Quasi-two Dimensional Hydrodynamics and Interaction of Vortex Tubes

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1 Introduction

This paper is long overdue. The most of the results presented here were obtained in 1986-87. Just a small portion of them (the equations for dynamics of the pair of counter-rotating vortices and their self-similar solutions) were published in time - in 1988 [1]. The publication was very brief and did not include any details of the calculations.

Nevertheless, it was noticed and then generously cited by R. Klein, A. Maida and K. Damodaran [2]. In this publication I would like to express my gratitude to these authors who reobtained an essential part of the results published below. In spite of the fact that some of the results published below (equations for the systems of almost parallel vortices) could be found in their article, I believe that my paper deserves to be published. Some of the results presented here are completely new, and, which is more important, the methodology published here is completely different from one used in [2].

In this article we develop a systematic approach to stydu stationary and nonstationary flows of ideal incompressivle fluid under assumption that the gradients in one preferred direction z are much less than the gradients in the orthogonal plane. Such flows could be called quazi-two dimentional. There are two motivations for paying a special attention to this class of fluid motion.

One is connected with the classical problem of the "blow-up" in the Euler equation. According to the most plausible scenario, (see, for instance, [3]), in the point of blow-up the vorticity becomes infinite. As far as vorticity is a vector, this assumption presumes that the flow near the blow-up point is almost two-dimentional and the velocity field is concentrated mostly in the plane orthogonal to the vorticity vector. An elaboration of this regular tool for description of this type of flow looks very timely.

Another motivation is the vortex dynamics. This is a subject which has a chance to become the backbone of the future theory of turbulence. Probably, there is no way to explain qualitatively and quantitatively the fundamental phenomenon of intermittency but a careful study of the dynamics of the vortex tubes or their systems in a real 3-dimentional nonstationary flow. "Vortices are the sinews of turbulence" said K. Moffatt (look at his lecture on the Seventh European Turbulence Conference [4]).

2 Quasi-two dimensional Hydrodynamics

Let us consider the dynamics of incompressible and inviscous fluid. We will describe the fluid motion by means of two stream functions Ψ and Φ. We put:

$$v_x = \Psi_y + \Phi_{xz}$$

$$
\begin{aligned}
v_y &= -\Psi_x + \Phi_{yz} \\
v_z &= -\Delta_\perp \Phi = -\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)\, \Phi
\end{aligned}
\tag{2.1}
$$

The condition of incompressibility $div\, v = 0$ is satisfied in virtue of (1) automatically. The components of the vorticity we present as follow:

$$
\operatorname{curl} \vec{v} = \Omega_1\, \vec{i} + \Omega_2\, \vec{j} + \Omega_3\, \vec{k} \tag{2.2}
$$

Here

$$
\Omega_1 = \Psi_{xz} - \frac{\partial}{\partial y}\, \Delta\, \Phi \tag{2.3}
$$

$$
\Omega_2 = \Psi_{yz} + \frac{\partial}{\partial x}\, \Delta\, \Phi \tag{2.4}
$$

$$
\Omega_3 = -\Delta_\perp \Psi \tag{2.5}
$$

The components of the vorticity satisfies the equation

$$
\frac{\partial \Omega_i}{\partial t} + (v\, \nabla)\, \Omega_i = (\Omega\, \Delta)\, v_i \tag{2.6}
$$

Putting in (6) $i = 3$ one can easily obtain the equation for Ψ:

$$
\begin{aligned}
-\frac{\partial}{\partial t}\Delta_\perp \Psi + \Big\{\Psi, \Delta_\perp \Psi\Big\} &= \operatorname{div}_\perp\Big(\Delta_\perp \Psi\, \nabla_\perp \Phi_z\Big) - \\
&- \operatorname{div}_\perp\Big(\Delta_\perp \Phi\, \nabla_\perp \Psi_z\Big) + \Big\{\Delta_\perp \Phi,\, \Phi_{zz}\Big\}
\end{aligned}
\tag{2.7}
$$

Here and further

$$
\Big\{A, B\Big\} = A_x\, B_y - A_y\, B_x
$$

To find the equation for Φ one can notice that the expression

$$
S = \frac{\partial \Omega_1}{\partial y} - \frac{\partial \Omega_2}{\partial x} = -\Delta_\perp\, \Delta \Phi \tag{2.8}
$$

does not include Ψ.

From (6), putting $i = 1, 2$ one can calculate dS/dt. Cumbersome calculations leads to the following result:

$$
\begin{aligned}
\Delta_\perp\Big(- \Delta\, \Phi_t + \{\Psi,\, \Delta \Phi\}\Big) &= \frac{\partial}{\partial z}\Big\{\frac{1}{2}\Delta_\perp (\nabla_\perp \Psi)^2 - \operatorname{div} \Delta_\perp \Psi\, \nabla_\perp \Psi\Big\} \\
&+ \Delta_\perp \{\Phi_z, \Psi_z\} - \frac{\partial}{\partial z}\{\Delta_\perp \Phi,\, \Phi_z\} + \\
&+ \Delta_\perp (\nabla_\perp \Phi_z,\, \nabla_\perp\, \Delta \Phi) - \frac{\partial}{\partial z} \operatorname{div} \Delta_\perp\, \Phi\, \nabla_\perp\, \Delta \Phi
\end{aligned}
\tag{2.9}
$$

The system of equations (7), (9) is just a little bit bizzare form of the Euler equations for incompressible fluid. They can be transformed to the Navier-Stokes system by the simple change

$$
\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - \nu\, \Delta, \tag{2.10}
$$

where ν is a viscosity coefficient.

