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DESCRIPTION OF THE n-ORTHOGONAL CURVILINEAR COORDINATE SYSTEMS AND HAMILTONIAN INTEGRABLE SYSTEMS OF HYDRODYNAMIC TYPE, I: INTEGRATION OF THE LAMÉ EQUATIONS

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1. Introduction. The problem of describing n-orthogonal curvilinear coordinate systems can be formulated as follows: Find in R^n all the coordinate systems

$$u^i = u^i(x^1, \ldots, x^n), \tag{1.1}$$

$$\det\left\|\frac{\partial u^i}{\partial x^j}\right\| \neq 0, \tag{1.2}$$

satisfying the condition of orthogonality

$$\sum_{k=1}^{n} \frac{\partial u^i}{\partial x^k}\frac{\partial u^j}{\partial x^k} = 0, \qquad i \neq j. \tag{1.3}$$

The problem can be formulated either locally (in the same domain Ω) or globally (in the whole R^n). In the latter case, one can admit that condition (1.2) can be violated on some manifold of dimension $m < n$, and the system of intersecting hypersurfaces may have a nontrivial topology. Coordinates $u^i(x)$ are defined up to an obvious transformation

$$u^i = f^i(\tilde{u}^i). \tag{1.4}$$

For $n = 2$, the problem can be solved very easily. Let us choose a function (u^1, for instance) in an arbitrary way and consider a system of its level lines on the plane x^1, x^2. Then one can construct the vector field of normals to the level lines. Integral curves of this vector field are the level lines for u^2, which can be reconstructed uniquely up to transformation (1.4).

For $n \geqslant 3$, the problem is much more difficult. The first nontrivial case $n = 3$ is known in differential geometry as the problem of triply orthogonal systems of surfaces. It was formulated in 1810 when Dupin and Binet found a family of

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confocal quadrics satisfying condition (1.3). Since that time, the problem became one of the classical and the most popular. The first general theorem, stating that the intersections of two orthogonal surfaces are the lines of curvature, was obtained by Dupin in 1813. During more than a century after, the problem was attacked by many first-class mathematicians—Gauss, Lamé, Bonnet, Cayley, and Darboux are the most famous among them. The total amount of published materials about this topic is enormous. The articles of Bianchi alone, devoted to the problem of triply orthogonal systems of surfaces, comprise a book of 850 pages (Volume 3 of his *Opere*, published in Rome in 1955 [2]). The milestone in the history of this problem was a fundamental monography, *Lecons sur les systems orthogonaux et les cordonees curvilineare*, by Darboux [5], printed in Paris in 1910. The work is really astonishing; much of this book is exciting to a person familiar with the modern mathematical theory of solitons.

After the First World War, the problem of n-orthogonal coordinate systems became less popular, and temporarily lost its conspicuous status. Nevertheless, it attracted the attention of Cartan [4] and others, most of them French mathematicians (see, for instance, [3] and [1]).

Let us summarize some basic achievements of the "classical" period. First of all, the problem of n-orthogonal systems of surfaces can be formulated as a problem of intrinsic geometry (Gauss, Lamé). Due to (1), (2), one can resolve

$$x^i = x^i(u^1, \dots, u^n). \tag{1.5}$$

Let us denote

$$H_i^2 = \sum_k \left(\frac{\partial x^i}{\partial u^k}\right)^2. \tag{1.6}$$

The metric tensor in R^n in the coordinate system u^i is diagonal:

$$ds^2 = \sum_{i=1}^{n} H_i^2 (du^i)^2. \tag{1.7}$$

Christoffel's coefficients for the Levi-Civita connection are

$$\Gamma^i_{lm} = 0, \qquad i \neq l \neq m,$$

$$\Gamma^i_{il} = \frac{1}{H_i}\frac{\partial H_i}{\partial u^l}, \tag{1.8}$$

$$\Gamma^i_{ll} = -\frac{H_l}{H_i^2}\frac{\partial H_l}{\partial u^i}, \qquad i \neq l. \tag{1.9}$$

The space R^n is flat; hence the Riemann's curvature tensor vanishes:

$$R_{il,jm} \equiv 0. \tag{1.10}$$

Because the metric tensor is diagonal, condition (1.10) is satisfied automatically if

$$i \neq l \neq j \neq m.$$

Conditions

$$R_{il,im} = 0, \qquad l \neq m, \tag{1.11}$$

impose on coefficients H_i (Lamé coefficients) the following system of equations:

$$\frac{\partial^2 H_i}{\partial u^l \partial u^m} = \frac{1}{H_l} \frac{\partial H_l}{\partial u^m} \frac{\partial H_i}{\partial u^l} + \frac{\partial H_m}{H_m \partial H_l} \frac{\partial H_i}{\partial u^m}. \tag{1.12}$$

The number of equations (1.12) is $n(n-1)(n-2)/2$.

Conditions

$$R_{il,il} = 0 \tag{1.13}$$

impose on H_i another system of $n(n-1)/2$ equations

$$\frac{\partial}{\partial u^l} \frac{\partial H_i}{H_l \partial u^l} + \frac{\partial}{\partial u^i} \frac{\partial H_l}{H_i \partial u^i} + \sum_{m \neq i \neq j} \frac{1}{(H^m)^2} \frac{\partial H_i}{\partial u^m} \frac{\partial H_l}{\partial u^m} = 0. \tag{1.14}$$

Systems (1.12) and (1.14) are heavily overdeterminated, but still has common solutions. Bianchi [2] and Cartan [4] showed that a general solution of both systems can be parametrized locally by $n(n-1)/2$ arbitrary functions of two variables.

If Lamé coefficients H_i are known, one can find $x^i(u^1, \dots, u^n)$ (i.e., solve the embedding problem) by solving another overdeterminated (but linear!) problem

$$\frac{\partial^2 x^i}{\partial u^k \partial u^l} = \Gamma^k_{kl} \frac{\partial x^i}{\partial u^k} + \Gamma^l_{lk} \frac{\partial x^i}{\partial u^l}, \tag{1.15}$$

$$\frac{\partial^2 x^i}{\partial (u^l)^2} = \sum_k \Gamma^k_{ll} \frac{\partial x^i}{\partial u^k}. \tag{1.16}$$

One can prove (see, for instance the book of Forsyth [10]) that the systems

(1.15) and (1.16) are compatible in virtue to (1.12) and (1.14), and they define n-orthogonal surfaces up to transition and orthogonal rotation in R^n. (Here, only the case $n = 3$ is considered, but the generalization is easy.) It is important to mention that system (1.15) alone has many more solutions, which can be parametrized by arbitrary functions of one variable.

An interest in the problem of n-orthogonal surfaces was reestablished a decade ago, when it was found that the problem has natural applications in mathematical physics. In 1983, Novikov and Dubrovin [8] developed a "geometrical" theory of quasilinear systems of hydrodynamic type in $1 + 1$-dimensions. These systems have the form

$$\frac{\partial u^i}{\partial t} = \sum V_k^i(u) \frac{\partial u^k}{\partial x}, \qquad u = u^1 \cdots u^n. \tag{1.17}$$

Novikov and Dubrovin showed that system (1.17) is a Hamiltonian system with a "local" Hamiltonian

$$H[u] = \int h(u)dx,$$

if the matrix V_k^i can be presented in the form

$$V_k^i = \sum_k \left(g^{il}(u) \frac{\partial^2 h}{\partial u^l \partial u^k} + b_k^{il}(u) \frac{\partial h}{\partial u^l} \right). \tag{1.18}$$

Here $g^{il}(u)$ are some metrics in a *flat* space R^n, while

$$b_k^{il}(u) = -\sum_s g^{ik} \Gamma_{sk}^l. \tag{1.19}$$

Here Γ_{sk}^l are the corresponding Christoffel's coefficients.

System (1.17) is a generalization of the Euler equations for ideal compressible fluids (in this case, $n = 2$). It was known since the time of Riemann that, for $n = 2$, the Hamiltonian system (1.17) is integrable by the hodograph method, and can be transformed to a diagonal form

$$\frac{\partial u^i}{\partial t} = V^i(u) \frac{\partial u^i}{\partial x}. \tag{1.20}$$

"Diagonal" variables u^i are called Riemann's invariants, and the coefficients $V^i(u)$ are "diagonal" velocities. In 1984, Tsarev, a student of Novikov, generalized classical Riemann's results to the case of arbitrary n [11]. Developing the ideas of Dubrovin and Novikov, he proved that the Hamiltonian system (1.17) can be integrated by some generalization of the hodograph method, only in the case that it can be transformed by a proper choice of variables u^i to the diagonal form (1.20). In this case, the flat metrics g^{ik} are diagonal, and the Hamiltonian h satisfies the system of equations

$$\frac{\partial^2 h}{\partial x^i \partial u^j} = \Gamma^i_{ij} \frac{\partial h}{\partial u^i} + \Gamma^j_{ji} \frac{\partial h}{\partial u^j} \tag{1.21}$$

coinciding to the first half of the embedding conditions (1.15).

