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Hamiltonian formalism for nonlinear waves

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Abstract. **The Hamiltonian description of hydrodynamic type systems in application to plasmas, hydrodynamics, and magnetohydrodynamics is reviewed with emphasis on the problem of introducing canonical variables. The relation to other Hamiltonian approaches, in particular natural-variable Poisson brackets, is pointed out. It is shown that the degeneracy of non-canonical Poisson brackets relates to a special type of symmetry, the relabeling transformations of fluid-particle Lagrangian markers, from which all known vorticity conservation theorems, such as Ertel's, Cauchy's, Kelvin's, as well as vorticity frozenness and the topological Hopf invariant, are derived. The application of canonical variables to collisionless plasma kinetics is described. The Hamiltonian structure of Benney's equations and of the Rossby wave equation is discussed. Davey – Stewartson's equation is given the Hamiltonian form. A general method for treating weakly nonlinear waves is presented based on classical perturbation theory and the Hamiltonian reduction technique.**

1. Introduction

The equations of hydrodynamics and their generalizations are among the most basic tools for the description of nonlinear waves in macroscopic physics. In studying them, an important question is whether these equations, in the absence of dissipation, have a Hamiltonian structure. This problem is primarily important in connection with the problem of quantization. However, in the classical case too, establishing that a given system is Hamiltonian allows one to hope (although this is not always a simple matter) to introduce explicitly canonical variables, after which all the variants of perturbation theory are considerably simplified and standardized (cf., for example, Refs [1 – 4]). In particular, this approach gives an opportunity to consider all nonlinear processes from the general point of view without fixing their proper peculiarities connected with a given medium. The Hamiltonian approach also gives certain advantages when approximations must be made. A classical example of this is a description of well-separated space or time scales, in particular, of high-frequency and low-frequency waves (for review, see the remarkable book of Whitham [5]). For continuous Hamiltonian systems the stability problem for stationary solutions as cnoidal waves, solitons, vortices, etc. is formulated more or less in the same manner and can be solved by studying the quadratic Hamiltonian for small perturbations or by taking the Hamiltonian in combination with other integrals (numbers of particles, momentum, etc.) as the Lyapunov functional if considering nonlinear stability (cf., for instance, Refs [6 – 8]).

Besides hydrodynamics, equations of the hydrodynamic type are widely used for the description of various processes in plasma physics as well as in magnetohydrodynamics (MHD). They combine the equation of medium motion and the Maxwell equations for electromagnetic fields. These models also play an essential role in solid state physics and nonlinear optics.

The problem of the Hamiltonian structure of the hydrodynamic equations has a long history. There are two traditional approaches to it. Firstly, one can try, for some system or other, to directly guess a complete set of canonical variables. Then the problem of calculating the Poisson brackets between any physical quantities is automatically solved, and one also succeeds in writing down a variational principle. Usually the Hamiltonian variables are expressed in

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terms of the natural variables (velocity, pressure) in a highly non-trivial fashion.

An alternative path is directly to find expressions for the Poisson brackets in 'natural' variables. This does not enable one to introduce a variational principle, but for many physical problems, including the problem of quantization, it appears to be sufficient. The hydrodynamic type equations have the same degree of nonlinearity (quadratic in the velocities) as the energy integral. It then follows that the expression for the Poisson brackets must be linear with respect to the variables (velocity, density, etc.) that enter these equations. It is easy to show that all such brackets are brackets of the Berezin – Kirillov – Kostant type on certain Lie groups. This quite important fact was understood relatively recently, apparently first by V I Arnold [9, 10] (see also, Ref. [11]) although Poisson brackets between velocity components had already been calculated in connection with the problem of quantization in a paper by L D Landau [12]. Also devoted to these notions were some papers by I E Dzyaloshinskii and G E Volovik [13], and S P Novikov. For the equations of magnetohydrodynamics the non-canonical Poisson brackets were first calculated by Greene and Morrison [15] and for the Vlasov – Maxwell equations for a plasma they were obtained by Morrison [16].

As for canonical variables, for the ideal hydrodynamics of a homogeneous incompressible fluid they had already been found in the previous century by Clebsch (cf., for example, Ref. [17]). The topological meaning of these variables was clarified in the paper by Kuznetsov and Mikhailov [18]. In 1932, H Bateman [19], and later independently B I Davydov [20], extended the result of Clebsch to a compressible barotropic liquid. In 1952 for nonbarotropic flows of an ideal liquid, the variables were found by I M Khalatnikov [21]. Later this result was rediscovered in another set of articles (see, for example, Ref. [22]).

From these results one can obtain the canonical variables for an incompressible fluid of variable density, including fluids with a free boundary, as was done by Kontorovich, Kravchik and Time [23]. However, the extremely important problem, from the point of view of surface waves, of the Hamiltonian description of a fluid with free surface was solved earlier by one of the authors of the present work (V E Zakharov). The canonical variables were introduced without proof in 1966 [24], and the complete proof was published in 1968 in Ref. [25]. In these papers only potential fluid flows were considered. A partial transfer of the results to the case of non-potential flow was accomplished by Voronovich [26] and Goncharov [27], who also solved the problem of the Hamiltonian description of internal waves in the ocean. A presentation of these results can be found in the monograph by Yu Z Miropolskii [28] as well as in a recent book by Goncharov and Pavlov [29], both written entirely from the point of view of Hamiltonian formalism.

Of especial interest is the Hamiltonian formalism for the Benney equations, describing non-potential long waves on shallow water. The system of Benney equations is completely integrable [30, 31], and the Hamiltonian formalism for them was formulated (in the language of Poisson brackets between moments of the longitudinal velocity) in a paper by Manin and Kupershmidt [32].