Equations (7), (9) preserve the integral of energy

$$E = \frac{1}{2}\int \Big\{(\nabla_\perp \Psi)^2 + (\Delta_\perp \Phi)^2\Big\} dr\, dz\,; \qquad \frac{dE}{dt} = 0. \tag{2.11}$$

The new form of Euler's equation is good for description of quasi-two dimensional stationary and nonstationary flows, when

$$\frac{\partial}{\partial z} \ll \frac{\partial}{\partial x},\ \frac{\partial}{\partial y}$$

Let us put for the beginning

$$\frac{\partial \Psi}{\partial z} = 0; \quad \frac{\partial \Phi}{\partial z} = 0.$$

Then

$$\frac{\partial}{\partial t}\Delta_\perp \Psi = \Big\{\Psi,\, \Delta_\perp \Psi\Big\} \tag{2.12}$$

$$\frac{\partial}{\partial t}\Delta_\perp \Phi = \Big\{\Psi,\, \Delta_\perp \Phi\Big\} \tag{2.13}$$

Equation (12) is a standard $2-D$ Euler equation for incompressible fluid. The passive scalar equation (13) describes a transport of z - the independent vertical velocity.

In the next step we keep in (7), (9) the terms linear in $\partial/\partial z$. One obtains the following system:

$$\frac{\partial}{\partial t}\Delta_\perp \Psi - \{\Psi,\, \Delta_\perp \Psi\} = \mathrm{div}_\perp\Big[\Delta_\perp \Phi \nabla_\perp \Psi_z - \Delta_\perp \Psi \nabla_\perp \Phi_z\Big] \tag{2.14}$$

$$\Delta_\perp\Big(- \Delta_\perp \Phi_t + \{\Psi, \Delta_\perp \Phi\}\Big) = \frac{\partial}{\partial z}\Big[\frac{1}{2}\Delta_\perp (\nabla_\perp \Psi)^2 - \mathrm{div}_\perp \Delta_\perp \Psi \nabla_\perp \Psi\Big]$$

$$\Delta_\perp(\nabla_\perp \Phi_z,\, \nabla_\perp \Delta_\perp \Phi) - \frac{\partial}{\partial z}\, div_\perp \Delta_\perp \Phi \nabla_\perp \Delta_\perp \Phi \tag{2.15}$$

The system (14), (15) describes a generic almost two-dimensional nonstationary flow of the imcompressible fluid. If one assumes that the vertical velocity is small ($v_z \ll v_x, v_y$), the equation (15) can be simplified up to the form:

$$\Delta_\perp\Big(- \Delta_\perp \Phi_t + \{\Psi, \Delta_\perp \Phi\}\Big) = \frac{\partial}{\partial z}\Big[\frac{1}{2}\Delta_\perp (\nabla_\perp \Psi)^2 - \mathrm{div}_\perp \Delta_\perp \Psi \nabla_\perp \Psi\Big] \tag{2.16}$$

This is remarkable that the both systems (14), (15) and (14), (16) preserve the energy integral (11). The both are Hamiltonian systems, having the same Hamiltonian (11), but different Poisson's structures. In this article we will not discuss this interesting question in more details.

It is important to explore how far the approximate systems (14), (15) and (14), (16) differ from the exact Euler equation. One can use the axial symmetric case as a test. In this case

$$v_\phi = -\frac{\partial \Psi}{\partial r}$$

$$v_z = -\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} \tag{2.17}$$

$$v_r = \Phi_{rz} \tag{2.18}$$

Plugging (17) into (14) we find that v_ϕ satisfies the exact equation

$$\frac{\partial v_\phi}{\partial t} + v_r\,\frac{\partial v_\phi}{\partial r} + v_z\,\frac{\partial v_\phi}{\partial z} + \frac{v_r\, v_\phi}{r} = 0 \tag{2.19}$$

Equations (15) and (16) transform to the following reduced equations:

$$\frac{\partial}{\partial r}\left(\frac{\partial v_z}{\partial t} + v_r\,\frac{\partial v_z}{\partial r} + v_z\,\frac{\partial v_z}{\partial z}\right) = \frac{1}{r}\,\frac{\partial}{\partial z}\,v_\phi^2 \tag{2.20}$$

$$\frac{\partial}{\partial r}\,\frac{\partial v_z}{\partial t} = \frac{1}{r}\,\frac{\partial}{\partial z}\,v_\phi^2 \tag{2.21}$$