Moreover, each solution $P(u)$ of this system generates an integral P of system (1.17) as

$$P = \int P(u)du,$$

and all of these integrals commute. So, classification of flat diagonal matrixes $ds^2 = H_i^2 du^{i^2}$ is an important preliminary step in classification of integrable Hamiltonian systems of hydrodynamic type. To accomplish the classification, one must find all solutions of the system coinciding with one of the embedding equations (1.15), (1.16). It is important to mention that the diagonal velocities $V^i(u)$ obey the following overdeterminated system (see [13]):

$$\frac{\partial}{\partial u^i}\left(\frac{1}{V^j - V^k}\frac{\partial V^k}{\partial u^j}\right) = \frac{\partial}{\partial u^j}\left(\frac{1}{V^i - V^k}\frac{\partial V^k}{\partial u^i}\right). \tag{1.22}$$

So, the problems of description of n-orthogonal surfaces and classification of the Hamiltonian of hydrodynamic-type systems are almost equivalent. The core of both problems is to find all solutions to the overdeterminated system (1.12), (1.14). It is important that the order of these systems can be reduced to one. Let us introduce the "rotation coefficients" (see, for instance, [2] and [5])

$$\beta_{ik} = \frac{1}{H_i}\frac{\partial H_k}{\partial u^i}. \tag{1.23}$$

From (1.12), one can find that β_{ik} satisfy the following first-order system of equations:

$$\frac{\partial \beta_{ij}}{\partial u^k} = \beta_{ik}\beta_{kj}, \tag{1.24}$$

$$\frac{\partial \beta_{ij}}{\partial u^i} + \frac{\partial \beta_{ji}}{\partial u^j} + \sum_{m \neq i,j} \beta_{mi}\beta_{mj} = 0. \tag{1.25}$$

If the solution of systems (1.21) and (1.22) is known, one can find the Lamé coefficients by solving the linear problem

$$\frac{\partial \Psi_i}{\partial u^k} = \beta_{ki} \Psi_k, \qquad i \neq k, \tag{1.26}$$

and putting $H_i = \Psi_i$. But a common solution of system (1.26) is far from being unique. Let $\tilde{\Psi}_i$ be another solution. Introducing

$$V_i = \frac{\tilde{\Psi}_i}{H_i},$$

we get the identities

$$\left(\frac{1}{V_k - V_i}\right) \frac{\partial V_i}{\partial u^k} = \frac{\partial \ln H_i}{\partial u^k}. \tag{1.27}$$

So the quantities V_i satisfy equations (1.22), and are diagonal velocities for some integrable Hamiltonian system of the hydrodynamic type.

Different solutions of system (1.26), affiliated to given rotation coefficients β_{ik}, describe different n-orthogonal coordinate systems, related by the so-called Combescure transformation. Suppose H_i and $\tilde{H}_i$ are two sets of Lamé coefficients, related by the Combescure transformation. Their quotient $W_i = H_i/\tilde{H}_i$ satisfies equations (1.22). The system of hydrodynamic type

$$\frac{\partial u^i}{\partial \tau} = W_i(u) \frac{\partial u^i}{\partial x} \tag{1.28}$$

is a symmetry of system (1.20). Any set $W_i(u)$ provides a solution $u = u(x, t)$ of (1.20) in an implicit form:

$$W^i(u) = V^i(u)t + x. \tag{1.29}$$

One purpose of this article is to show that systems (1.24) and (1.25) can be integrated by the inverse scattering method (ISM). We use a version of ISM known as the dressing method, formulated by Shabat and by the author of this paper in 1974 [16] (see also [13]). The starting point of the dressing method is construction of a certain integral equation of Marchenko type. Its solution gives exact solutions of (1.24) and (1.25), together with a fundamental solution of the linear system (1.26). So it becomes possible to find a set of β_{ik} parametrized by $n(n-1)/2$ functions of two variables, and to construct, for a given β_{ik}, all n-orthogonal systems related by a Combescure transformation. Each solution of (1.24) and (1.25) describes a Hamiltonian system of the hydrodynamic type, together with all its symmetries.

Integrability of systems (1.24) and (1.25) is not an astonishing fact. In the simplest case, $n = 3$, system (1.24) is nothing but a well-known "three-wave system" (see [13]) on an algebra of real 3×3 matrices l_3. The similiar system on the symmetric space of complex-valued Hermitian matrices is widely used in nonlinear optics. In the general case, system (1.24) is a generalization of the three-wave system. It was found in articles [16] and [14].

Thus, construction of the solution of system (1.24) is a relatively easy problem. The really new and difficult problem is to separate those special solutions of system (1.24) that satisfy system (1.25) as well. This problem is solved by imposing on the "dressing matrix function" a certain differential relation, which connects the dressing matrix with its transponent. We hope that invention of this new type of reduction will allow us to find new classes of integrable equations in the future.

Integrability of systems (1.24) and (1.25), though in a very restricted sense, was known to the classics in a form of the so-called Ribaucour transformation (see [2]). If a given solution of systems (1.24), (1.25), and (1.26) is known, one can find a new solution by the formula

$$\hat{\beta}_{ik} = \beta_{ik} - \frac{\Psi_i}{A}\left(\frac{\partial \Psi_k}{\partial u^k} + \sum_{s\neq k} \beta_{sk}\Psi_s\right). \tag{1.30}$$

Here $A = \sum_p (\Psi_p)^2$. We show that the Ribaucour transformation is a very special case of the dressing procedure. It allows us to use for dressing a complete set of $n(n-1)/2$ arbitrary functions of two variables.

Mathematicians of the classical period found a number of special solutions of systems (1.12), (1.14), (1.24), and (1.25). One of the most remarkable is the so-called Egorov's solution

$$H_i^2 = \frac{\partial \Phi}{\partial u^i}. \tag{1.31}$$

Here Φ is some scalar function of u^i. In this case, the matrix of rotational coefficients is symmetric,

$$\beta_{ik} = \beta_{ki},$$

and system (1.22) can be reduced to the form

$$\sum_{k=1}^{n} \frac{\partial}{\partial u^k} \beta_{ij} = 0, \qquad i \neq j. \tag{1.32}$$

Also, systems (1.21) and (1.22) can be reduced to an n-wave system on the algebra of real matrices in $1+1$-dimensional space. This fact has been established by Dubrovin [9], who also applied Egorov's metrics to the classification of Frobenius manifolds in topological quantum field theory [15]. One can check that in Egorov's case (1.31), formula (1.30) indeed presents a new solution of the system (1.24)–(1.25).

In addition, we must remark that the problem of n-orthogonal surfaces can be formulated in a loosened form. Namely, we can impose on the Lamé coefficients only equations (1.12) and drop out equations (1.14). The obtained system describes n-orthogonal metrics in a Riemann space of a special type, defined by the condition of "diagonality" of the Riemann's curvature tensor

$$R_{iklm}(1-\delta_{il}\delta_{km})=0. \tag{1.33}$$

We call such Riemann spaces *spaces of diagonal curvature*. In hydrodynamics, they correspond to a system of hydrodynamic type, which can be diagonal (and hence is integrable), but has no local Hamiltonian structure. This is the *semi-Hamiltonian system*. We show that constructing such systems is easier than constructing integrable Hamiltonian systems.

In this article, we consider construction of n-orthogonal systems as a problem of intrinsic geometry, and basically do not touch the problem of embedding. According to this attitude, we do not try to find Hamiltonians and conservation laws to find systems of the hydrodynamic type. We discuss this problem in the next article. Also we construct n-orthogonal systems locally in some domains in R^n, and do not discuss so far the problem of globalization. All obtained results can be easily expanded to the case of pseudo-Euclidean space $R^{(p,q)}$.

2. The dressing method and the abstract n-wave system. In this section, we describe a method of solution of system (1.12) and the equivalent systems (1.24) and (1.25). System (1.24) is a special case of a much more general integrable system, which can be written in a rather abstract form.

Let A be an associative algebra over a field of real numbers R or complex numbers C. Let Ω be a domain in R^n, and let $u=(u^1,\dots,u^n)$ be coordinates in R^n. Then $\Omega(u)$ is an A-valued function on Ω. We introduce n A-valued functions on Ω, $I_k(u)$, which commute

$$[I_i,I_k]=0, \tag{2.1}$$

and obey the condition

$$\frac{\partial I_k}{\partial u^i}=0, \qquad i\neq k. \tag{2.2}$$

So I_k depends only on u^k. In typical cases, I_k are constants.

We consider now the system of $n(n-1)(n-2)/6$ overdeterminated systems of nonlinear equations imposed on Q:

$$\sum_{perm} \varepsilon_{ijk}\left(I_i \frac{\partial Q}{\partial u^j} I_k - I_i Q I_j Q I_k\right) = 0. \tag{2.3}$$

Here ε_{ijk} is antisymmetric with respect to all permutations, and

$$\varepsilon_{ijk} = 1, \qquad i > j > k. \tag{2.4}$$

Summation in (2.3) goes over all possible (six) permutations. We call system (2.3) the *abstract n-wave system.* It was introduced in [16] and [14] for matrix algebras as a natural generalization of the well-known n-wave system.

If I_i are constant in A, they are defined up to an arbitrary linear transformation

$$I_k = \sum_k q_{ik}\,\tilde{I}_k \qquad det\|q_{ik}\| \neq 0.$$

The system (2.3) is invariant with respect to the transformation

$$Q \to Q + J \qquad [I_k, J] = 0.$$

Let $A_0 < A$ be the maximum commutative subalgebra containing all I_k. One can decompose A in a sum of linear spaces

$$A = A_0 + A_1.$$

Without loss of generality, one can consider that $Q \in A_1$.

Suppose that E is the unit element in A and $I_n = E$. (If the algebra A has no unit element, one can interpret E as a unit operator $E : A \to A$.) Then the system of $(n-1)(n-2)/2$ equations in (2.3), associated with I_n, can be written in the form

$$\frac{\partial}{\partial u^j}[I_i, Q] - \frac{\partial}{\partial u^i}[I_j, Q] + I_i \frac{\partial Q}{\partial u^n} I_j - I_j \frac{\partial Q}{\partial u^n} I_i - [[I_i, Q], [I_j, Q]] = 0, \tag{2.5}$$

for $i \neq j \neq n$. This is a "standard form" of the n-wave system (see, for instance, [16]).

It is important to mention that in solving system (2.5), it is enough to solve all other equations, including (2.3). To be sure of this, one can multiply (2.5) by

I_k, $k \neq i \neq j \neq n$, from the left side, and do the cyclic permutation. As a result, one achieves all remaining equations in (2.3). We proved the following simple theorem.