Canonical variables enabling one to calculate Poisson brackets between any quantities were found for the Benney equations in Ref. [30]. This question unexpectedly turned out to be related to the question of the Hamiltonian description of plasma, which had earlier attracted attention. A Hamiltonian description of magnetohydrodynamics was achieved by the authors of the present work in 1970 [33]. Canonical variables in a two-fluid hydrodynamic model were introduced in Ref. [34], and were used later in various papers describing nonlinear processes in plasma (cf., for example, Ref. [35]). This did not solve the question of introducing canonical variables in the collisionless kinetics of a plasma, although, after paper [31], it became clear that such variables must exist. In the present survey we introduce such variables, using the equivalence of the Vlasov equations to an infinite system of hydrodynamic equations. This equivalence, which was noted by one of the authors (E A Kuznetsov), is established by a Radon transform, and was essentially used in Ref. [30], where it was shown that the Benney equations are equivalent to one variant of the Vlasov equations.

In the present survey we also give a systematic description of the result recalled above. In addition we discuss the interesting question of the Hamiltonian structure for two-dimensional incompressible hydrodynamics, and for the Charny – Obukhov – Hasegawa – Mima equation describing Rossby waves. In these systems, there has not yet been success in introducing suitable canonical variables, although the existence of a Hamiltonian structure is a proven fact. Recently Piterbarg [39], generalizing the results of papers [38], proved that the non-canonical Poisson brackets for such systems for arbitrary flows with closed stream lines can be reduced to the Gardner – Zakharov – Faddeev brackets appearing at first for the integrable equations [40] and suggested a constructive scheme for finding a canonical basis. Finally we consider some general properties of Hamiltonian systems with a continuous number of degrees of freedom.

The basis of this survey was a paper by the authors [1], published in 1986 in English in a sufficiently rare journal and therefore unknown to a wide audience, both Russian and abroad. The text of this survey has been revised and broadened significantly from [1]. First of all, the problems of the non-canonical Poisson brackets and their degeneracy were revised and supplemented. For systems of the hydrodynamic type this degeneracy is connected with a hidden symmetry of the equations, in fact, the gauge symmetry. This symmetry has a Lagrangian origin; it relates to the group of transformations *relabeling* the Lagrangian variables marking each fluid particle. Evidently no changes in markers may influence the system dynamics. This fact was first understood completely by R Salmon [41] in 1982 although Eckart in 1938 and then in 1960 [44] and later Newcomb [46] understood the role of this symmetry. In particular, all known theorems on vorticity conservation, i.e., the Ertel theorem about the existence of the Lagrangian (material) invariants [42] (see also Ref. [54], p. 31), the Cauchy theorem of frozenness of vorticity into a fluid [17] and the Kelvin theorem about the conservation of the velocity circulation (see, for instance, [54]), as well as the conservation of the topological Hopf invariant [56, 57] characterizing the flow knottiness, are a consequence of this symmetry. This symmetry is also connected with introducing the canonical Clebsch variables and their gauge symmetry.

One should note that introducing canonical variables of the Clebsch kind into systems of the hydrodynamic type allows one to find expressions for all the non-canonical Poisson brackets known up to now, starting from the canonical one. This fact was first demonstrated [47, 1] by the authors of the given survey for the equations of ideal hydrodynamics and for the kinetic Vlasov–Maxwell equations for plasma. However, passing to the opposite direction, i.e., finding canonical brackets from non-canonical brackets, entails some difficulties in the general situation, connected with the degeneracy of non-canonical brackets.

In this survey we consider all these questions for the hydrodynamic equations in more detail. Here we don't discuss the role of this symmetry for other models, except the MHD equations (about this subject see the recent paper [48]). Now this question for systems of the hydrodynamic type is far from well studied and requires additional investigations. In our opinion, it has a principle meaning for understanding many nonlinear phenomena which take place in fluids and plasma. First of all these are the processes of reconnection of vortex lines for fluids or magnetic field lines in plasma, namely, the processes which change the system topology.

2. General remarks

We recall some elementary facts. The most naive definition of a finite-dimensional Hamiltonian system reads as follows. One considers a system of an even number of differential equations for the time-dependent functions $q_k(t)$, $p_k(t)$ $(k = 1, \ldots, N)$, having the form

$$\dot{q}_k = \frac{\partial H}{\partial p_k}\,, \qquad \dot{p}_k = -\frac{\partial H}{\partial q_k}\,. \tag{2.1}$$

Here $H(p_1, \ldots, p_N, q_1, \ldots, q_N)$, which is a given function of the variables, is the Hamiltonian.

The definition presented here is far from being always satisfactory, since it assumes implicitly that the domain of variation of the p_i and q_i (the phase space) is a domain in the real vector space R^{2N}. However, for the case of the mathematical pendulum, where the generalized coordinate is an angle, one must identify its values that differ by 2π. Thus the phase space of the pendulum is a cylinder, which is extremely significant, since functions defined uniquely on the cylinder must be periodic functions of the angular coordinate. The situation becomes even more complicated when we consider a spherical pendulum or the motion of a rigid body with one point fixed. All of these examples deal with the next class of Hamiltonian systems in degree of nontriviality, where one can, with a sufficient degree of definition, distinguish two groups of variables: generalized coordinates $q_1, \ldots, q_N$ and generalized momenta $p_1, \ldots, p_N$. The separation is based on the fact that the generalized coordinates give a point on an arbitrary N-dimensional manifold (configuration space) M, while the momenta can have arbitrary values in the vector space of momenta R^N. In this case the phase space of the system, $G = T^*(M)$, is the tangent bundle of the manifold M. Hamiltonian systems of this type preserve the basic properties of 'naive' Hamiltonian systems. In particular, the variational principle in the Hamiltonian form is valid and one can go over to a Lagrangian description.