At the same time the exact Euler equation after excluding the pressure takes a form

$$\begin{aligned}&\frac{\partial}{\partial r}\left(\frac{\partial v_z}{\partial t} + v_r\,\frac{\partial v_z}{\partial r} + v_z\,\frac{\partial v_z}{\partial z}\right) - \\ &\frac{\partial}{\partial z}\left(\frac{\partial v_r}{\partial t} + v_r\,\frac{\partial v_r}{\partial r} + v_z\,\frac{\partial v_r}{\partial z}\right) = \frac{1}{r}\,\frac{\partial}{\partial z}\,v_\phi^2\end{aligned} \tag{2.22}$$

There is one more modification of the quasi-two dimensional Hydrodynamics. Instead of (2.16) one can use a more exact equation

$$\Delta_\perp\Big(-\Delta\Phi_t + \{\Psi, \Delta\Phi\}\Big) = \frac{\partial}{\partial z}\left[\frac{1}{2}\,\Delta_\perp(\nabla_\perp\Psi)^2 - \mathrm{div}\Delta_\perp\Psi\nabla_\perp\Psi\right] \tag{2.23}$$

In the axial symmetric case it leads to the following modification of equation (2.22):

$$\frac{\partial}{\partial r}\,\frac{\partial v_z}{\partial t} - \frac{\partial}{\partial z}\,\frac{\partial v_r}{\partial z} = \frac{1}{r}\,\frac{\partial}{\partial z}\,v_\phi^2 \tag{2.24}$$

The difference between $\Delta\Phi$ and $\Delta_\perp\Phi$

$$\Delta\Phi - \Delta_\perp\Phi = \frac{\partial^2\Phi}{\partial z^2}$$

is of the second order in $\partial/\partial z$ and looks to be negligible. We will see further that this is not quite right. In some cases the approximation (2.16) is too crude, and more exact equation (2.23) should be used to obtain the correct results.

Comparison of (2.19) and (2.21) demonstrates the fact that in the quasi-two dimensional equation one takes into account only the lowest order in $\partial/\partial z$.

3 Dynamics of the isolated vortex tube

In this chapter we apply the derived quasi-two dimensional Hydrodynamic equations for description of the dynamics of a single vortex tube. We will assume that the core of the tube is axially symmetric and small with respect to its characteristic curvature radius.

Let the central line of the tube is given by the formula

$$\begin{aligned}\chi &= \chi_0 = a(t, z) \\ y &= y_0 = b(t, x)\end{aligned} \tag{3.1}$$

One can introduce polar coordinate in the coordinate frame attached with the vortex line

$$\chi = a + r\cos\phi$$
$$y = b + r\sin\phi \tag{3.2}$$

Now

$$\{A, B\} = \frac{1}{r}\Big(A_r\, B_\phi - A_\phi\, B_r\Big) \tag{3.3}$$

$$\frac{\partial}{\partial t} \to D_t = \frac{\partial}{\partial t} - (\dot a\sin\phi + \dot b\cos\phi)\frac{\partial}{\partial r} - \frac{1}{r}(\dot a\cos\phi - \dot b\sin\phi)\frac{\partial}{\partial\phi}$$
$$\frac{\partial}{\partial z} \to D_z = \frac{\partial}{\partial z} - (a'\sin\phi + b'\cos\phi)\frac{\partial}{\partial r} - \frac{1}{r}(a'\cos\phi - b'\sin\phi)\frac{\partial}{\partial\phi} \tag{3.4}$$

In the polar coordinates

$$\begin{aligned} v_\phi &= -\frac{\partial\Psi}{\partial r} + \frac{1}{r}\frac{\partial}{\partial\phi}D_z\,\Phi \\ v_r &= \frac{1}{r}\frac{\partial\Psi}{\partial\phi} + \frac{\partial}{\partial r}D_z\,\Phi \\ v_z &= -U = -\frac{1}{r}\frac{\partial}{\partial r}r\,\Phi - \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2} \end{aligned} \tag{3.5}$$

$$\begin{aligned} \Omega_z &= -\Omega = \Delta_\perp\Psi = -\frac{1}{r}\frac{\partial}{\partial r}r\,\Psi_r - \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\phi^2} \\ \Omega_r &= D_z\,\Psi_r - \frac{1}{r}\frac{\partial}{\partial\phi}\Delta_\perp\,\Phi \\ \Omega_\phi &= -\frac{1}{r}D_z\,\Psi_\phi + \frac{\partial}{\partial r}\Delta_\perp\Phi \end{aligned} \tag{3.6}$$

In new variables equation (2.14) can be rewritten as follow:

$$\begin{aligned} \frac{1}{r}\Big(\Psi_r\Omega_\phi - \Psi_\phi\Omega_r\Big) &= D_t\,\Delta_\perp\Psi - \frac{1}{r}\frac{\partial}{\partial r}r\left(U\frac{\partial}{\partial r}D_z\Psi - \Omega_z\frac{\partial}{\partial r}D_z\,\Phi\right) \\ &- \frac{1}{r}\frac{\partial}{\partial r}\left(U\frac{\partial}{\partial\phi}D_z\Psi - \Omega_z\frac{\partial}{\partial\phi}D_z\,\Phi\right) \end{aligned} \tag{3.7}$$

