \tilde{I}_{1256}\delta_{34}) +\\ +N_5(\tilde{I}_{2346}\delta_{15} + \tilde{I}_{1346}\delta_{25} + \tilde{I}_{1246}\delta_{35}) +\\ +N_6(\tilde{I}_{2345}\delta_{16} + \tilde{I}_{1345}\delta_{26} + \tilde{I}_{1245}\delta_{36}) + \tilde{I}_{123456}\end{aligned} \tag{8.5}$$

The formulae (8.1)–(8.4) are exact. There $\tilde{I}_{1234}$ and $\tilde{I}_{123456}$ are the cumulants, irreducible parts of the correlators.

Substituting (8.5) into (8.3) and using (8.1) we obtain

$$\frac{\partial}{\partial t}\tilde{I}_{1234} + (i\tilde{\Delta} + T)\tilde{I}_{1234} = T_{1234}(N_2N_3N_4 + N_1N_3N_4 - N_1N_2N_3 - N_1N_2N_4) + \hat{L}\, I + Q \tag{8.6}$$

Here Q is the right part of the equation (8.2) where the six-point correlator is replaced by a corresponding cumulant, for instance, $I_{256347} \to \tilde{I}_{256347}$.

$$\tilde{\Delta} = -\tilde{\omega} - \tilde{\omega_2} + \tilde{\omega_3} + \tilde{\omega_4}, \tag{8.7}$$

where $\tilde{\omega}(k)$ is a renormalized dispersion relation

$$\tilde{\omega}(k) = \omega(k) + \int T(\vec{k}, \vec{k}_1) N_{k_1}\, dk_1\,, \quad T(\vec{k}, \vec{k}_1) = T(\vec{k}, \vec{k}_1, \vec{k}, \vec{k}_1) \tag{8.8}$$

$\hat{L}I$ is a linear operator:

$$(\hat{L}I)_{1234} = M_{1234} + M_{2134} - M_{3412} - M_{4312} \tag{8.9}$$

$$\begin{aligned} M_{1234} = & -\frac{i}{2} N_2 \int T_{1256}\, I_{5634}\, \delta(1+2-5-6)\, dk_5 dk_6 \\ & -iN_3 \int T_{1546}\, I_{2645}\, \delta(1+5-4-6)\, dk_5 dk_6 - iN_4 \int T_{1536}\, I_{2635}\, \delta(1+5-3-6) dk_5 dk_6 \end{aligned} \tag{8.10}$$

The system (8.1),(8.6) becomes closed by putting $\tilde{I}_{123456} = 0$. It is still very complicated. For further simplification one has to neglect $\hat{L}I$. Sending $\Gamma \to 0$, we finally get

$$I_m\, \tilde{I}_{1234} = \pi\, T_{1234} (N_2 N_3 N_4 + N_1 N_3 N_4 - N_1 N_2 N_3 - N_1 N_2 N_4)\, \delta(\tilde{\Delta}) \tag{8.11}$$

Substituting (8.9) into (8.1) leads to the final result

$$\begin{aligned} & \frac{\partial N_k}{\partial t} + 2\gamma_k\, N_k = st(N,N,N) \\ & st(N,N,N) = \pi \int |T_{1234}|^2\, (N_2 N_3 N_4 + N_1 N_3 N_4 - N_1 N_2 N_3 - N_1 N_2 N_4) \times \\ & \times \delta_{1+2-3-4}\, \delta(\tilde{\omega_1} + \tilde{\omega_2} - \tilde{\omega_3} - \tilde{\omega_4})\, dk_2 dk_3 dk_4 \end{aligned} \tag{8.12}$$

Due to the inclusion of the frequency normalization, the equation (8.12) is more exact than the "common" wave kinetic equation.

To get the quantum kinetic equation we can use the same procedure, assuming that a_k, a_k^+ are noncommutative operators of annihilation and creation of quasiparticles.