Only systems of this type are usually described in the standard textbooks of theoretical physics.

It is important, however, to consider Hamiltonian systems of a more general form, in which it is impossible to make a unique separation of variables into coordinates and momenta. Such systems are conveniently described in terms of generalized coordinates, that are generally speaking not canonical. Let G, the phase space of the system, be a manifold covered by some system of maps. We assume that on the manifold G there is a given symplectic structure — a nondegenerate closed two-form Ω. This means that at each point a twice covariant anti-symmetric tensor $\Omega_{ij} = -\Omega_{ji}$ is defined. Suppose that x_i are the local coordinates at some point. The closure condition implies that Ω_{ij} obeys the system of equations

$$\frac{\partial \Omega_{ij}}{\partial x_k} + \frac{\partial \Omega_{jk}}{\partial x_i} + \frac{\partial \Omega_{ki}}{\partial x_j} = 0 \tag{2.2}$$

with $\det \Omega_{ij} \neq 0$.

A system of differential equations defined on G is said to be Hamiltonian if there exists a function H on G such, that in the neighborhood of each point identified by x_i one has

$$\Omega_{ij}\dot{x}_j = \frac{\partial H}{\partial x_i}\,. \tag{2.3}$$

It is easy to see that under the changes of coordinates $x_i = x_i(\tilde{x}_1, \ldots, \tilde{x}_N)$, for which the Jacobian $\partial(x_1, \ldots, x_N) \times [\partial(\tilde{x}_1, \ldots, \tilde{x}_N)]^{-1} \neq 0$, equation (2.3) remains invariant. In this case the matrix Ω transforms as follows:

$$\tilde{\Omega}_{lm} = \frac{\partial x_i}{\partial \tilde{x}_l}\,\Omega_{ij}\,\frac{\partial x_j}{\partial \tilde{x}_m}\,.$$

A manifold with an assigned symplectic structure is said to be symplectic. It necessarily has even dimension (otherwise $\det \Omega_{ij} = 0$).

Within each simply connected region Eqn (2.2) can be integrated to read

$$\Omega_{ij} = \frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i}\,, \tag{2.4}$$

where $A_i(x)$ are the 'potentials' of the form Ω_{ij}. If the solutions of the system (2.3) do not extend beyond the limits of this region, the variational principle $\delta S = 0$ is valid, where

$$S = \int (A_i \dot{x}_i + H)\,\mathrm{d}t\,. \tag{2.5}$$

It has been noted that the variational principle (2.5) exists globally only if the form Ω_{ij} is exact, i.e., if relation (2.4) can be continued to the whole manifold G. Generally speaking, the A_i are multivalued functions on G, that acquire a non-zero addition in going around any cycle not homologous to zero. Locally, in each simply-connected region one can, by a suitable change of variables, bring the system to canonical coordinates, i.e., to form (2.1) (Darboux's theorem). However, globally (over all G) this is generally not possible, even if differential form (2.4) is exact. Due to the assumption of the nondegeneracy of the form Ω_{ij}, Eqn (2.3) can be written in the form

$$\dot{x}_i = R_{ij}\,\frac{\partial H}{\partial x_j}\,. \tag{2.6}$$

Here $R_{ij} = -R_{ji}$ is the matrix reciprocal to Ω_{ij}, such as $R^{-1} = \Omega$. It is then easily verified that the relations (2.2) are equivalent to the relations

$$R_{im}\,\frac{\partial}{\partial x_m}\,R_{jk} + R_{km}\,\frac{\partial}{\partial x_m}\,R_{ij} + R_{jm}\,\frac{\partial}{\partial x_m}\,R_{ik} = 0\,. \tag{2.7}$$

Next, by means of the matrix R one defines the Poisson brackets between any functions A and B given on G:

$$\{A, B\} = \sum R_{ij} \frac{\partial A}{\partial x_i} \frac{\partial B}{\partial x_j} . \tag{2.8}$$

From the antisymmetry of R_{ij} it follows that

$$\{A, B\} = -\{B, A\} ,$$

while the relations (2.7) guarantee that the Jacobi identity

$$\{\{A, B\}, C\} + \{\{B, C\}, A\} + \{\{C, A\}, B\} = 0 \tag{2.9}$$

is satisfied. Because of the non-degeneracy of the matrix Ω_{ij}, in each coordinate system the matrix R_{ij} is also non-degenerate. The matrix R is called the symplectic operator, and it plays the same role as the metric tensor g_{ij} in Euclidean geometry. The condition (2.7) is analogous to the vanishing of the curvature tensor for Euclidean space, and, respectively, the canonical form

$$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

has the same meaning as

$$g = I$$

in Euclidean space.

The next step for generalizing a Hamiltonian system is to drop the requirement of nonsingularity of R. This variant of Hamiltonian mechanics is called Poisson mechanics.