THEOREM 2.1. *Equations (2.3), $i \neq j \neq k \neq n$, are compatibility conditions for equations (2.5).*

In the general case, none of I_k are the unit in A. But we can use the proved theorem to construct solutions of (2.3) at arbitrary I_k. We just make Q dependent on an auxilliary variable $u^{n+1} = s$, $-\infty < s < \infty$, and put $I_{n+1} = E$. Then we consider the system

$$\frac{\partial}{\partial u^j}[I_i, Q] - \frac{\partial}{\partial u^i}[I_j, Q] + I_i \frac{\partial Q}{\partial s} I_j - I_j \frac{\partial Q}{\partial s} I_i - [[I_i, Q], [I_j, Q]] = 0. \tag{2.6}$$

We know that: *Any solution of system (2.8) at any fixed s is automatically a solution of system (2.3).*

System (2.6) can be solved by the dressing method. The main tool here is the integral equation

$$K(s, s', u) = F(s, s', u) + \int_s^\infty K(s, q, u) F(q, s', u) dq. \tag{2.7}$$

Here $K, F \in A$ and $-\infty < s < \infty$, $-\infty < s' < \infty$. Suppose that $F(s, s', u)$ is a given function satisfying the following two conditions.

(1) Equation (2.7) is uniquely resolved.

(2) $F(s, s', u)$ obeys the set of equations

$$D_i F = \frac{\partial F}{\partial u^i} + I_i \frac{\partial F}{\partial s} + \frac{\partial F}{\partial s'} I_i = 0. \tag{2.8}$$

Then $Q = K(s, s, u)$ obeys system (2.8), and consequently system (2.3).

The proof of this fact is straightforward. One can write equation (2.7) in a symbolic form

$$K = F + K * F, \tag{2.9}$$

and denote

$$D_i F = \frac{\partial F}{\partial u^i} + I_i \frac{\partial F}{\partial s} + \frac{\partial F}{\partial s'} I_i. \tag{2.10}$$

Applying D_i to (2.10), one gets, after simple transformations,

$$\tilde{D}_i K = D_i F + \tilde{D}_i K * F + K * D_i F, \tag{2.11}$$

$$\tilde{D}_i K = \frac{\partial K}{\partial u^i} + I_i \frac{\partial K}{\partial s} + \frac{\partial K}{\partial s'} I_i + [I, Q]K. \tag{2.12}$$

As far as $D_i F = 0$, we have

$$\tilde{D}_i K = \tilde{D}_i(K) * F. \tag{2.13}$$

Then, by virtue of the unique resolvability of equation (2.7), one receives

$$\tilde{D}_i K = 0. \tag{2.14}$$

Here,

$$\tilde{D}_i K = \frac{\partial K}{\partial u^i} + I_i \frac{\partial K}{\partial s} + \frac{\partial K}{\partial s'} I_i + [I_i, Q]K, \tag{2.15}$$

$$Q = K(s, s, u).$$

Then we set

$$[\tilde{D}_i, \tilde{D}_j]K = 0. \tag{2.16}$$

An operator $[\tilde{D}_i, \tilde{D}_j]$ is a multiplication from the left to some element $R_{ij}(s, u) \in A$. Equation (2.16) holds identically for all s'; hence one can cancel K. We get

$$[\tilde{D}_i, \tilde{D}_j] = 0. \tag{2.17}$$

So operators $\tilde{D}_i$ commute. One can check that condition (2.17) coincides identically with (2.6).

Let Ψ_0 be any solution of the system

$$\frac{\partial \Psi_0}{\partial u^i} + I_i \frac{\partial \Psi_0}{\partial s} = 0. \tag{2.18}$$

Then $\Psi(s, u)$ is defined as

$$\Psi = \Psi_0(s, u) + \int_s^\infty K(s, s', u)\, \Psi_0(s', u) ds'. \tag{2.19}$$

A direct calculation shows that Ψ satisfies the system

$$L_i\Psi = 0,$$

$$L_i\Psi = \frac{\partial\Psi}{\partial u^i} + I_i\frac{\partial\Psi}{\partial s} + [I_i, Q]\Psi. \tag{2.20}$$

Moreover, L_i commute

$$[L_i, L_j] = 0. \tag{2.21}$$

It is just another notation of identity (2.17).

The system of linear equations (2.20) is compatible. Its compatibility conditions are equations (2.6). Thus, (2.20) gives a "Lax representation" for system (2.6). One can construct the Lax representation for system (2.3) as well. It is given by linear equations

$$(I_jL_i - I_iL_j)\Psi = 0 \tag{2.22}$$

or

$$I_j\frac{\partial\Psi}{\partial u^i} - I_i\frac{\partial\Psi}{\partial u^j} = (I_iQI_j - I_jQI_i)\Psi. \tag{2.23}$$

The procedure for constructing the exact solution of system (2.3) is a generalization of the dressing method introduced in [13]. This is the simplest case of dressing—dressing against a "trivial background." We can essentially generalize this procedure, considering the dressing against an arbitrary solution of equations (2.6). Let $Q_0(u, s)$ be such a solution. Suppose that K and F are connected as before by relation (2.9), and F satisfies the system of equations

$$D_i[Q_0]F = \frac{\partial F}{\partial u^i} + I_i\frac{\partial F}{\partial s} + \frac{\partial F}{\partial s'}I_i + [I_i, Q_0(s,u)]F - F[I_i, Q_0(s',u)] = 0. \tag{2.24}$$

By virtue of (2.3), this system is compatible. Applying operators $D_i[Q_0]$ to equation (2.9), we get

$$\tilde{D}_i[Q]K = 0,$$

$$\tilde{D}_i[Q]K = \frac{\partial K}{\partial u^i} + I_i\frac{\partial K}{\partial s} + \frac{\partial K}{\partial s'}I_i + [I_i, Q_0(s,u)]K - K[I_i, Q_0(s',u)] = 0, \tag{2.25}$$

$$\tilde{Q} = Q_0 + K(s, s', u). \tag{2.26}$$

Equation (2.25) can be solved if one can find a fundamental solution of the linear system

$$\frac{\partial \Psi_0}{\partial u^i} + I_i \frac{\partial \Psi_0}{\partial s} + [I_i, Q_0(s,u)]\Psi_0 = 0. \tag{2.27}$$

Then

$$F = \Psi_0(s,u)\, F_0(s,s',u)\, \Psi_0^{-1}(s',u). \tag{2.28}$$

Here F_0 satisfies system (2.8). To find Ψ, one must apply transformation (2.19) to any solution of (2.27). Apparently

$$\delta Q(s,u) = F(s,s,u) \tag{2.29}$$

at any s satisfies the linearized system (2.3)

$$\sum \varepsilon_{ijk} \left(I_i \frac{\partial}{\partial u^j} \delta Q\, I_k - I_i\, Q_0\, I_j \delta\, Q\, I_k - I_i \delta\, Q\, I_j\, Q_0\, I_k \right) = 0, \tag{2.30}$$

and the linearized system (2.6):

$$\begin{aligned} &\frac{\partial}{\partial u^i}[I_i, \delta Q] - \frac{\partial}{\partial u^i}[I_i, \delta Q] + I_i \frac{\partial}{\partial s} \delta Q I_j - I_j \frac{\partial}{\partial s} \delta Q I_i \\ &\quad + [[I_i, \delta Q], [I_j, Q_0]] + [[I_i, Q_0], [I_j, \delta Q]] = 0. \end{aligned} \tag{2.31}$$

Finding Ψ, if $Q_0(s,u)$ is known, is a solution of "the direct scattering problem" (in terms of the theory of solitons).

The dressing method gives explicit solutions of system (2.3) if the kernel F is degenerative:

$$F = \sum_{q=1}^{N} f_q(s,u) g_q(s',u). \tag{2.32}$$

Here f_q, g_q satisfy the equations

$$L_i f_q = 0, \qquad L^+ g_q = 0, \tag{2.33}$$

and L_i^+ is an adjoint to the L_i operator

$$Ł_i^+ g = \frac{\partial g}{\partial u^i} + \frac{\partial g}{\partial s'} I_i - g\,[I_i, Q_0] = 0. \tag{2.34}$$

If $N = 1$ and

$$F(s, s', u) = f(s, u)g(s', u), \tag{2.35}$$

then

$$K(s, s', u) = K(s, u)g(s', u),$$

$$K(s, u) = f(s, u)\left[1 - \int_s^\infty g(s', u)f(s', u)\, ds'\right]^{-1}, \tag{2.36}$$

$$K(s, s, u) = f(s, u)\left[1 - \int_s^\infty g(s', u)f(s', u)ds'\right]^{-1} g(s, u). \tag{2.37}$$

The presumed unique solvability of equation (2.9) is a guarantee that the inversion in (2.36) and (2.37) is possible.

3. *n*-Orthogonal systems in spaces of diagonal curvature and semi-Hamiltonian systems of hydrodynamic type. Let us show how the construction of Section 2 works for systems (1.12), (1.23), and (1.24). In this case, A is $l_n(R)$, which means the algebra of $n \times n$ matrices with real coefficients. I_k are diagonal matrices. They can be choosen as

$$I_k = diag(\overbrace{\underbrace{0, \dots, 1}_{k}\, 0, \dots, 0}^{n}). \tag{3.1}$$

Obviously,

$$I_i I_k = 0 \tag{3.2}$$

if $i \neq k$.