In this article we will use only simplified equation (2.16). In polar coordinates it reads

$$\begin{aligned} &\Delta_\perp\Big(\Psi_r\,U_\phi - \Psi_\phi\,U_r\Big) = \Delta_\perp\,D_t\,\Phi + \\ &D_z\left(\frac{1}{2r}\frac{\partial}{\partial r}\Psi_r^2 + \frac{1}{r^2}\Big(\Psi_{rr}\Psi_{\phi\phi} - \Psi_{r\phi}^2\Big) - \frac{1}{r^3}\Psi_\phi\,\Psi_{r\phi} - \frac{1}{r^4}\Psi_\phi^4\right) \end{aligned} \tag{3.8}$$

If $a = b = 0$, systems (3.6), (3.7) and (3.6), (3.8) have a trivial solution

$$\Psi = \Psi_0(r)\,; \quad \Phi = 0, \tag{3.9}$$

describing a solitary stationary vortex tube. In this case only the angular component of velocity exists

$$v_\phi = -\frac{\partial\Psi_0}{\partial r}, \tag{3.10}$$

while the vorticity is presented by only one vertical component

$$\Omega_3 = -\Omega_0 = \frac{1}{r}\frac{\partial}{\partial r}\, r\,\Psi_{0r}, \tag{3.11}$$

where $\Omega(r)$ is an arbitrary function. We will assume that this function has a finite support. In another words:

$$\Omega_0(r) \neq 0 \quad if \;\; 0 < r < \rho$$

$$\Omega_0(r) = 0 \quad if \quad r > \rho$$

Here ρ can be interpreted as a size of the tube's core.

Let us denote

$$\Gamma = -2\pi \int_0^\infty r\Omega_0(r)\, dr \tag{3.12}$$

This is a total vorticity of the tube. At $r \to \infty$:

$$\Psi_{0r} \to \frac{\Gamma}{2\pi r}, \quad \Psi_0 \simeq \frac{\Gamma}{2\pi}\ln r + c \tag{3.13}$$

The constant $c = c(z,t)$ is indefinite so far. In presence of a, b (3.8) is not any more an exact solution. One has to seek the solution in a form

$$\Psi = \Psi_0(r,z,t) + \Psi'(r,z,\phi,t) \tag{3.14}$$

$$\Phi = \Phi_0(r,z,t) + \Phi'(r,z,\phi,t) \tag{3.15}$$

Here Ψ', Φ' are periodic functions on phi:

$$\Psi' = \sum_{n\neq 0} \Psi_n\, e^{in\phi}, \quad \Psi_{-n} = \Psi_n^*$$

$$\Phi' = \sum_{n\neq 0} \Phi_n\, e^{in\phi}, \quad \Phi_{-n} = \Phi_n^* \tag{3.16}$$

Let us introduce the complex coordinate of the vortex line (the centre of the vortex tube) as

$$w = a + ib \tag{3.17}$$

The derivative $w' = \partial w/\partial z$ is a dimensionless parameter characterizing the angle between the tangent to the vortex line and the vertical axis. We assume that this angle is small,

$$w' \simeq \epsilon \ll 1. \tag{3.18}$$

Another small parameter,

$$\mu \sim \rho w'', \tag{3.19}$$

is a ratio of the size of the vortex line to its curvature radius. We assume that this parameter is small too,

$$\mu \ll 1. \tag{3.20}$$

All components of the Fourier series (3.15) are small and can be expanded in powers of ϵ and μ. The question about order of magnitudes averaged in angle stream functions Ψ_0 and Φ_0 is more delicate. In principle, a shape of tube described by the function $\Psi_0(r,z,t)$ can essentially depend on z.

In a long run this dependence can cause to development of intensive vertical flow, describing by a relative high value of Φ_0. But all these effects are out of a scope of this article. We will assume that Φ_0, as well as the difference $\Psi_0(r,z,t) - \Psi_0(r)$ are small, and in the first approximation can be neglected.

In this theory the leading terms in the expansions (3.15) are:

$$\Phi \simeq \Phi_1\, e^{i\phi} + \Phi_1^*\, e^{-i\phi} \tag{3.21}$$
$$\Psi' \simeq \Psi_1\, e^{i\phi} + \Psi_1^*\, e^{-i\phi} \tag{3.22}$$

In the terms of w, w^* the derivatives (3.4) can be rewritten as follow:

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + \frac{i}{2}\Big(\dot{w}\, e^{i\phi} - \dot{w}^*\, e^{-i\phi}\Big)\frac{\partial}{\partial r} - \frac{1}{2r}\Big(\dot{w}\, e^{i\phi} + \dot{w}^*\, e^{-i\phi}\Big)\frac{\partial}{\partial \phi} \\ D_z &= \frac{\partial}{\partial z} + \frac{i}{2}\Big(w'\, e^{i\phi} - w'^*\, e^{-i\phi}\Big)\frac{\partial}{\partial r} - \frac{1}{2r}\Big(w'\, e^{i\phi} + w'^*\, e^{-i\phi}\Big)\frac{\partial}{\partial \phi} \end{aligned} \tag{3.23}$$