If $\det R_{ik} = 0$, then a return to form (2.3) is impossible. Suppose that vectors ξ_i^α ($\alpha = 1, \ldots, k$) form a basis of the co-kernel of the operator R_{ij} (i.e., $\xi_i R_{ij} = 0$). Then, from Eqn (2.6), it follows that the relations

$$\xi_i^\alpha \dot{x}_i = 0 , \qquad \alpha = 1, \ldots, k \tag{2.10}$$

hold. In a simply-connected domain in which the rank of the matrix R is constant, due to the Frobenious theorem, Eqns (2.10) can be integrated:

$$f^\alpha(x_1, \ldots, x_n) = \text{const} , \qquad \alpha = 1, \ldots, k .$$

In turn, these relations are connected with the vectors ξ_i^α by the evident formulae:

$$\xi_i^\alpha = \frac{\partial f^\alpha}{\partial x_i} .$$

The constants f^α are called Casimirs. Moreover, the Frobenious theorem and relations (2.7) guarantee that all these k invariants are functionally independent. They are evidently integrals of motion for our Hamiltonian system. These integrals split G into manifolds invariant under system (2.6) (symplectic leaves). On each of them one can introduce the usual Hamiltonian mechanics. From our remarks it is clear that the possibility of introducing Poisson brackets implies the system under consideration to be Hamiltonian in the weakest sense.

Of special interest is the case where the metric elements R_{ij} are linearly dependent on the coordinates as follows:

$$R_{ij} = e_{ij,m} x_m . \tag{2.11}$$

From condition (2.7) it now follows that the $e_{ij,m}$ are subject to the relations

$$e_{ik,m}\, e_{jm,l} + e_{ji,m}\, e_{km,l} + e_{kj,m}\, e_{im,l} = 0 ,$$

i.e., they are the structure constants of some Lie algebra L. Calculating the brackets between quantities x_i, x_j, it can be checked that

$$\{x_i, x_j\} = R_{ij} = e_{ij,m}\, x_m . \tag{2.12}$$

Thus, the space G itself is now a Lie algebra L.

The matrix R_{ij} is in general degenerate. However, relations (2.10) are always integrable. Consider the algebra L^*, dual to L, and the corresponding Lie group l^*. Here the algebra L forms the co-adjoint representation of the group l^*. Relations (2.10) are invariant under the action of the group l^*, and so conditions (2.12) hold, and define the orbits of the action of the group l^* on L. On these orbits (cf. Kirillov [36], Kostant [37]) a fully valid Hamiltonian mechanics exists.

If the Hamiltonian is polynomial in its variables, then the equations are also polynomials in the canonical coordinates, and they have a nonlinearity that is one degree lower. If the degree of nonlinearity of the investigated system coincides with the degree of nonlinearity of the Hamiltonian, then the matrix is linear in the coordinates, and the symplectic manifold G is the orbit of some Lie group in its co-adjoint representation. This currently happens for equations of the hydrodynamic type.

Another interesting case is the situation when the Poisson structure R depends on the coordinates x_i quadratically. In this case it can be regarded as the classical R-matrix which plays an important role in the theory of Hamiltonian systems integrable by the inverse scattering transform. This theory, however, is far from a scope of this survey and we shall not further touch this question.

3. Hamiltonian formalism in continuous media

The introduction of a Hamiltonian structure for conservative nonlinear media is essentially a generalization of the Hamiltonian formalism for systems with a finite number of degrees of freedom to systems with a continuous number of degrees of freedom. We shall basically understand this to give a description of the dynamics of waves evolving in a continuous medium by means of canonical variables. There are no general recipes for the introduction of canonical variables in continuous media. To solve this problem it is sometimes useful to make use of a Lagrangian with constraints, which one takes in the form of some equations of motion. This method, which apparently arose in the work of B I Davydov [20], is justified when the expression for the Lagrangian without the constraints comes directly from mechanics or field theory. Such a procedure, in particular, applies to the hydrodynamic type systems that will be considered in this survey, and is widely used for describing nonlinear waves in plasma, in hydrodynamics and magnetohydrodynamics. For this purpose, we shall find the canonical variables for all of these systems.

Suppose that the medium is described by a pair of canonical variables — the generalized coordinate $q(\mathbf{r}, t)$ and the generalized momentum $p(\mathbf{r}, t)$, whose evolution is given by the Hamiltonian equations:

$$\frac{\partial p}{\partial t} = -\frac{\delta H}{\delta q}, \qquad \frac{\partial q}{\partial t} = \frac{\delta H}{\delta p} . \tag{3.1}$$

Here the Hamiltonian is some functional of p and q. Formally it can be written as a series in powers of the canonical variables:

$$H = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \int G_n^k(\mathbf{r}_1, \ldots, \mathbf{r}_k, \mathbf{r}_{k+1}, \ldots, \mathbf{r}_n) \times p(\mathbf{r}_1) \ldots p(\mathbf{r}_k) q(\mathbf{r}_{k+1}) \ldots q(\mathbf{r}_n) \, d\mathbf{r}_1 \ldots d\mathbf{r}_n . \quad (3.2)$$

This expansion, in the absence of external forces, begins with quadratic terms in p and q. For spatially homogeneous media the structure functions G_n^k are functions of the differences $(\mathbf{r}_i - \mathbf{r}_j)$. In particular, for such media the quadratic term H_0 in the expansion has the form

$$H_0 = \frac{1}{2} \int \big[A(\mathbf{r} - \mathbf{r}') p(\mathbf{r}) p(\mathbf{r}') + 2B(\mathbf{r} - \mathbf{r}') p(\mathbf{r}) q(\mathbf{r}') + C(\mathbf{r} - \mathbf{r}') q(\mathbf{r}) q(\mathbf{r}') \big] \, d\mathbf{r} \, d\mathbf{r}' , \quad (3.3)$$

whose diagonalization solves the problem of stability of a homogeneous medium against small perturbations.