It is immediately clear that system (2.3) now has a form

$$\frac{\partial Q_{ik}}{\partial u^j} = Q_{ij}Q_{jk}, \tag{3.3}$$

which formally coincides with (1.24). Here $i \neq j \neq k$. In the future, we will identify $Q_{ik} = \beta_{ki}$. Then

$$\left(\frac{\partial}{\partial u^i} + \frac{\partial}{\partial u^j} + \frac{\partial}{\partial s}\right)Q_{ij} - \sum_{k \neq ij} Q_{ik}Q_{kj} = 0, \qquad i \neq j. \tag{3.4}$$

The linear equation (2.22) is simplified up to the form

$$I_i \frac{\partial \Psi}{\partial u^i} = I_i Q I_j \Psi. \tag{3.5}$$

Let Ψ_i be any column in the matrix Ψ. Equation (3.2) gives

$$\frac{\partial \Psi_i}{\partial u^j} = Q_{ij} \Psi_j, \qquad i \neq j. \tag{3.6}$$

For $i = j$ one gets, from (2.20),

$$\frac{\partial \Psi_i}{\partial u^i} + \frac{\partial \Psi_i}{\partial s} + \sum_{k \neq j} Q_{ik} \Psi_k = 0. \tag{3.7}$$

Equation (2.8) can be solved as

$$F_{ij}(s, s', u) = f_{ij}(s - u^i, s' - u^j). \tag{3.8}$$

Here $f_{ij}(\xi, \eta)$ are n^2 arbitrary functions of two variables.
Substituting (3.3) with (3.4) and (3.6) with (3.7) yields

$$\frac{\partial Q_{ij}}{\partial s} + D Q_{ij} = 0, \qquad \frac{\partial \Psi_i}{\partial s} + D \Psi_i = 0. \tag{3.9}$$

Here

$$D = \sum_{k=1}^{n} \frac{\partial}{\partial u^k}. \tag{3.10}$$

So

$$\Psi_i = \Psi(u^1 - s, \ldots, u^n - s),$$

$$Q_{ij} = Q_{ij}(u^1 - s, \ldots, u^n - s). \tag{3.11}$$

In this case, dependence of the auxilliary parameter s is very simple.

Now we can describe a procedure for the implementation of the dressing method to the formulated problem. It consists of the following steps.

(1) Choose arbitrarily a real matrix function of two variable $f_{ij}(\xi, \eta)$'s and solve the integral equation (2.9). The only restriction on f_{ij} is that equation (2.9) must be uniquely resolvable for $-\infty < s < \infty$, $a \in \Omega$. Any choice of f_{ij} produces a

one-parameter family of solutions of system (3.3) by the formula

$$Q_{ij}(s,u) = K_{ij}(s,s,u). \tag{3.12}$$

The solutions are parametrized by $-\infty < s < \infty$. Dependence upon s, according to (3.11), is just a shift of arguments $u^i \to u^i - s$.

(2) Choosing arbitrarily a solution of system (2.18), one constructs one solution of system (3.6) using the dressing formula (2.19). Each column of Ψ_0 in (2.19) is dressed independently, so one can parametrize Ψ_i by an arbitrary vector solution of (2.18). In our case they have a very simple form:

$$\Phi_{0i} = \phi_i(s - u^i). \tag{3.13}$$

Here $\phi_i(\xi)$ is an arbitrary vector function of one variable.

(3) Now we can identify

$$H_i = \Psi_i. \tag{3.14}$$

We receive a solution of system (1.12) describing an intrinsic geometry of some space of diagonal curvature. Another choice of $\phi_i(\xi)$ produces another solution to (1.12), $\tilde{H}_i$, related to (3.14) by a Combescure transformation. All possible quotients

$$V_i(u) = \frac{\tilde{H}_i}{H_i} \tag{3.15}$$

define an integrable semi-Hamiltonian system (1.20). The other choice of H_i and $\tilde{H}_i$, that is, H_i' and $\tilde{H}_i'$, generates another semi-Hamiltonian system,

$$W_i(u) = \frac{\tilde{H}_i'}{H_i'}. \tag{3.16}$$

The systems

$$\begin{aligned} \frac{\partial u^i}{\partial t} &= V_i(u)\frac{\partial u^i}{\partial x}, \\ \frac{\partial u^i}{\partial \tau} &= W_i(u)\frac{\partial u^i}{\partial x} \end{aligned} \tag{3.17}$$

are compatible, and they are symmetries of each other. Each $W_i(u)$ generates in implicit form an exact solution of systems (1.20) (see [12]):

$$V_i(u) = tW_i(u) + x. \tag{3.18}$$

To accomplish this, let us write the complete set of equations imposed on K_{ij}:

$$\frac{\partial K_{ij}}{\partial u^k} = Q_{ik}K_{kj}, \qquad k \neq i \neq j, \tag{3.19}$$

$$\frac{\partial K_{ij}}{\partial u^i} + \frac{\partial K_{ij}}{\partial s} + \sum_{k\neq i} Q_{ik}K_{kj} = 0, \qquad i \neq j, \tag{3.20}$$

$$\frac{\partial K_{ij}}{\partial u^j} + \frac{\partial K_{ij}}{\partial s'} - Q_{ij}K_{ji} = 0, \qquad i \neq j, \tag{3.21}$$

$$\frac{\partial K_{ii}}{\partial u^i} + \frac{\partial K_{ii}}{\partial s} + \frac{\partial K_{ii}}{\partial s'} + \sum_{k\neq i} Q_{ik}K_{ki} = 0, \tag{3.22}$$

$$\frac{\partial K_{ii}}{\partial u^j} = Q_{ij}K_{ji}, \qquad i \neq j.$$

From (3.19), (3.20), one derives

$$\frac{\partial K_{ij}}{\partial s} + DK_{ij} = 0, \quad i \neq j, \tag{3.23}$$

$$\frac{\partial K_{ii}}{\partial s} + \frac{\partial K_{ii}}{\partial s'} + DK_{ii} = 0.$$

Hence

$$K_{ij}(s,s',u) = K_{ij}(s', u^1 - s, \dots, u^n - s), \qquad i \neq j,$$

$$K_{ii}(s,s',u) = K_{ii}(s' - s, u^1 - s, \dots, u^n - s). \tag{3.24}$$

Now we can study the dressing against an arbitrary background. Suppose one solution $H_{0i}(u)$ of (1.12) is known. Then one can find $Q_{0ij}(u)$ by

$$Q_{0ij} = \frac{1}{H_{0j}} \frac{\partial H_{0i}}{\partial u^j}, \tag{3.25}$$

and extend it for all values of s by relation (3.11):

$$Q_{0ij}(u^1, \dots, u^n) \to Q_{0ij}(u^1 - s, \dots, u^n - s) = Q_0(u - s). \tag{3.26}$$

Equation (1.25) now takes the form

$$\frac{\partial F}{\partial u^i} + I_i \frac{\partial F}{\partial s} + \frac{\partial F}{\partial s'} I_i + [I_i, Q_0(u-s)]F - F[I_i, Q_0(u-s')] = 0. \tag{3.27}$$

To solve this equation, one must solve equation (2.27). It is enough to find a *fundamental* solution Ψ_0 of the system

$$\frac{\partial \Psi_{0i}}{\partial u^j} = Q_{0ij}^{(u)} \Psi_{0j}, \qquad i \neq j, \tag{3.28}$$

and to extend it to all s by (3.11):

$$\Psi_0(u^1, \ldots, u^n) \to \Psi_0(u^1 - s, \ldots, u^n - s) = \Psi_0(u-s). \tag{3.29}$$

Then one must use (2.28), where

$$F_{0ij}(s, s', u) = f_{0ij}(s - u^i, s' - u^j) \tag{3.30}$$

and $f_{0ij}(\xi, \eta)$ are arbitrary. One column in the matrix Ψ_0 is given by H_{0i}. All others are connected to H_{0i} by a Combescure transformation. So, finding all Combescure-equivalent metrics to a given one is actually the solution of the direct scattering problem.

4. Algebraic reductions. The machinery built up in previous chapters makes it possible to construct exact solutions of system (2.3). It is unclear so far how dense the set of the found solution is, and how efficiently one can approximate by this solution a generic solution of (2.3). This question is especially difficult if the conditions of periodicity or quasi-periodicity are imposed. We do not address this really important question here. In a sense, we constructed too many solutions, and we concentrate our efforts in retrieving some interesting special classes of them.

Let * mean an involution in A,

$$(a^*)^* = a, \qquad a, b \in A, \qquad (ab)^* = b^* a^*. \tag{4.1}$$

If A is l_n, this involution might be

$$a^* = R^{-1} a^{tr} R. \tag{4.2}$$

Here a^{tr} is the matrix transponent to a. R is an arbitrary matrix satisfying the condition $R^{tr} = \pm R$. The following theorem holds.

THEOREM 4.1. *Let F and K be connected by the relation*

$$K = F + K \times F, \tag{4.3}$$

and let $\varepsilon(s)$ be an A-valued function on s satisfying the condition $\varepsilon^*(s) = \pm\varepsilon(s)$. Let

$$F(s, s')\varepsilon(s') = \varepsilon(s)F^*(s', s). \tag{4.4}$$

Then

$$K(s, s)\varepsilon(s) = \varepsilon(s)K^*(s, s). \tag{4.5}$$

Proof. Let us expand K in powers of F and present

$$K(s, s) = F(s, s) + \int_s^\infty F(s, q)F(q, s)dq + \cdots. \tag{4.6}$$

Then

$$K^*(s, s) = F^*(s, s) + \int_s^\infty F^*(q, s)F^*(s, q)dq + \cdots. \tag{4.7}$$

From (4.4), one can see that

$$F(s, s)\varepsilon(s) = \varepsilon(s)F^*(s, s).$$

Let us multiply (4.6) by the right and (4.7) by the left to $\varepsilon(s)$, and compare the results. Applying (4.4) to any term in expansion (4.7), one can realize that they coincide.