Using (3.20), (3.21) one finds:

$$\begin{aligned} U &\simeq U_1\, e^{i\phi} + U_1^*\, e^{-i\phi} \\ \Omega' &\simeq \Omega_1\, e^{i\phi} + \Omega_1^*\, e^{-i\phi} \end{aligned} \tag{3.24}$$

Here

$$\begin{aligned} U_1 &= \left(\frac{1}{r}\frac{\partial}{\partial r}\, r\, \frac{\partial}{\partial r} - \frac{1}{r^2}\right)\Phi_1 = L\,\Phi_1 \\ \Omega_1 &= \left(\frac{1}{r}\frac{\partial}{\partial r}\, r\, \frac{\partial}{\partial r} - \frac{1}{r^2}\right)\Psi_1 = L\,\Psi_1 \end{aligned} \tag{3.25}$$

and

$$L = \frac{1}{r}\frac{\partial}{\partial r}\, r\, \frac{\partial}{\partial r} - \frac{1}{r^2}$$

In the leading order in ϵ, μ the equation (3.7) can be rewritten as follow:

$$L\,\frac{1}{r}\,\Psi_{0r}\, U_1 = \frac{1}{4}\, w'\, \frac{1}{r}\frac{\partial}{\partial r}\, r\, \frac{\partial}{\partial r}\,\Psi_{0r}^2 \tag{3.26}$$

This equation can be integrated twice. The result is

$$U_1 = \frac{1}{2}\, w'\, \Psi_{0r} \tag{3.27}$$

This result has a very simple physical explanation. In the leading order the vertical component of the velocity is proportional to the local angular velocity. It is just a result of tilting the planes where the fluid rotates causing by bending of the vortex line.

This point need to be commented. Actually, formula (3.21) cannot be valid for very large values of r. It is correct only for $r \ll l$, where l is a characteristic scale along the vortex line. To find v_z at $r \geq l$ one should use the more exact equation (2.23). Now one obtain instead of (3.26):

$$-\Delta\Phi = \frac{1}{2}\, w'\, \Psi_{0r} \tag{3.28}$$

This equation should be accomplished by the boundary condition

$$\Phi \to 0 \qquad \text{at} \quad r \to \infty$$

The solution of (3.28) decays exponentially at $r \to \infty$:

$$U_1 \simeq l^{\frac{r}{l}} \qquad r \to \infty$$

This fact makes possible to perform a choice of an unknown constant $c = c(z,t)$. The explicit formula for $c(z,t)$ is not important. One have to introduce a new stream function $\tilde{\Psi}_0(r)$ defined as follow:

$$\begin{array}{ll} \tilde{\Psi}_{0r}(r) = \Psi_{0r} & r < l \\ \tilde{\Psi}_0(r) \to 0 & r \to \infty \end{array} \tag{3.29}$$

A correct definition of u_1 is

$$u_1 = \frac{1}{2}\, w'\, \tilde{\Psi}_{0r} \tag{3.30}$$

By integrating of (3.28) one obtains

$$\Phi_1 = \frac{1}{2}\, w'\, \frac{1}{r} \int_0^r r\, \tilde{\Psi}_0(r)\, dr \tag{3.31}$$

Hence in the leading order $\Phi_1 \sim \epsilon$.

For finding Ψ_1 one should use equation (3.7). After simple transformation in the leading order one obtains

$$\begin{aligned} &-\frac{i}{r}\Big[\Psi_{0r}\, L\, \Psi_1 - \Psi_1\, L\, \Psi_{0r}\Big] = \\ &= \frac{i}{2}\, \dot{w}\, \frac{\partial}{\partial r}\, \Omega_0 + \frac{1}{2}\, w''\Big[- \frac{1}{r^2}\, \frac{\partial}{\partial r}\, r^2\, \Omega_0\, \Phi_1 + \frac{1}{r}\, \frac{\partial}{\partial r}\, r\, \Omega_0\, \tilde{\Psi}_0 \Big] \end{aligned} \tag{3.32}$$

One can multiply (3.28) to r^2 and integrate over r from zero to infinity. It is easy to check that the left hand vanishes. The right hand gives the equation describing the dynamics of the vortex line

$$i\, \dot{w} + \lambda\, w'' = 0 \tag{3.33}$$

Here

$$\lambda = -\frac{\pi}{\Gamma} \int_0^\infty r\, \Omega_0\, \tilde{\Psi}_0(r)\, dr \tag{3.34}$$

The vorticity Γ is defined by (3.11). Integrating by parts in (3.32) leads to the result

$$\begin{aligned} \lambda &= \frac{E}{\Gamma} \\ E &= 2\pi \int_0^\infty r\, \tilde{\Psi}_{0r}^2\, dr \end{aligned} \tag{3.35}$$

Integral (3.33) diverges logarithmically

$$E \simeq \ln L/\rho \tag{3.36}$$

The equation (5.31) is not a new one, of course. It can be obtained from so called "local induction approximation" (see, for instance [ ]).

The equation (3.31) can be solved analytically. The correction $\Psi_1 \sim \mu$, but the explicit expression for Ψ_1 is cumbersome and will not be used further in the paper.