9. Renormalized dispersion relation

Frequency renormalization is described by the diagonal part of the four-wave interaction coefficient

$$T(\vec{k}_1, \vec{k}_2) = T(\vec{k}_1, \vec{k}_2, \vec{k}_1, \vec{k}_2) = T_{12} \tag{9.1}$$

This "naive" formula presumes the existence of the limit:

$$T(\vec{k}_1, \vec{k}_2) = \lim_{|\vec{q}| \to 0} T(\vec{k}_1, \vec{k}_2, \vec{k}_1 + \vec{q}, \vec{k}_2 - \vec{q}) \tag{9.2}$$

This limit exists and does not depend on the direction of the vector $\vec{q}$ only in deep water. In the general case, we can obtain from (4.9)

$$\begin{aligned} T_{12} = & -\frac{1}{16\pi^2} \left(\frac{A_1 A_2}{B_1 B_2}\right)^{1/2} \left[2k_1^2 B_1 + 2k_2^2 B_2 - (\omega_1 + \omega_2)^2 A_{1+2} - (\omega_1 - \omega_2)^2 A_{1-2}\right] \\ & - \frac{1}{32\pi^2} \left(\frac{B_1 B_2}{A_1 A_2}\right)^{1/2} \left\{ \frac{1}{B_{1+2}} \left[L_{12}^2 + \frac{u_{12}^2}{\omega_{1+2}^2 - (\omega_1 + \omega + 2)^2}\right] + \frac{1}{B_{1+2}} \left[L_{-1,2}^2 + \frac{u_{-1,2}^2}{\omega_{1-2}^2 - (\omega_1 - \omega_2)^2}\right] \right\} \end{aligned} \tag{9.3}$$

In the absence of capillarity in deep water the expression (9.3) becomes

$$\begin{aligned} T_{12} &= -\frac{1}{8\pi^2}\frac{1}{(k_1k_2)^{1/2}}\left\{3k_1^2k_2^2 + (\vec{k}_1\cdot\vec{k}_2)^2 - 4\omega_1\omega_2(\vec{k}.\vec{k}_2)(k_1+k_2) + \right. \\ & \left. +2\frac{(\omega_1+\omega_2)^2\left[(\vec{k}_1\cdot\vec{k}_2)^2 - k_1^2k_2^2\right]}{\omega_{1+2}^2 - (\omega_1+\omega_2)^2} + 2\frac{(\omega_1-\omega_2)^2\left[(\vec{k}_1\cdot\vec{k}_2)^2 + k_1^2k_2^2\right]}{\omega_{1-2}^2 - (\omega_1-\omega_2)^2}\right\} \end{aligned} \tag{9.4}$$

In the one-dimensional case the formula (9.4) becomes remarkably simple

$$T_{12} = \frac{1}{2\pi^2}\begin{cases} k_1^2k_2 & k_1 < k_2 \\ k_1k_2^2 & k_1 > k_2 \end{cases} \tag{9.5}$$

The function T_{12} is continuous at $k = k_1$, but its first derivative has a jump. This result was published by the author in 1992 (Zakharov, 1992). At $k_2 = k_1$

$$T_{12} \to T_{11}, \quad T_{11} = \frac{1}{2\pi^2}k^3. \tag{9.6}$$

In the presence of capillarity

$$T_{11} = \frac{k^3}{4\pi^2}\frac{2-\sigma k^2}{1-2\sigma k^2}. \tag{9.7}$$

For monochromatic waves we have:

$$b = F\,\delta(k-k_0), \quad \delta\omega = \frac{1}{2}T_{11}\,|F|^2 \tag{9.8}$$

In natural variables

$$\eta = a\cos(k_0x - \omega t - \phi), \quad a^2 = \frac{1}{2\pi^2}\frac{k_0}{\omega_{k_0}}|F|^2$$

and

$$\frac{\delta\omega}{\omega} = \frac{1}{4}\frac{2-\sigma k^2}{1-2\sigma k^2}(ka)^2 \tag{9.9}$$

It is in agreement with the classical results of Stokes and other authors. In shallow water the limiting procedure (9.2) needs some accuracy and falls beyond the framework of this article.

10. Kolmogorov spectra

Let us look now for stationary solutions of the kinetic wave equation (8.12). They satisfy the equation

$$st(N,N,N) = 0 \tag{10.1}$$

This equation has an ample array of solutions describing direct and inverse cascades of energy, momentum, and wave action. A full description of these solutions has not been done so far. Only very special, isotropic solutions could be found analytically in the case when ω_k is a power function

$$\omega_k = a|k|^\alpha, \tag{10.2}$$

and $T(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4)$ is a homogeneous function:

$$T(\epsilon\vec{k}_1, \epsilon\vec{k}_2, \epsilon\vec{k}_3, \epsilon\vec{k}_4) = \epsilon^\beta\, T(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) \tag{10.3}$$

It is assumed that the function $T(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4)$ is invariant with respect to rotation in $\vec{k}$-space.