Identities (4.4) and (4.5) present an *algebraic reduction* imposed on the Marchenko equation (4.3). In the simplest case, it takes the form

$$F(s, s') = F^*(s', s). \tag{4.8}$$

Then

$$K(s, s) = K^*(s, s). \tag{4.9}$$

Let us study algebraic reduction in the abstract n-wave system. Starting from the simplest reductions (4.8) and (4.9), we assume that the involution * leaves all I_k unchanged. We have

$$I_k^* = I_k. \tag{4.10}$$

Then constraint (4.4) is compatible with equation (2.8). Imposing this constraint provides that, in (2.3),

$$Q^* = Q. \tag{4.11}$$

This reduction defines special classes of solutions in (2.3).

To make a degenerative kernel F satisfy reduction (4.8), one must put

$$F = F_1 + F_2, \tag{4.12}$$

$$F_1 = \sum_{q=1}^{N_1} f_q(s,u) f_q^*(s',u), \tag{4.13}$$

$$F_2 = \sum_{p=1}^{N_2} \Big(f_p(s,u) g_p^*(s',u) + g_p(s,u) f_p^*(s',u) \Big). \tag{4.14}$$

If $A = l_n$ is defined by (3.1), one can take involution in the form (4.2), where R is a diagonal matrix:

$$R = diag\, \varepsilon_i, \qquad \varepsilon_i = \varepsilon_i(u^i). \tag{4.15}$$

In particular, one can put

$$R = E, \qquad Q^* = Q^{tr}.$$

Now

$$Q_{ik} = Q_{ki}, \qquad \frac{1}{H_i}\frac{\partial H_k}{\partial u^i} = \frac{1}{H_k}\frac{\partial H_i}{\partial u^k}, \tag{4.16}$$

and

$$H_i^2 = \frac{\partial \Lambda}{\partial u^i}. \tag{4.17}$$

Here $\Lambda = \Lambda(u)$ is a scalar function obeying the following overdeterminated system of third-order equations:

$$\frac{\partial \Lambda}{\partial u^i}\frac{\partial \Lambda}{\partial u^l}\frac{\partial \Lambda}{\partial u^k}\frac{\partial^3 \Lambda}{\partial u^i \partial u^l \partial u^k} = \frac{1}{4}\left(\frac{\partial \Lambda}{\partial u^i}\frac{\partial^2 \Lambda}{\partial u^l \partial u^i}\frac{\partial^2 \Lambda}{\partial u^k \partial u^i} + \frac{\partial \Lambda}{\partial u^l}\frac{\partial^2 \Lambda}{\partial u^i \partial u^l}\frac{\partial^2 \Lambda}{\partial u^k \partial u^k} \right.$$
$$\left. + \frac{\partial \Lambda}{\partial u^k}\frac{\partial^2 \Lambda}{\partial u^l \partial u^k}\frac{\partial^2 \Lambda}{\partial u^k \partial u^i} \right). \tag{4.18}$$

It is clear that system (4.18) is compatible and integrable. Its integrating can be done by the dressing method described in Sections 2 and 3. To provide fulfilling of reduction (4.15), one must impose that the matrix f_{ij} in (3.7) satisfies the symmetry conditions

$$f_{ij}(\xi,\eta) = f_{ji}(\eta,\xi). \tag{4.19}$$

Let us study what kind of general algebraic reductions (4.4) and (4.5) are possible in systems (2.3) and (2.6). Suppose that condition (4.10) is satisfied. The reduction must be compatible with the basic equation (2.6). It is done if

$$[I_k, \varepsilon] = 0 \tag{4.20}$$

and

$$\left(\frac{\partial}{\partial u^i} + I_i \frac{\partial}{\partial s}\right)\varepsilon = 0. \tag{4.21}$$

For the n-orthogonal systems (3.3) and (3.4), it is implied that ε is a diagonal matrix

$$\varepsilon_{ij} = diag\, \varepsilon_i,$$

$$\varepsilon_i = \varepsilon_i(u^i - s),$$

and

$$f_{ij}(\xi,\eta) = \frac{\varepsilon_i(\eta)}{\varepsilon_j(\xi)} f_{ji}(\eta,\xi). \tag{4.22}$$

Now

$$Q_{ik}(u) = \frac{\varepsilon_i(u^j)}{\varepsilon_k(u^i)} Q_{ki}(u). \tag{4.23}$$

It is easy to check that reduction (4.19) is compatible with system (2.3). The described algebraic reductions are common in the theory of solitons (see, for instance, [16] and [6]).

5. Differential reductions. In this section, we introduce a new class of reductions in integrable systems—the differential reductions. These reductions are essential for integration of n-orthogonal curvilinear coordinate systems, and for the theory of integrable systems of hydrodynamic type. We start with the following theorem.

THEOREM 5.1. *Let A be an associative algebra, * be an involution in A, $F(s, s')$ be an A-valued function of two variables $-\infty < s < \infty$, $-\infty < s' < \infty$, and the equation*

$$K(s, s') = F(s, s') + \int_s^\infty K(s, q)F(q, s')dq \tag{5.1}$$

be resolvable uniquely for all s. Let F satisfy the equation

$$\frac{\partial F(s, s')}{\partial s'} + \frac{\partial F^*(s', s)}{\partial s} = 0. \tag{5.2}$$

Then, on the diagonal $s' = s$,

$$\left(\frac{\partial K}{\partial s'} + \frac{\partial K^*}{\partial s}\right)\bigg|_{s=s'} = K(s, s)\, K^*(s, s). \tag{5.3}$$

Proof. Expand $K(s, s')$ in a series in powers of F:

$$K(s, s') = F(s, s') + \cdots + \int_s^\infty \cdots \int_s^\infty F(s, q_1)F(q_1, q_2) \cdots F(q_n, s')dq_1 \cdots dq_n + \cdots. \tag{5.4}$$

Then

$$K^*(s', s) = F^*(s', s) + \cdots + \int_{s'}^\infty \cdots \int_{s'}^\infty F^*(q_1, s)F^*(q_1, q_2) \cdots F^*(q_n, s')dq_1 \cdots dq_n + \cdots. \tag{5.5}$$

(We renamed $q_1, \ldots, q_n \to q_n, \ldots, q_1$ in (5.4).)

Let us denote

$$R(s) = \lim_{s' \to s}\left(\frac{\partial K(s, s')}{\partial s'} + \frac{\partial K^*(s', s)}{\partial s}\right). \tag{5.6}$$

Expanding R in powers of F, F^*, one gets

$$R = R_2 + \cdots + R_n + \cdots, \tag{5.7}$$

$$R_n = \int_s^\infty \cdots \int_s^\infty F(s, q_1) \cdots F(q_{n-2}, q_{n-1}) \frac{\partial F(q_{n-1}, s')}{\partial s'} dq_1 \cdots dq_{n-1}$$
$$+ \int_{s^*}^\infty \cdots \int_{s^*}^\infty \frac{\partial F^*(q_1, s)}{\partial s} F^*(q_1, q_2) \cdots F^*(q_{n-1}, s)dq_1 \cdots dq_{n-1}. \tag{5.8}$$

The first term in (5.7) drops out due to (5.2).

Now one can apply identity (5.2) to the nth term in (5.7), $n-1$ times, and perform $n-1$ integrations by parts. One can see that the terms containing $n-1$ integrations cancel. Terms, including $n-2$ integrations, can be collected in the sum

$$R_n = F(s,s)\int_s^\infty \cdots \int_s^\infty F^*(q_1,s)\cdots F^*(q_{n-1},s)dq_2\cdots dq_{n-1}$$
$$+\cdots+\int_s^\infty \cdots \int_s^\infty F(s,q_1)\cdots F(q_{n-2},s)dq_1\cdots dq_{n-2}F^*(s,s). \qquad (5.9)$$

Then all the sums in (5.7) present the expansion for the product $K(s,s)K^*(s,s)$. The theorem is proved.

The constraint in (5.2) is the simplest example of a differential reduction. If F is a generative kernel, one can present it (see (4.10) and (4.12)) as

$$F = F_1 + F_2, \qquad (5.10)$$

$$F_1 = \sum_{q=1}^{N_1} \frac{\partial p_q^*(s,u)}{\partial s}\Lambda f_q(s',u), \qquad (5.11)$$

$$F_2 = \sum_{p=1}^{N_2}\left(\frac{\partial h_p^*(s,u)}{\partial s}g_p(s',u) - \frac{\partial g_p^*(s,u)}{\partial s}h_p(s',u)\right), \qquad (5.12)$$

$$\Lambda^* = -\Lambda.$$

Here f_p, g_p, h_p are arbitrary A-valued functions.

Theorem 5.1 can be generalized further. Let L be an A-valued differential operator

$$LF = \sum_{n=0}^{N} \frac{\partial^n F(s,s')}{\partial s'^n}U_n(s'), \qquad (5.13)$$

while $\tilde{L}$ is the adjoint operator

$$\tilde{L}F = \sum_{n=0}^{N}(-1)^n \frac{\partial^n}{\partial s^n}U_n(s)F. \qquad (5.14)$$

Suppose that $F(s,s')$ satisfies the equation

$$LF(s,s') = \tilde{L}F^*(s',s). \qquad (5.15)$$

Applying the involution * to (5.15), and permutating $s \leftrightarrow s'$, one derives

$$\tilde{L}^* F(s, s') = L^* F^*(s', s). \tag{5.16}$$

Here

$$L^* F^* = \sum_{n=0}^{N} U_n^* \frac{\partial^n F^*(s', s)}{\partial s^n}, \tag{5.17}$$

$$\tilde{L}^* F = \sum_{n=0}^{\infty} (-1)^N \frac{\partial^n}{\partial s'^n} F(s, s') U_n^*(s'). \tag{5.18}$$

Comparing (5.15) and (5.16), one derives

$$\tilde{L}^* = \pm L. \tag{5.19}$$

Condition (5.19) imposes $n + 1$ relations on coefficients of L:

$$\begin{aligned} U_n^*(s) &= \pm U_n(s), \\ U_{n-1}^*(s) &= \mp \big(U_{n-1} + U_n'(s)\big). \end{aligned} \tag{5.20}$$

Relation (5.15) can be called a differential reduction of the order n. Algebraic reduction is a trivial case of differential reductions (the order of the operator L is zero). Let us consider

$$\big(LK(s, s') - \tilde{L}K^*(s', s)\big)\big|_{s=s'}. \tag{5.21}$$

One can show that R can be expressed through

$$K(s, s')|_{s=s'}, \qquad K^*(s', s)|_{s'=s},$$

and a finite number of derivatives

$$\frac{\partial^q K(s, s')}{\partial s^q}\bigg|_{s=s'}, \quad \frac{\partial^q K(s, s')}{\partial s_n'^q}\bigg|_{s=s'}, \quad \frac{\partial^q K^*(s', s)}{\partial s'^q}\bigg|_{s=s'}, \quad \frac{\partial^q K^*(s', s)}{\partial s^q}\bigg|_{s=s'} \qquad (q < n).$$

Moreover, R is a bilinear operator (linear with respect to $K(s, s')$ and $K^*(s', s)$, separately).