Assuming $w \simeq e^{-i\Omega t+ikx}$ one finds that (3.32) describes the waves with the dispersion relation

$$\Omega = \lambda\, k^2, \tag{3.37}$$

actually $\lambda \simeq \Gamma \ln k\rho$. The waves are circularry polarized and rotate in the direction opposite to rotation of the fluid in the vortex tube.

Equation (3.32) is just the leading term in expansion of a more accurate nonlinear equation describing the dynamics of a free vortex line. The developed procedure makes possible to find this equation up to any desirable accuracy. These calculations will be published separately.

4 Vortex tube in an external flow

In this chapter we study a behavior of the vortex tube in a two-dimensional flow. We will assume that a characteristic scale of the flow is much bigger that the size of the tube core ρ.

In this case one can separate

$$\Psi = \Psi_0 + \Psi' + \tilde{\Psi} \tag{4.1}$$

Here Ψ_0 and Ψ' are defined from (3.13), while $\tilde{\Psi}$ is a stream function of the external flow.

In the leading order the most important contribution of the external flow appears in the equation (2.14), which should be replaced to the equation

$$\begin{aligned} \frac{\partial}{\partial t}\Delta_\perp \Psi \quad &- \quad \{\Psi, \Delta_\perp \Psi\} = \{\tilde{\Psi}, \Delta_\perp \Psi\} + \\ &+ \quad \mathrm{div}_\perp \Big[\Delta_\perp \Phi\, \Delta_\perp \Psi_z - \Delta_\perp \Psi\, \nabla_\perp \Phi_z\Big] \end{aligned} \tag{4.2}$$

Omitting the details, we just present the result of influence of the large scale flow. The equation (3.32) should be replaced by the nonlinear Schrodinger equation

$$i\dot{w} + \lambda w'' = i\, F(w, \bar{w}) \tag{4.3}$$

Here

$$F = \frac{\partial\tilde{\Psi}}{\partial y} - i\frac{\partial\tilde{\Psi}}{\partial x}\bigg|_{x-iy=w} = \frac{\partial\tilde{\Psi}}{\partial\bar{w}}\bigg|_{x-iy=w} \tag{4.4}$$

Equation (4.3) has a very clear physical meaning - it just describes the transport of the point vortex by the external flow having the velocity components

$$v_x = \frac{\partial\tilde{\Psi}}{\partial y}, \quad v_y = -\frac{\partial\tilde{\Psi}}{\partial x}$$

The system (4.3) is Hamiltonian. It can be written as follow

$$i\,\dot{w} = \frac{\delta H}{\delta\bar{\omega}} \tag{4.5}$$

Here

$$H = \int \Big\{\lambda|\omega'|^2 + \tilde{\Psi}(\omega, \bar{\omega})\Big\} dz \tag{4.6}$$

The function $\tilde{\Psi}$ is a stream function of the external flow. One can admit that it depends slowly on z and t.

We can mention now that

$$F = \bar{U}$$

Here $U = \tilde{v}_x - i\,\tilde{v}_y$ is a complex velocity of the external flow. Suppose, that the external flow is potential

$$\frac{\partial \tilde{v}_x}{\partial y} = \frac{\partial \tilde{v}_y}{\partial x}$$

In this case U is an analytic function on w. Hence, F is anti-analytic. In a potential case

$$F = F(\bar{w}). \tag{4.7}$$

If the external flow is axially symmetric, then

$$\begin{aligned} \tilde{\Psi} &= A(|w|^2) \\ F &= -i\,w\,A'(|w|^2) \end{aligned} \tag{4.8}$$

Equation (4.3) takes the form, which is standard for the nonlinear optics

$$i\,\dot{w} + \lambda w'' = w\,A'(|w|^2) \tag{4.9}$$

Equations (4.3), (4.9) may have solitonic solutions, which are very interesting from the physical view-point. Suppose, $\Gamma > 0$, $\lambda < 0$. By the rescaling

$$\lambda \frac{\partial^2}{\partial z^2} \to \frac{\partial^2}{\partial z^2}$$

one can transform the equation to the standard form, neglecting the logarithmic difference,

$$i\,\dot{w} + w'' - w\,A'(|w|^2) = 0 \tag{4.10}$$

Let us consider the Taylor expansion

$$-A'(\xi) = \alpha + \beta\xi + \cdots \qquad \xi \to 0 \tag{4.11}$$

Then we obtain the cubic Nonlinear Schrodinger equation

$$i\,\dot{w} + w'' + \Big(\alpha + \beta|w|^2\Big)w = 0 \tag{4.12}$$

The vorticity of the external flow

$$\Omega_3 = -\frac{1}{r}\frac{\partial}{\partial r}\,r\,\frac{\partial \tilde{\Psi}}{\partial r}$$

can be expanded at small r as follow

$$\Omega_3 \simeq 2(\alpha + 2\beta\,r^2) \tag{4.13}$$

Equation (4.12) is focusing, if $\beta > 0$, and defocusing, if $\beta < 0$. The sign of α defines the direction of rotation of the external flow. If $\alpha < 0$, this direction is opposite to the rotation of the vortex tube (the case of counter-rotation). For $\alpha > 0$ the signs of rotations are the same (the case of co-rotation). In a typical case the absolute maximum of external vorticity is posed in the centre at $r = 0$. It presumes $\alpha\beta < 0$.