To solve it we first carry out a Fourier transformation in the coordinates:

$$p(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \int p_k \exp(i\mathbf{k}\mathbf{r}) \, d\mathbf{k} , \qquad p_k = p_{-k}^* ,$$

$$q(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \int q_k \exp(i\mathbf{k}\mathbf{r}) \, d\mathbf{k} , \qquad q_k = q_{-k}^* .$$

As a result, equation (3.3) is rewritten in the form

$$H_0 = \frac{1}{2} \int \big[A_k p_k p_k^* + 2B_k p_k q_k^* + C_k q_k q_k^* \big] \, d\mathbf{k} .$$

The Fourier transforms of the structural functions that enter here have the following properties:

$$A_k = A_k^* = A_{-k} , \qquad C_k = C_k^* = C_{-k} ,$$

$$B_k = B_{1k} + iB_{2k} = B_{1-k} - iB_{2-k} .$$

In the k-representation Eqns (3.1) then take the form

$$\frac{\partial p_k}{\partial t} = -\frac{\delta H}{\delta q_k^*} , \qquad \frac{\partial q_k}{\partial t} = \frac{\delta H}{\delta p_k^*} .$$

The equations for small perturbations are obtained from this by varying the Hamiltonian H_0. Analysis of these equations shows that waves with frequencies

$$\omega_{1,2} = -B_{2k} \pm \sqrt{A_k C_k - B_{1k}^2}$$

can propagate in the medium. The medium will be stable with respect to small perturbations if

$$A_k C_k - B_{1k}^2 > 0 , \quad (3.4)$$

and unstable in the opposite case. The latter case, for instance, can be realized in a cold plasma with a monochromatic electron beam when the plasma electrons and beam electrons can be considered as two independent fluids.

In the following we shall assume that the stability condition (3.4) is satisfied. For media that are invariant under reflection [i.e., $B(\mathbf{r}) = B(-\mathbf{r})$], one obtains

$$B_{2k} = 0 , \qquad \omega_k^2 = A_k C_k - B_{1k}^2 .$$

We further carry out a canonical transformation

$$p_k = U_k a_k + U_k^* a_{-k}^* \qquad (U_k = U_{-k}) ,$$

$$q_k = V_k a_k + V_k^* a_{-k}^* \qquad (V_k = V_{-k}) \quad (3.5)$$

from the variables p_k and q_k to normal variables a_k and a_k^*, in which the quadratic Hamiltonian is

$$H_0 = \int \omega_k a_k^* a_k \, d\mathbf{k} , \quad (3.6)$$

and the equations of motion have the form

$$\frac{\partial a_k}{\partial t} = -i \, \frac{\delta H}{\delta a_k^*} . \quad (3.7)$$

Here ω_k denotes one of the functions $\omega_{1,2}$.

Substituting the transforms (3.5) into Eqn (3.3), and from a comparison with Eqn (3.6) we get a system of equations for determining U_k and V_k. By requiring that this transformation is canonical, we get

$$U_k V_k^* - U_k^* V_k = -i ,$$

and find from this system

$$U_k = i \, \frac{B_{1k} - i\omega_{0k}}{\sqrt{2 A_k \omega_{0k}}} \exp(i\varphi_k) ,$$

$$V_k = -i \sqrt{\frac{A_k}{2\omega_{0k}}} \exp(i\varphi_k) .$$

In the above expressions ω_{0k} is the quantity $\operatorname{sign}(A_k)(A_k C_k - B_{1k}^2)^{1/2}$, and φ_k is an arbitrary phase factor, which we shall set equal to zero from now on [this corresponds to a simple redefinition of a_k: $a_k \to a_k \exp(i\varphi_k)$].

Let us now explicitly consider the complete frequency

$$\omega_k = -B_{2k} + \operatorname{sign}(A_k)(A_k C_k - D_{1k}^2)^{1/2}$$

that is the dispersion law for the waves. It is essential that the sign of the frequency coincides with the sign of the wave energy in the nonlinear medium†. Accordingly all waves can be divided into two big classes: waves with positive energy and waves with negative energy. All well-known waves (gravity and capillary waves on a fluid surface, acoustic and electromagnetic waves, and so on) belong to the first class. Waves with a negative energy typically appear in media with some current (it may be electron or ion beams in plasma, or flow of one fluid with respect to another, etc.) and in this case the origin of a negative frequency is connected with the Doppler effect. One should say that there is no principle difference in the nonlinear interaction between waves within their respective classes. This arises for the interaction between waves with positive and negative energies.

† Here we assume that the nonlinear interaction is weak so that the energy sign of the nonlinear medium coincides with the sign of its quadratic Hamiltonian.

In order to classify the nonlinear interaction between waves, let us consider the next terms in the expansion in powers of a and a^*, which can be obtained after substitution of Eqn (3.5) into (3.2). In particular, the cubic term H_1 has the form

$$H_1 = \int (V_{kk_1k_2} a_k^* a_{k_1} a_{k_2} + \text{c.c.}) \delta_{k-k_1-k_2} \, \mathrm{d}\mathbf{k} \, \mathrm{d}\mathbf{k}_1 \mathrm{d}\mathbf{k}_2 + \frac{1}{3} \int (U_{kk_1k_2} a_k^* a_{k_2}^* a_{k_2}^* + \text{c.c.}) \delta_{k+k_1+k_2} \, \mathrm{d}\mathbf{k} \, \mathrm{d}\mathbf{k}_1 \mathrm{d}\mathbf{k}_2 \,, \quad (3.8)$$

where the matrix elements U and V have the following symmetry properties:

$$U_{kk_1k_2} = U_{kk_2k_1} = U_{k_2k_1k} \,, \qquad V_{kk_1k_2} = V_{kk_2k_1} \,.$$