We do not prove this fact in its general form here. We just display the two simplest examples.

(1) Let

$$LF = \frac{\partial F(s,s')}{\partial s'} U(s') + F(s,s')V(s')$$

and

$$U^*(s) = U(s),$$
$$V(s) + V^*(s) = U'(s). \tag{5.22}$$

Now

$$R = K(s,s)U(s)K^*(s,s), \tag{5.23}$$

and relation (5.21) takes the form

$$\left[\frac{\partial K(s,s')}{\partial s'} U(s') + \frac{\partial}{\partial s} U(s) K^*(s',s)\right]_{s=s'} + K(s,s)V(s) - V(s)K^*(s,s)$$
$$= K(s,s)U(s)K^*(s,s). \tag{5.24}$$

In particular, if

$$U = 1 \qquad V^*(s) = -V(s), \tag{5.25}$$

one derives

$$\lim_{s'\to s}\left(\frac{\partial K(s,s')}{\partial s'} + \frac{\partial}{\partial s} K_m^*(s',s)\right) + K(s,s)V(s) - V(s)K^*(s,s)$$
$$= K(s,s)K^*(s,s). \tag{5.26}$$

(2) Let us choose

$$LF = \frac{\partial^2 F(s,s')}{\partial s'^2}.$$

Using the same technique of expansion in a series, one can show that in this case, relation (5.21) becomes

$$\left.\left(\frac{\partial^2 K(s,s')}{\partial s'^2} - \frac{\partial^2 K^*(s',s)}{\partial s^2}\right)\right|_{s=s'}$$
$$= \left.\left(\frac{\partial K(s,s')}{\partial s} K(s,s) - K(s,s)\frac{\partial K^*(s,s')}{\partial s}\right)\right|_{s=s'}. \tag{5.27}$$

Now we show how the simplest differential reductions (5.2) and (5.3) work for the abstract n-wave system. Suppose that $I_k^* = I_k$, and consider the identity

$$I_j D_i K - I_i D_j K = 0, \tag{5.28}$$

which is

$$I_j \frac{\partial K(s,s')}{\partial u^i} - I_j \frac{\partial K(s,s')}{\partial u^j} + I_j \frac{\partial K(s,s')}{\partial s'} I_i - I_i \frac{\partial K(s,s')}{\partial s'} I_j$$

$$+ (I_j Q I_i - I_i Q I_j) K(s,s') = 0. \tag{5.29}$$

Applying the involution $*$ to (5.29), and permutating $s \leftrightarrow s'$, one gets

$$\frac{\partial K^*(s,s')}{\partial u^j} I_i - \frac{\partial K(s',s)}{\partial u^i} I_j + I_j \frac{\partial K^*(s',s)}{\partial s} - I_i \frac{\partial K^*(s',s)}{\partial s} I_j$$

$$+ K^*(s',s)(I_j Q^* I_i - I_i Q^* I_j) = 0. \tag{5.30}$$

Adding (5.27) and (5.28), putting $s' \to s$, and using relation (5.3), one gets, after a simple calculation,

$$I_j \frac{\partial Q}{\partial u^i} - I_j \frac{\partial Q}{\partial u^j} + \frac{\partial Q^*}{\partial u^j} I_i - \frac{\partial Q^*}{\partial u^i} I_j + I_j Q Q^* I_i - I_i Q Q^* I_j$$

$$- I_j Q I_i Q + I_i Q I_j Q - Q^* I_j Q^* I_i + Q^* I_i Q^* I_j = 0. \tag{5.31}$$

6. n-Orthogonal coordinate systems in the flat space-dressing against a Cartesian background. Now we can apply the dressing method to the Lamé equations (1.12) and (1.14). Again, A is $l_n(R)$, and I_k are given by (3.1). According to Section 3, the dressing function $F(s,s',u)$ is given by expression (3.8). Let us assume that it also satisfies the simplest differential reduction (5.2), which can be written as follows:

$$\frac{\partial F_{ij}(s,s',u)}{\partial s'} + \frac{\partial F_{ji}(s',s,u)}{\partial s} = 0. \tag{6.1}$$

To find additional equations imposed by (6.1) on Q_{ij}, one can use the general formula (5.29), or just apply the involution $*$, and the permutation $s \leftrightarrow s'$ to equation (3.21). One can read

$$\frac{\partial K_{ij}(s,s',u)}{\partial u^j} + \frac{\partial K_{ji}(s',s,u)}{\partial u^i} + \frac{\partial K_{ij}(s,s',u)}{\partial s'} + \frac{\partial K_{ji}(s',s)}{\partial s}$$

$$- Q_{ij}(s) K_{jj}(s,s') - K_{ii}(s,s') Q_{ij}(s') = 0. \tag{6.2}$$

Relation (5.3) now has the form

$$\left(\frac{\partial K_{ij}(s,s')}{\partial s'}+\frac{\partial K_{ji}(s',s)}{\partial s}\right)\bigg|_{s=s'}=\sum_l K_{il}(s,s)K_{jl}(s,s). \tag{6.3}$$

Putting $s'=s$ in (6.2) and using (6.3) yields immediately

$$\frac{\partial Q_{ij}}{\partial u^j}+\frac{\partial Q_{ji}}{\partial u^i}+\sum_{l\neq i,j} Q_{il}Q_{jl}=0. \tag{6.4}$$

Equation (6.4) is identical to (1.25). (Remember that $Q_{ij}=\beta_{ji}$!) To resolve equations (3.8) and (6.1), one can introduce $n(n-1)/2$ functions of two variables $\Phi_{ij}(\xi,\eta)$, $i<j$, and put

$$F_{ij}=\frac{\partial \Phi_{ij}(s-u^i,s'-u^j)}{\partial s},\qquad i<j,$$

$$F_{ji}=\frac{\partial \Phi_{ji}(s'-u^i,s-u^j)}{\partial s}. \tag{6.5}$$

Equations (3.8) and (6.1) are satisfied now for all off-diagonal elements $i\neq j$. To satisfy the diagonal elements, one has to introduce n diagonal, antisymmetric functions of two variables

$$\Phi_{ii}(\xi,\eta)=-\Phi_{ii}(\eta,\xi)$$

and put

$$F_{ii}=\frac{\partial \Phi_{ii}(s-u^i,s'-u^i)}{\partial s}. \tag{6.6}$$

We found that our solution is parametrized by $n(n-1)/2$ functions of two variables $\Phi_{ij}(\xi,\eta)$ together with n additional, antisymmetric functions $\Phi_{ii}(\xi,\eta)$. So, the total number of functional parameters participating in the dressing procedure is even more than the needed number $n(n-1)/2$. This means that in reality, we constructed certain classes of equivalent dressings. This equivalence will be considered in another article.

Expansion (5.10) now has the form

$$F=F^{(1)}+F^{(2)},$$

$$F_{ij}^{(1)}=\sum_{p=1}^{N_1}\frac{\partial f_{ki}^{(p)}(s-u^i)}{\partial s}\Lambda_{kl}^{(p)}f_{lj}^{(p)}(s'-u^j), \tag{6.7}$$

$$F_{ij}^{(2)}=\sum_{p=1}^{N_2}\left(\frac{\partial h_{ki}^{(p)}(s-u^i)}{\partial s}g_{kj}^{(p)}(s'-u^j)-\frac{\partial g_{ki}^{(p)}(s-u^i)}{\partial s}h_{kj}^{(p)}(s'-u^j)\right).$$

Here $f^{(p)}, g^{(p)}$, and $h^{(p)}$ are arbitrary real matrix functions of one variable, while

$$\Lambda_{kl}^{(p)} = -\Lambda_{lk}^{(p)} \tag{6.8}$$

are arbitrary constant antisymmetric matrixes.

The simplest solution, obtained by the dressing method, appears if $F^{(2)} = 0$, $N_1 = 1$, $f^{(1)}$ is diagonal, and $f_{ij} = f_{ij}^{(1)} = f_i(s - h^i)\delta_{ij}$.

Now

$$F_{ij} = \frac{\partial f_i(s - u^i)}{\partial s}\Lambda_{ij} g_j(s' - u^j), \qquad \Lambda_{ij} = -\Lambda_{ji}. \tag{6.9}$$

Assuming

$$K_{ij} = w_{ij}(s, u) f_j(s' - u^j), \tag{6.10}$$

we observe that integration in (2.7) can be done explicitly. One derives

$$w_{ij}(s, u) = \frac{\partial f_i}{\partial s}(s - u^i) \sum_k \Lambda_{ik}(T^{-1})_{kj}. \tag{6.11}$$

Here T^{-1} is the inverse matrix to

$$T_{ij} = \delta_{ij} + \frac{1}{2} f_i^2 \Lambda_{ij}, \tag{6.12}$$

$$Q_{ij} = w_{ij}(s, u) f_j(s - u^j). \tag{6.13}$$

Now we can put

$$H_i(s, u) = \phi(s - u^i) + \sum_j w_{ij}(s, u) \int_s^\infty f_j(s' - u^j)\phi_j(s' - u^j)\,ds'. \tag{6.14}$$

Here $\phi_i(\xi)$ are n arbitrary functions of one variable. Expression (6.14) presents the simplest explicit solution of Lamé equations, while (6.13) gives exact solutions of systems (1.24) and (1.25). A different choice of $\phi_i(\xi)$ provides the Combescure covariance.