In the general case of water of finite depth ω_k is *not* a homogeneous function. As a result, all known analytical methods are unable to construct any nontrivial (non-thermodynamic) solution of equation (10.1). But in two limiting cases, deep water and very shallow water, some solutions can be found. On deep water

$$\omega_k = \sqrt{gk}, \quad \alpha = 1/2, \tag{10.4}$$

and $T(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4)$ is given by the expression (5.1). Apparently, $\beta = 3$. On very shallow water

$$\omega_k = s|k|, \quad \alpha = 1, \tag{10.5}$$

and $T(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4)$ is given by formula (6.2). As far as singularities in (6.2) are cancelled, it is a regular continuous function on the resonant manifold (6.4). Now $\beta = 2$. On a flat bottom the isotropy with respect to rotation is satisfied.

It is well known (see, for instance, Zakharov, Falkovich and Lvov, 1992) that under conditions (10.2), (10.3) the equation (10.1) has powerlike Kolmogorov solutions

$$\begin{aligned} n_k^{(1)} &= a_1\, P^{1/3}\, k^{-\frac{2\beta}{3}-d} \\ n_k^{(2)} &= a_2\, Q^{1/3}\, k^{-\frac{2\beta-\alpha}{3}-d} \end{aligned} \tag{10.6}$$

Here d is a spatial dimension ($d = 2$ in our case).

The first one is a Kolmogorov spectrum, corresponding to a constant flux of energy P to the region of small scales (direct cascade of energy). The second one is a Kolmogorov spectrum, describing inverse cascade of wave action to large scales, and Q is the flux of action. In both cases a_1 and a_2 are dimensionless "Kolmogorov's constants". They depend on the detailed structure of $T(k, k_1, k_2, k_3)$ and are represented by some three-dimensional integrals.

It is known since 1966 (Zakharov and Filonenko, 1966) that on deep water

$$n_k^{(1)} = a_1\, P^{1/3}\, k^{-4}. \tag{10.7}$$

For the energy spectrum

$$I_\omega d\omega = \omega_k\, n_{\vec{k}}\, d\vec{k} \tag{10.8}$$

one obtains

$$I_\omega \simeq P^{1/3}\, \omega^{-4}. \tag{10.9}$$

This result is supported now by many observational data as well as numerical simulations.

In the same way on deep water (Zakharov and Zaslavsky, 1982):

$$n_k^{(2)} = a_2\, Q^{1/3}\, k^{-23/6}, \quad I_\omega \simeq Q^{1/3}\, \omega^{-11/3}. \tag{10.10}$$

On a very shallow water $\alpha = 1$, $\beta = 2$, and we obtain:

$$n_k^{(1)} = \tilde{a}_1\, P^{1/3}\, k^{-10/3}\, h^{2/3}, \quad I_\omega^{(1)} \simeq P^{1/3}\, \omega^{-4/3} \tag{10.11}$$

$$n_k^{(2)} = \tilde{a}_2\, Q^{1/3}\, k^{-3}\, h^{2/3}, \quad I_\omega^{(2)} \simeq Q^{1/3}\, \omega^{-1} \tag{10.12}$$

Formulae (10.11), (10.12) are new. We must keep in mind that they are applicable only if the condition $\mu \ll \delta^5$ is satisfied.

11. Conclusions

The weakly nonlinear theory of gravity waves has some window of applicability on shallow water. But this window shrinks dramatically when the parameter $\delta = kh$ tends to zero. For $\delta \simeq 0.5$ the window is relatively wide, $\mu \leq 10^{-2}$, but for $\delta \simeq 0.2$ it barely exists, $\mu \ll 10^{-4}$.

On deep water we can neglect the difference between the observed, n_k, and renormalized, N_k, wave action. On shallow water the difference could be very important for correct interpretation of observed data. We have to remember that the kinetic equation is written not for real, but for "renormalized" wave action.

Many problems pertaining to the statistical theory of gravity waves on shallow water are still unresolved. The most important problem is finding a Kolmogorov spectra for a fluid of arbitrary depth. From dimensional consideration we can conclude that it has the form

$$N_k^{(1)} = P^{1/3}\,k^{-4}\,F(kh), \quad F \to a_1, \quad kh \to \infty\,, \quad F \to \tilde{a_1}\,(kh)^{2/3} \quad kh \to 0 \tag{11.1}$$

The function $F(\xi)$ is unknown and should be found numerically.

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