Among the fourth-order terms, we shall be interested in the term of the form

$$H_2 = \frac{1}{2} \int T_{k_1k_2k_3k_4} a_{k_1}^* a_{k_2}^* a_{k_3} a_{k_4} \delta_{k_1+k_2-k_3-k_3} \prod_i \mathrm{d}\mathbf{k}_i \,.$$

Each term in the expansion of H in powers of a and a^* has a simple physical meaning. The equation of motion in the form (3.7) is the limit of the corresponding quantum equations for the Bose operators in the case of a classical wave field, where the variables a^* and a appear as analogs of the creation and annihilation operators. Thus the cubic term in the expansion of the Hamiltonian describes three-wave processes (the first term in H_1 is responsible for the processes of decay of one wave into three waves, the second corresponds to the simultaneous creation of three waves), the next term describes four-wave processes, etc.

It is necessary to say that a calculation of matrix elements in this scheme assumes a pure algebraic procedure that consists in a substitution of the transformation (3.5) into the corresponding Hamiltonian, a forthcoming simplification and a symmetrization of the final result.

For a medium described by several pairs of canonical variables and when H_0 is diagonalized, several wave branches can appear, with dispersion laws $\omega_i(k)$ and amplitudes $a_i(k)$. In this case a summation over all types of waves in each term of the expansion is needed.

In the next sections we show how both canonical variables are introduced and matrix elements are calculated on concrete examples.

4. Canonical variables in hydrodynamics

As a first example we consider the equations of potential flow of an ideal compressible barotropic fluid, in which the pressure p is a single-valued function of the density ρ. These equations can be written in the following form:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \nabla \varphi) = 0 \,, \quad (4.1)$$

$$\frac{\partial \varphi}{\partial t} + \frac{(\nabla \varphi)^2}{2} + w(\rho) = 0 \,. \quad (4.2)$$

Here φ is the velocity potential, $\omega(\rho) = \partial \varepsilon / \partial \rho$ is the enthalpy, where $\varepsilon(\rho)$ denotes the internal energy density. These equations conserve the energy

$$H = \int \left[\frac{\rho (\nabla \varphi)^2}{2} + \varepsilon(\rho) \right] \mathrm{d}\mathbf{r} \,. \quad (4.3)$$

It can be checked that the equation set (4.1) and (4.2) can be represented in the form of the Hamiltonian equations:

$$\frac{\partial \varphi}{\partial t} = \frac{\delta H}{\delta \rho} \,, \qquad \frac{\partial \rho}{\partial t} = -\frac{\delta H}{\delta \varphi} \,.$$

Thus the density ρ is a generalized coordinate, and φ is the generalized momentum.

This result can also be obtained from a Lagrangian approach. In this case one makes use of the well-known expression for the Lagrangian of a mechanical system, generalized to the continuous case, supplementing it by the constraint

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0 \,.$$

Then the action is

$$S = \int L \, \mathrm{d}t = \int \left\{ \frac{\rho \mathbf{v}^2}{2} - \varepsilon(\rho) + \varphi \left[\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) \right] \right\} \mathrm{d}\mathbf{r} \, \mathrm{d}t \,.$$

Its variation with respect to the variable $\mathbf{v}$ leads to the potential condition for the flow, $\mathbf{v} = \nabla \varphi$, and variations with respect to the variables ρ and φ lead to Eqns (4.1) and (4.2). Here the transition to the Hamiltonian form is accomplished by the standard formula

$$H = \int \varphi \frac{\partial \rho}{\partial t} \, \mathrm{d}\mathbf{r} - L$$

and leads us to Eqn (4.3).

We give the expression for the coefficients of the Hamiltonian expansion. The diagonalization of

$$H_0 = \int \left[\frac{1}{2} \rho_0 (\nabla \varphi)^2 + c_\mathrm{s}^2 \frac{\delta \rho^2}{2\rho_0} \right] \mathrm{d}\mathbf{r}$$

can be made by the transformation

$$\varphi_k = -\frac{\mathrm{i}}{k} \left(\frac{\omega_k}{2\rho_0} \right)^{1/2} (a_k - a_{-k}^*) \,,$$

$$\delta \rho_k = k \left(\frac{\rho_0}{2\omega_k} \right)^{1/2} (a_k + a_{-k}^*) \,. \quad (4.4)$$

Here $\omega_k = kc_\mathrm{s}$ refers to the eigenfrequency, $\delta \rho = \rho - \rho_0$ is the deviation of the density from the equilibrium ρ_0, $c_\mathrm{s} = (\partial p / \partial \rho_0)^{1/2}$ is the velocity of sound. Substitution of this transformation into the next term H_1 of the expansion,

$$H_1 = \int \left[\frac{1}{2} \delta \rho (\nabla \varphi)^2 + c_\mathrm{s}^2 g \frac{\delta \rho^3}{2\rho_0^2} \right] \mathrm{d}\mathbf{r} \,,$$

gives the following expression for $U_{kk_1k_2}$ and $V_{kk_1k_2}$:

$$U_{kk_1k_2} = V_{kk_1k_2} = \frac{1}{16(\pi^3 \rho_0)^{1/2}} \left[3g c_\mathrm{s}^2 \frac{kk_1k_2}{(\omega_k \omega_{k_1} \omega_{k_2})^{1/2}} + \left(\frac{\omega_k \omega_{k_1}}{\omega_{k_2}} \right)^{1/2} k_1 \frac{(\mathbf{k}\mathbf{k}_1)}{kk_1} + \left(\frac{\omega_k \omega_{k_{24}}}{\omega_{k_1}} \right)^{1/2} k_1 \frac{(\mathbf{k}\mathbf{k}_2)}{kk_2} + \left(\frac{\omega_{k_2} \omega_{k_1}}{\omega_k} \right)^{1/2} k \frac{(\mathbf{k}_2 \mathbf{k}_1)}{k_2 k_1} \right] . \quad (4.5)$$