In the simplest case $n = 3$,

$$w_{ij} = -\frac{1}{\Delta}\frac{\partial f_i}{\partial u^i}\left(\Lambda_{ij} + \frac{1}{2}\Lambda_{ik}\Lambda_{jk} f_k^2\right), \qquad k \neq i, j. \tag{6.15}$$

Here

$$\Delta = \det \|T_{ij}\| = 1 + \frac{1}{4}\sum_{i<j} \Lambda_{ij}^2 f_i^2 f_j^2. \tag{6.16}$$

Now one can change $f_i \to N f_i$ and put $N \to \infty$. Q_{ij} remains finite in this limit. Now

$$Q_{ij} \to -2\frac{\partial f_i}{\partial u^i} f_i \Lambda_{ik}\Lambda_{jk} f_k^2 \left(\sum_{i<j} \Lambda_{ij}^2 f_i^2 f_j^2\right)^{-1}. \tag{6.17}$$

Let us denote

$$\Psi_k = (\Lambda_{ik}\Lambda_{jk} f_k)^{-1}, \qquad i \neq j \neq k, \quad i<j. \tag{6.18}$$

Expression (6.17) can be rewritten as

$$Q_{ij} = -\frac{2}{A}\frac{\partial \Psi_i}{\partial u^i}\Psi_j, \qquad A = \sum_p \Psi_p^2. \tag{6.19}$$

Expression (6.19) can be interpreted as follows. Suppose an n-orthogonal system is a Cartesian system of orthogonal planes. Now

$$Q_{ik} = 0, \qquad H_i = \Psi_i(u^i),$$

and Ψ_i is an arbitrary function of one variable.

Comparing (6.19) to (1.30), one can see that we constructed the Ribaucour transformation against the simplest Cartesian background. The entire procedure described above is dressing on the Cartesian background.

Indeed, at $s \to \infty$ we have $Q_{in} \to 0$, and our n-orthogonal coordinate system goes to a system of orthogonal hyperplanes.

7. Dressing on an arbitrary background. Suppose now that the array $H_{0i}(u)$ satisfies both Lamé equations (1.12) and (1.14). To realize the dressing procedure, starting from this solution, one has to find Q_{0ij} by (3.25) and extend it to all s by (3.26). According to Section 3, any solutions of system (3.27) give a solution of (1.12). To also satisfy system (1.14), one has to find a generalization of differential reductions (5.2) and (6.1) compatible with (3.27). The answer is given by the following theorem.

THEOREM 7.1. *The dressing function $F(s, s', u)$, satisfying condition (3.27),*

$$\frac{\partial F}{\partial u^i} + I_i\frac{\partial F}{\partial s} + \frac{\partial F}{\partial s'} I_i + [I_i, Q_0(s)]F - F[I_i, Q_0(s')] = 0, \tag{7.1}$$

gives the solution of system (1.14) if it satisfies also the differential reduction

$$\frac{\partial F(s,s',u)}{\partial s'}+\frac{\partial F^{tr}(s',s,u)}{\partial s}+F(s,s',u)[Q_0(u,s')-Q^{tr}(u,s')]$$
$$[Q_0(u,s)-Q_0^{tr}(u,s)]F^{tr}(s',s)=0. \tag{7.2}$$

To prove the theorem, one must first check the compatability of (3.27) and (7.1). It is enough to apply the operator D_i to (7.1) and put the result zero in virtue of (3.27) and (7.1). It imposes on Q_0 the equation

$$\frac{\partial}{\partial u^k}(Q_0-Q_0^{tr})+I_k\frac{\partial Q_0}{\partial s}-\frac{\partial Q_0^{tr}}{\partial s}I_k$$
$$=Q_0^{tr}I_kQ_o^{tr}+Q_0I_kQ_0-Q_0Q_0^{tr}I_k+I_kQ_0Q_0^{tr}+(Q_0^{tr})^2I_k-I_kQ_0^2. \tag{7.3}$$

Put $Q_0=Q_{0ij}$ in (7.2). Condition (7.2) is antisymmetric and satisfied automatically for $i=j$. Let $i\neq j\neq k$. Now (7.2) is satisfied due to the fact that Q_{0ij} satisfies equation (3.3). Let $i\neq k=j$. Then

$$\frac{\partial}{\partial u^k}(Q_{0ik}-Q_{0ki})+\frac{\partial Q_{0ki}}{\partial s}-Q_{0kk}(Q_{0ki}-Q_{0ik})-\sum_l Q_{0kl}Q_{0il}+\sum_l Q_{0kl}Q_{0li}$$
$$=-\sum_{l\neq k\neq j}Q_{0kl}Q_{0il}+\sum_{l\neq k\neq j}Q_{0kl}Q_{0li}=-\sum_{l\neq k\neq j}Q_{0kl}Q_{0il}+\sum_{l\neq k\neq j}\frac{\partial Q_{0ki}}{\partial u^l} \tag{7.4}$$

or

$$\frac{\partial Q_{0ik}}{\partial u^k}+\frac{\partial Q_{0ki}}{\partial s}-\sum_{l\neq k}\frac{\partial Q_{0ki}}{\partial u^l}+\sum_{l\neq k\neq i}Q_{0kl}Q_{0il}=0. \tag{7.5}$$

Now, using condition (3.9),

$$\frac{\partial Q_{0ki}}{\partial s}=\sum_l\frac{\partial Q_{0ki}}{\partial u^l}, \tag{7.6}$$

one gets

$$\frac{\partial Q_{0ik}}{\partial u^k}+\frac{\partial Q_{0ki}}{\partial u^i}+\sum_{l\neq k\neq i}Q_{0kl}Q_{0il}=0. \tag{7.7}$$

Equation (7.3) is identical to (6.4).

The rest of the proof is straightforward. Due to the results of Section 5, on the diagonal $s = s'$,

$$\left(\frac{\partial K_{ij}(s,s')}{\partial s'} + \frac{\partial K_{ji}(s',s)}{\partial s}\right)\Bigg|_{s=s'} + \sum_l (K_{il}(Q_{0lj} - Q_{0jl} - (Q_{0il} - Q_{0li})K_{jl})$$
$$= \sum_l K_{il}K_{jl}. \tag{7.8}$$

From (3.27), one can find that $K(s,s')$ satisfies the equation

$$\frac{\partial K}{\partial u^j} + I_j \frac{\partial K}{\partial s} + \frac{\partial K}{\partial s'} I_j + [I_j Q(s,u)]K - K[I_j Q_0(u,s)] = 0, \tag{7.9}$$

where

$$Q = Q_0 + K(s,s).$$

From (7.9), one derives

$$\frac{\partial K_{ij}(s,s')}{\partial u^j} + \frac{\partial K_{ij}(s,s')}{\partial s'} - Q_{ij}(s)K_{ij}(s,s') + \sum_{l\neq j} K_{il}(s,s')Q_{0lj}(s') = 0. \tag{7.10}$$

Permutating $i \leftrightarrow j$ and $s \leftrightarrow s'$ in (7.10), we have

$$\frac{\partial K_{ji}(s,s')}{\partial u^i} + \frac{\partial K_{ji}(s',s)}{\partial s} - Q_{ji}(s')K_{ii}(s',s) + \sum_{l\neq i} K_{jl}(s',s)Q_{0li}(s) = 0. \tag{7.11}$$

Now we can put $s = s'$ in (7.11) and (7.10). Combining then (7.9), (7.11), (7.10), and (7.7), after elementary calculations one receives

$$\frac{\partial Q_{ij}}{\partial u^j} + \frac{\partial Q_{ji}}{\partial u^i} + \sum_{l\neq i\neq j} Q_{il}Q_{jl} = 0. \tag{7.12}$$

This accomplishes the proof. We mention that

$$F = \Psi_0 F_0 \Psi_0^{-1}$$

is a common solution of equations (3.27) and (7.1) if F_0 realizes dressing against a Cartesian background, Ψ_0 is orthogonal,

$$\Psi_0^{-1} = \Psi_0^{tr},$$

and

$$\frac{\partial \Psi_0}{\partial s} = (Q_0 - Q_0^{tr})\Psi_0. \tag{7.13}$$

It is unclear if Ψ_o can be chosen as orthogonal for a general case.