Assuming that this combination is satisfied one can conclude that the vortex is described by the focusing NSL in the co-rotational case, while in the counter-rotational case one should use the defocusing NSL. This leads to another important conclusion: in the counter-rotational case the position of the vortex tube outside of the centre is stable, while in the co-rotational case it is unstable. Let us recall again that this is correct only if the absolute value of the external vorticity reaches the maximum at $r = 0$. If the absolute value of the vorticity grows with r, the situation is quite opposite.

One can easily study solitonic solutions of the equation (4), (9).

5 Interaction of vortex tubes

Let us study the interaction of a system of n almost parallel vortex tubes with vorticities $\Gamma_1, \ldots, \Gamma_n$. The system of vortex tubes can be described by the following systems of Nonlinear Schrodinger equations

$$i\,\dot{w}_k + \lambda_k\, w_k^{''} = -\sum_{l \neq k} \frac{\Gamma_l}{\bar{w}_k - \bar{w}_l} \tag{5.1}$$

(See [2]). The right hand of (5.1) is anti-analytic in all w_k.

It is interesting to study the interaction of two identical vorticies. We can put $n = 2$, $\Gamma_1 = \Gamma_2 = \Gamma$ and assume that the vorticies $\lambda_1 = \lambda_2$ are posed antisymmetrically with respect to the origin of coordinate. Thus, $\bar{\bar{w}}_2 = -\bar{\bar{w}}_1$, and equation (5.1) can be transformed to the following NSL equation

$$i\,\dot{w} + w_{zz} + \frac{w}{|w|^2} = 0 \tag{5.2}$$

Another interesting problem is interaction of two antiparallel vorticies. In this case $\Gamma_2 = -\Gamma_1 = -\Gamma$, $\lambda_1 = -\lambda_2$, and one can assume

$$w_2 = \bar{w}_1 = w$$

Equation (5.1) reads now (see [1])

$$i\,\dot{w} + w^{''} = \frac{1}{w - \bar{w}} \tag{5.3}$$

or

$$\begin{aligned} \dot{x} &= -y^{''} + \frac{1}{2y} \\ \dot{y} &= x^{''} \end{aligned} \tag{5.4}$$

One can study also the continual limit of system (5.1). Suppose, that we have a congruency of almost parallel vorticies. Each vortex is marked by two Lagrangian labels p, q. We can introduce $s = p + iq$ and consider

$$w = w(t, z, s, \bar{s}), \qquad \lambda = \lambda(s, \bar{s}), \qquad \Gamma = \Gamma(s, \bar{s})$$

Equation (5.1) can be replaced by the system

$$i\,\frac{\partial w}{\partial t} + \lambda(s, \bar{s})\,\frac{\partial^2 \omega}{\partial z^2} + \int \frac{\Gamma(s', \bar{s}')ds\, d\bar{s}'}{\bar{w}(s, \bar{s}) - \bar{w}(s', \bar{s}')} = 0 \tag{5.5}$$

Integration in (5.5) is understood as a principle value. We can interpret w as a displacement of a vortex from its initial position

$$w\Big|_{t=0} = s \tag{5.6}$$

6 Instability and collapse of traveling vortex pair

Equation (5.4) has a trivial solution

$$\begin{aligned} y &= a = const \\ x &= \frac{1}{2a}t \end{aligned} \tag{6.1}$$

describing the pair of antiparallel vortex tubes moving along the x axis.

In the moving frame

$$x' = x - \frac{1}{2a}t$$

System (5.4) reads

$$\begin{aligned} \dot{x} &= -y'' + \frac{1}{2y} - \frac{1}{2a}x' \\ \dot{y} &= x'' \end{aligned} \tag{6.2}$$

Linearization of (6.2)

$$\omega = x' + iy = a + xe^{i\Omega t + ipz}$$

leads to the result

$$\Omega^2 = -\frac{1}{2a^2}p^2 + p^4 \tag{6.3}$$

The counterrotating pair of vortices is unstable if

$$p^2 < \frac{1}{2a^2} \tag{6.4}$$

This is the so-called "Crow instability" [5]. It is interesting to study the nonlinear stage of this instability. One can look for the self-similar solutions of the system (5.4).

$$\begin{aligned} x &= \sqrt{t_0 - t}A\left(\frac{z}{\sqrt{t_o - t}}\right) \\ y &= \sqrt{t_0 - t}B\left(\frac{z}{\sqrt{t_o - t}}\right) \end{aligned} \tag{6.5}$$

A and B satisfy the equations

$$\begin{aligned} \frac{1}{2}(-A + xA') &= -B'' + \frac{1}{2B} \\ \frac{1}{2}(-B + xB') &= A'' \end{aligned} \tag{6.6}$$

Asymptotically as $z \to \infty$

$$A \to \alpha z \qquad\qquad B \to \beta z \tag{6.7}$$

at $z \to -\infty$

$$A \to -\alpha z \qquad B \to \beta z$$

Here α, β - some constants. Solution (6.5) describes the collapse. Vortex lines converge ($\|\omega\| \sim \sqrt{t_0 - t}$) and meet at $z = 0$ in the moment of time $t = t_0$. In this moment the asymptotic (6.7) is valid for all z, and the vortex tubes are the straight lines angled at $z = 0$. It is natural to make a conjecture that the collapse leads to the reconnection of the vortex tubes. But the system (6.4) is applicable only if the distance between the tubes is much greater than the size of their cores. ($\|\omega\| \gg \rho$). Numerical experiments [6] show that at $|w| \sim \rho$ vortex cores deform and lose their round shape. They become flat, and the process of collapse slows down.