The equations describing nonlinear sound waves in media with dispersion belong to the same type of system as (4.1) and (4.2). These equations can be derived when considering the internal energy of the system $\mathcal{E}_{\rm in}$ as a functional of the density. This functional can be represented as a power series in $\boldsymbol{\nabla}\rho$:

$$\mathcal{E}_{\rm in}=\int\left[\varepsilon(\rho)+\frac{v}{2}(\boldsymbol{\nabla}\rho)^2+\ldots\right]\mathrm{d}\mathbf{r}\,. \tag{4.6}$$

Classical hydrodynamics corresponds to keeping only the first term in the above series. If we now include the second term, we get the Boussinesq system:

$$\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho\boldsymbol{\nabla}\varphi)=0\,,$$

$$\frac{\partial\varphi}{\partial t}+\frac{1}{2}(\boldsymbol{\nabla}\varphi)^2=-\frac{\delta\mathcal{E}_{\rm in}}{\delta\rho}=-w(\rho)-v\Delta\rho\,.$$

The Hamiltonian in this case coincides with the total energy of the system, i.e., with a sum of kinetic energy and internal energy given by Eqn (4.6), while ρ and φ remain the canonical conjugated variables.

Introduction of canonical variables is possible also when we include vortex motion in an ideal fluid [17, 19, 20]. To this aim we must start from the full Euler equations of hydrodynamics:

$$\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho\mathbf{v})=0\,, \tag{4.7}$$

$$\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\boldsymbol{\nabla})\mathbf{v}=-\frac{\boldsymbol{\nabla}p(\rho)}{\rho}=-\boldsymbol{\nabla}w(\rho)\,. \tag{4.8}$$

We know that, for the Euler equations in accordance with the Kelvin theorem, the circulation of the fluid velocity around any closed contour moving together with the fluid is conserved. In other words, in such a system there is a certain scalar function $\mu(\mathbf{r},t)$ which is convected by the fluid and described by the following equation:

$$\frac{\mathrm{d}\mu}{\mathrm{d}t}=\left(\frac{\partial}{\partial t}+\mathbf{v}\boldsymbol{\nabla}\right)\mu=0\,. \tag{4.9}$$

Therefore, in formulating the variational principle we should include this equation as a constraint which implies setting

$$L=\int\left[\frac{\rho\mathbf{v}^2}{2}-\varepsilon(\rho)+\varphi\left(\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho\mathbf{v})\right)-\lambda\left(\frac{\partial\mu}{\partial t}+\mathbf{v}\boldsymbol{\nabla}\mu\right)\right]\mathrm{d}\mathbf{r}\,. \tag{4.10}$$

The variation of L with respect to the variables $\mathbf{v}$, ρ and μ leads to the following equations:

$$\mathbf{v}=\frac{\lambda}{\rho}\boldsymbol{\nabla}\mu+\boldsymbol{\nabla}\varphi\,, \tag{4.11}$$

$$\frac{\partial\varphi}{\partial t}+(\mathbf{v}\boldsymbol{\nabla}\varphi)-\frac{\mathbf{v}^2}{2}+w(\rho)=0\,, \tag{4.12}$$

$$\frac{\partial\lambda}{\partial t}+\operatorname{div}(\lambda\mathbf{v})=0\,. \tag{4.13}$$

Here the first equation is the well-known change to the Clebsch variables λ and μ; the second represents the generalization of the Bernoulli equation to the non-potential flows and the last governs the dynamics of a new variable λ. The choice of λ and μ for a given value of $\mathbf{v}$ is not unique.

Let us consider two sets of potentials λ, μ, φ and λ', μ', φ', giving the same value for the velocity $\mathbf{v}$ with the help of Eqn (4.11). Multiplying Eqn (4.11) by the differential $\mathrm{d}\mathbf{r}$ (for a fixed time t), we get a relation

$$\mathrm{d}\varphi+\frac{\lambda}{\rho}\,\mathrm{d}\mu=\mathrm{d}\varphi'+\frac{\lambda'}{\rho}\,\mathrm{d}\mu'$$

between the new and old variables, or

$$\mathrm{d}f\equiv\mathrm{d}(\varphi-\varphi')=-\frac{\lambda}{\rho}\,\mathrm{d}\mu-\frac{\lambda'}{\rho}\,\mathrm{d}\mu'\,. \tag{4.14}$$

The last relation shows that $\varphi'-\varphi$ is the generating function f of a gauge transformation, depending on μ and μ'. The old and new canonical coordinates are then expressed in terms of the generating function by means of the formulae [21]

$$\lambda=-\rho\,\frac{\partial f}{\partial\mu}\,,\qquad\lambda'=\rho\,\frac{\partial f}{\partial\mu'}\,, \tag{4.15}$$

determining the non-uniqueness in the choice of Clebsch variables.