8. Egorov's metrics. Let us find what kind of algebraic reductions are possible in the equation describing n-orthogonal systems. According to Section 4, all algebraic reductions in a space of diagonal curvature are given by (4.20). Substituting (4.21) to (6.4) and using (3.3) yields

$$\begin{aligned}
&\frac{\partial}{\partial u^k} Q_{ik} + \varepsilon_k(u^k)\frac{\partial}{\partial u^i}\frac{1}{\partial \varepsilon_i(u^i)} Q_{ik} + \sum_{l \neq i,n} Q_{il}Q_{kl} \\
&= \left(\frac{\partial}{\partial u^k} + \varepsilon_k \frac{\partial}{\partial u^i}\frac{1}{\partial \varepsilon_i}\right) Q_{ik} + \varepsilon_k \sum \frac{1}{\varepsilon_l} Q_{il} Q_{lk} \\
&= \left(\frac{\partial}{\partial u^k} + \varepsilon_k \frac{\partial}{\partial u^i}\frac{1}{\varepsilon_i} + \varepsilon_k \sum \frac{1}{\varepsilon_l}\frac{\partial}{\partial u^l}\right) Q_{ik} = 0.
\end{aligned} \tag{8.1}$$

Replacing $i \leftrightarrow k$ and using (4.21), we find

$$\left(\frac{\partial}{\partial u^i} + \varepsilon_i \frac{\partial}{\partial u^k}\frac{1}{\varepsilon_k} + \varepsilon_i \sum \frac{1}{\varepsilon_l}\frac{1}{\partial u_l}\right)\frac{\varepsilon_k}{\varepsilon_i} Q_{ik} = 0. \tag{8.2}$$

Comparing (8.1) and (8.2), we derive

$$\varepsilon_i = \varepsilon_k = const.$$

Then the only possible algebraic reduction is

$$Q_{ik} = Q_{ki}. \tag{8.3}$$

In this case,

$$\frac{\partial Q_{ik}}{\partial s} = 0, \qquad \sum_l \frac{\partial Q_{ik}}{\partial u^l} = 0. \tag{8.4}$$

Reduction (8.3) defines so-called Egorov's metrics. In this case,

$$H_i^2 = \frac{\partial \Lambda}{\partial u^i} \tag{8.5}$$

and

$$\Omega_{ik} = \frac{1}{2}\frac{\partial^2 \Lambda}{\partial u^i \partial u^k}\left(\frac{\partial \Lambda}{\partial u^i}\frac{\partial \Lambda}{\partial u^k}\right)^{-1/2}. \tag{8.6}$$

Here $\Lambda(u)$ is a scalar function, satisfying equation (4.16) together with the system of equations (8.4). The dressing function F in Egorov's case satisfies condition

$$f_{ij}(\xi,\eta) = f_{ji}(\eta,\xi),$$

$$\frac{\partial f_{ij}(\xi,\eta)}{\partial \eta} + \frac{\partial f_{ji}(\eta,\xi)}{\partial \xi} = 0,$$

or

$$\left(\frac{\partial}{\partial \eta} + \frac{\partial}{\partial \xi}\right) f_{ij}(\xi,\eta) = 0. \tag{8.7}$$

Finally, we have

$$f_{ij} = f_{ij}(\xi - \eta)$$

or

$$f_{ij} = f_{ji} = f_{ij}(s - s' - u^i + u^j). \tag{8.8}$$

We can assume $F_{ii} = 0$. So Egorov's metrics are parametrized by $n(n-1)/2$ real functions of one variable.

In Egorov's case, the problem of n-orthogonal coordinate systems is reduced to the following system of first-order equations:

$$\frac{\partial Q_{ij}}{\partial u^k} = Q_{ik}Q_{kj}, \qquad Q_{ki} = Q_{ik},$$

$$\sum_l \frac{\partial Q_{ik}}{\partial u^l} = 0 \tag{8.9}$$

In the case $n = 3$, it is a system of three equations

$$\frac{\partial Q_{12}}{\partial u^1} + \frac{\partial Q_{12}}{\partial u^2} = -Q_{13}Q_{23},$$

$$\frac{\partial Q_{13}}{\partial u^2} = Q_{12}Q_{23}, \tag{8.10}$$

$$\frac{\partial Q_{23}}{\partial u^1} = Q_{12}Q_{13}.$$

This is the classical system of "three-waves" in the case of exact resonance (Q_{ij} are real). It has standard solitonic solutions (see, for instance, [16]). In a general case, (8.9) is a system of n-waves (actually $n(n-1)/2!$). This fact was established first by Dubrovin [9].

In Egorov's case, the kernel F in the integral equation (2.7) depends only on the difference $s - s'$. So it belongs to the Wiener-Hopf class, and its solution is equivalent to the solution of a certain matrix Riemann-Hilbert's problem. Now $Q = \lim_{s\to s'} K(s, s')$ does not depend on s.

One can use Egorov's metrics as a background for dressing. Now Q_0 in (3.27) does not depend on s, and equation (1.2) is reduced to (6.1). The result of dressing is Egorov's metrics itself if the dressing function is symmetric.

9. Pseudo-Euclidean metrics. All the results obtained above can be easily extended to n-orthogonal, curvilinear coordinate systems in $(p+q)$-dimensional flat pseudo-Euclidean space $R^{p,q}$ with the metrics

$$ds^2 = \sum_{i=1}^{p+q} \varepsilon_i (du^i)^2. \tag{9.1}$$

Here

$$\begin{aligned} &\varepsilon_i = -1, \quad i = 1, \dots, p, \\ &\varepsilon_i = 1, \quad i = p+1, \dots, p+q. \end{aligned} \tag{9.2}$$

To find the equations describing n-orthogonal metrics, one can use the following formal trick. First we can consider H_i, U_i, Q_{ij} complex. Then we return to real numbers, assuming

$$\begin{aligned} &u^k \to iu^k \quad Q_{jk} \to -iQ_{jk}, \qquad k = 1, \dots, p, \\ &u^k \to u^k \quad Q_{jk} \to Q_{jk}, \qquad k = p+1, \dots, p+q. \end{aligned}$$

Now

$$Q_{jk} = \varepsilon_k \frac{\partial H_j}{H_k \partial U_k}. \tag{9.3}$$

System (3.3) is invariant with respect to transformation (9.3). System (6.4), after the transformation, takes the form

$$\varepsilon_j \frac{\partial Q_{ij}}{\partial u^j} + \varepsilon_i \frac{\partial Q_{ji}}{\partial u^i} + \sum_{l \neq i,j} Q_{il} Q_{jl} \varepsilon_l = 0. \tag{9.4}$$

As far as system (3.3) is untouched, the dressing is realized by a matrix function F, satisfying equation (2.8), and having a form (3.8). The only difference with the Euclidean case is the differential reduction. It must be taken in the form

$$\frac{\partial F(s,s')}{\partial s'}R + R\frac{\partial F^{tr}(s',S)}{\partial s} = 0. \tag{9.5}$$

Here R is the diagonal matrix $R = diag\, \varepsilon_i$.

Indeed, from (5.24), one gets

$$\left(\frac{\partial K_{ij}(s,s')}{\partial s'}\varepsilon_j + \varepsilon_i \frac{\partial K_{ji}(s',s)}{\partial s}\right)_{s=s'} = \sum_l K_{il}K_{jl}\varepsilon_l. \tag{9.6}$$

Now, expressing derivatives by s' and s in (9.7) from (3.21), we get system (9.4). Reduction (9.4) realizes the dressing on a Cartesian background. To perform the dressing on an arbitrary background, one must change the reduction to

$$\begin{aligned}&\frac{\partial F(s,s')}{\partial s'}R + R\frac{\partial F^{tr}(s',s)}{\partial s} + F(s,s')\big(RQ_0(u,s') - Q_0^{tr}(u,s')R\big)\\ &\quad - \big(RQ_0(u,s) - Q_0^{tr}(u,s)R\big)F^{tr}(s',s) = 0.\end{aligned} \tag{9.7}$$

Systems (3.3), (9.4), and (9.8) allow the algebraic reduction

$$RQ = Q^{tr}R$$

or

$$\varepsilon_i Q_{ij} = \varepsilon_j Q_{ji}. \tag{9.8}$$

From (9.4), now one derives

$$\frac{\partial H_j}{H_k \partial u^k} = \frac{\partial H_k}{H_j \partial u^j}$$

and

$$H_i^2 = \frac{\partial \Phi}{\partial u^i}.$$

This is Egorov's metrics in the pseudo-Euclidean space.

10. Conclusion. We see that the Lamé equation, describing n-orthogonal, curvilinear coordinate systems in flat Euclidian and pseudo-Euclidian spaces, can be solved efficiently by using the simplest known version of the dressing method. By solving these equations, we achieve also a description of integrable Hamiltonian systems of hydrodynamic type, together with their symmetries. So far, the problem of embedding obtained metrics to the coordinate space (finding $x^i(u)$), as well as the equivalent problem of constructing Hamiltonians and conserving quantities for the corresponding system of hydrodynamic type, is unsolved. We hope to solve this problem by using more sophisticated versions of the dressing method, based on implementation of the $\bar{\partial}$-problem on the complex plane [7]. It is a subject of part II of this paper. Here our results can be generalized in several natural ways. Let us outline the most obvious directions of this generalization and the program of future research. The following problems can be solved in the near future.

(1) The exploration of highest differential reductions, allowed by the systems of the Lamé equation. These reductions are

$$L\left(\frac{\partial}{\partial s'}\right)F(s,s') = L^{+}\left(\frac{\partial}{\partial s}\right)F^{tr}(s',s). \tag{10.1}$$

Here

$$L = \sum_{n=0}^{N} a_n \frac{\partial^{2n}}{\partial s^{2n}}, \qquad L^{+} = L,$$

or

$$L = \sum_{n=1}^{N} a_n \frac{\partial^{2n+1}}{\partial s^{2n+1}}, \qquad L^{+} = -L,$$

are some operators with constant scalar coefficients. Reductions (10.1) are further generalizations of Egorov's reductions. They impose on Q complicated systems of nonlinear differential relations, which deserve to be studied.

(2) The transition to other associative algebras A. The algebra of complex-valued matrices is the closest natural object. Other interesting objects are infinite-dimensional algebras of integral and differential operators. Reductions (both algebraic and differential) in such algebras can generate very interesting new classes of integrable systems.

(3) All the developed techniques can be extended to the noncommutative version of the Lamé equations (see [7]):

$$\frac{\partial^2 H_i}{\partial u^l \partial u^m} = \frac{\partial H_l}{\partial u^m} H_l^{-1} \frac{\partial H_i}{\partial u^l} + \frac{\partial H_m}{\partial u^l} H_m^{-1} \frac{\partial H_i}{\partial u^m}. \tag{10.2}$$

Here H_i are noncommuting matrices. The classification of reduction in system (10.2) is especially interesting.

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