The system (5.4) has another class of approximate solutions. Let us suppose that

$$y'' \ll \frac{1}{2y} \tag{6.8}$$

Then y satisfies the equation

$$\ddot{y} - \left(\frac{1}{y}\right)'' = 0 \tag{6.9}$$

This is the elliptic equation, the Cauchy problem for this equation is ill-posed. Anyway, it has a family of self-similar solutions:

$$\begin{aligned} x &= (t_0 - t)^{1-\alpha} F\left(\frac{z}{(t_0 - t)^{1-\alpha}}\right) \\ y &= (t_0 - t)^{\alpha} G\left(\frac{z}{(t_0 - t)^{1-\alpha}}\right) \end{aligned} \tag{6.10}$$

To satisfy the condition (6.8) one has to put

$$1 > \alpha > \frac{1}{2} \tag{6.11}$$

Solution (6.10) describes more slow collapse than (6.5).

7 Solitons on the co-rotating vortex pair

It was shown in section 5 that the co-rotating pair of identical vortices is described by equation (5.2). The solution

$$W = A e^{\frac{i}{A^2} t} \tag{7.12}$$

corresponds to the uniform rotation. This solution is stable. But equation (5.2) has interesting solitonic solutions ("dark solutions"). By separating the phase and the amplitude

$$W = A(z,t) e^{\frac{i}{2}\Phi} \tag{7.13}$$

One obtains the system

$$\begin{aligned} \frac{\partial}{\partial t} A^2 + \frac{\partial}{\partial z} A^2 \phi_z &= 0 \\ \frac{A}{2}\left(\frac{\partial \Phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \Phi}{\partial z}\right)^2\right) &= A_{zz} + \frac{1}{A} \end{aligned} \tag{7.14}$$

In the long-wave limit $A_{zz} \ll \frac{1}{A^2}$ system (7.3) transforms th the hyperbolic system

$$\begin{aligned} \frac{\partial}{\partial t}A^2 + \frac{\partial}{\partial z}A^2\Phi_z &= 0 \\ \frac{\partial \Phi}{\partial t} + \frac{1}{2}(\frac{\partial \Phi}{\partial z})^2 - \frac{1}{A^2} &= 0 \end{aligned} \quad (7.15)$$

This is the system of gas-dynamic equations with the exotic dependence of pressure on density ($P = \ln \rho$). It was derived independently by V. Rulan [7].

To find the solitonic solutions one has to put

$$\begin{aligned} \frac{\partial}{\partial t} &= c\frac{\partial}{\partial z} \\ \Phi &\simeq e^{\frac{it}{A_0^2}}\Phi_0(z - ct) \end{aligned}$$

Then

$$\Phi_{0z} = c(-1 + \frac{A_0^2}{A^2}) \quad (7.16)$$

$$A_{zz} + \frac{\bar{\bar{c}}^2}{4}(A - \frac{A_0^4}{A^3}) + \frac{1}{A} - \frac{A}{A_0^2} = 0 \quad (7.17)$$

Equation (7.6) has solitonic solutions if

$$c^2 < \frac{2}{A_0^2} \quad (7.18)$$

The solitons are the "necks" - moving domains of "closing in" of the vortex tubes. Equation (7.6) has the integral

$$\frac{1}{2}A_z^2 + \frac{\bar{\bar{c}}^2}{8}(A^2 + \frac{A_0^4}{A^2}) + \ln\frac{A}{A_0} + \frac{A_0^2 - A^2}{2A_0^2} = 0 \quad (7.19)$$

The maximum of the amplitude A $A_{max} = A_0$ is reached in infinity. The minimum of A is the second solution of the transcendent equation

$$\frac{\bar{\bar{c}}^2}{8}(A_{min}^2 + \frac{A_0^2}{A_{min}^2}) + \ln\frac{A_{min}}{A_0} + \frac{A_0^2 - A_{min}^2}{2A_0^2} = 0 \quad (7.20)$$

If $\bar{\bar{c}}A_0 \ll 1$,

$$A_{min} \simeq \bar{\bar{c}}A_0^2 \ll A_0$$

In the limiting case of $c = 0$ equation (7.8) describes the merging of the vortex pair into a single vortex.

One should remember that if $\bar{\bar{c}} \to 0$ $A_z \to \infty$ at $A \to A_{min}$, and the conditions of applicability of the used model are no longer valid.

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