Substituting the velocity $\mathbf{v}$ expressed in terms of the variables λ, μ and φ directly into the Euler equation (4.8), we verify that

$$\frac{\lambda}{\rho}\,\boldsymbol{\nabla}\left[\frac{\partial\mu}{\partial t}+(\mathbf{v}\boldsymbol{\nabla})\mu\right]+\boldsymbol{\nabla}\mu\left[\frac{\partial}{\partial t}\frac{\lambda}{\rho}+(\mathbf{v}\boldsymbol{\nabla})\frac{\lambda}{\rho}\right]+\boldsymbol{\nabla}\left[\frac{\partial\varphi}{\partial t}+(\mathbf{v}\boldsymbol{\nabla})\varphi-\frac{\mathbf{v}^2}{2}+w(\rho)\right]=0\,.$$

Thus this equation is satisfied if Eqns (4.12), (4.13) are also imposed. If it is so the system of equations of hydrodynamics can be said to be equivalent to the system (4.7), (4.9), (4.12), and (4.13). This is based on the uniqueness of the solution of the Cauchy problem for the original system and that obtained (that is, rigorously speaking, an assumption). In doing this we must in addition, by means of the velocity $\mathbf{v}$ given at the initial time, construct some set of functions λ_0, μ_0 and φ_0, appearing as initial conditions for the system (4.9), (4.12), and (4.13).

Now changing to a Hamiltonian description, we get

$$\frac{\partial\rho}{\partial t}=\frac{\delta H}{\delta\varphi}\,,\qquad\frac{\partial\varphi}{\partial t}=-\frac{\delta H}{\delta\rho}\,;$$
$$\frac{\partial\lambda}{\partial t}=\frac{\delta H}{\delta\mu}\,,\qquad\frac{\partial\mu}{\partial t}=-\frac{\delta H}{\delta\lambda}\,, \tag{4.16}$$

where the Hamiltonian

$$H=\int\left[\frac{\rho\mathbf{v}^2}{2}+\varepsilon(\rho)\right]\mathrm{d}\mathbf{r}$$

coincides with the total energy of the system. For potential flows ($\lambda=\mu=0$) we again arrive at a pair of canonical variables (ρ,φ).

The canonical variables for the equations of relativistic hydrodynamics,

$$\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho\mathbf{v})=0\,,$$

$$\left[\frac{\partial}{\partial t}+(\mathbf{v}\boldsymbol{\nabla})\right]\mathbf{p}+m\boldsymbol{\nabla}w(\rho)=0\,,$$

$$\mathbf{p}=m\mathbf{v}\left(1-\frac{v^2}{c^2}\right)^{-1/2}\,,$$

are introduced in analogy to Eqn (4.11). In this case

$$\frac{\mathbf{p}}{m} = \frac{\lambda}{\rho}\,\nabla\mu + \nabla\varphi\,.$$

Just as in the preceding example, the variables (λ, μ) and (ρ, φ) form pairs of canonically conjugate quantities, subjected to Eqns (4.16), with the Hamiltonian

$$H = \int \left[\frac{\rho}{m}(m^2c + p^2c^2)^{1/2} + \varepsilon(\rho) \right] \mathrm{d}\mathbf{r}\,.$$

A natural generalization of the Clebsch formulation (4.8) is the introduction of canonical variables for nonbarotropic flows [22], when ε depends on the density ρ as well as on the entropy S. For this the equations of motion (4.9) and (4.11) are supplemented by the equation for the entropy advected by the fluid,

$$\left[\frac{\partial}{\partial t} + (\mathbf{v}\nabla) \right] S = 0\,,$$

and the thermodynamic relation

$$\mathrm{d}\varepsilon = \rho T\,\mathrm{d}S + w\,\mathrm{d}\rho$$

with T as the temperature. In this case the transition to the new variables is accomplished by the formula

$$\mathbf{v} = \nabla\varphi + \frac{\lambda}{\rho}\,\nabla\mu + \frac{\beta}{\rho}\,\nabla S\,. \tag{4.17}$$

For such flows (φ, ρ), (λ, μ) and (S, β) are pairs of canonically conjugate quantities:

$$\frac{\partial\rho}{\partial t} = \frac{\delta H}{\delta\varphi} = -\mathrm{div}\,(\rho\mathbf{v})\,,$$

$$\frac{\partial\varphi}{\partial t} = -\frac{\delta H}{\delta\rho} = \frac{\mathbf{v}^2}{2} - \mathbf{v}\nabla\varphi - w\,,$$

$$\frac{\partial\lambda}{\partial t} = \frac{\delta H}{\delta\mu} = -\mathrm{div}\,(\lambda\mathbf{v})\,,$$

$$\frac{\partial\mu}{\partial t} = -\frac{\delta H}{\delta\lambda} = -\mathbf{v}\nabla\mu\,,$$

$$\frac{\partial\beta}{\partial t} = \frac{\delta H}{\delta S} = -\mathrm{div}\,(\beta\mathbf{v}) + \rho T\,,$$

$$\frac{\partial S}{\partial t} = -\frac{\delta H}{\delta\beta}\,,$$

where $H = \int \left[\rho(\mathrm{v}^2/2) + \varepsilon(\rho, S) \right] \mathrm{d}\mathbf{r}$. The equivalence of these equations to the equations of hydrodynamics is verified by direct substitution of the velocity into the Euler equation (4.8). Thus, in comparison with the barotropic case the number of canonical variables increases by two.

Now let us ask the natural question: what is the minimal number of canonical conjugated pairs for describing any flow? As we saw above, introducing new canonical variables in the framework of the Lagrangian approach was connected with the addition of new constraints into the Lagrangian. For example, for the Lagrangian (4.11) they were the continuity equation for the density and the equation for the Lagrangian (material) invariant μ advected by the fluid. In the nonbarotropic case a new Lagrangian invariant, i.e, the entropy S, was added.

To describe the fluid in terms of the Lagrangian (material) variables it is enough to give three values $(a_1, a_2, a_3) = \mathbf{a}$ which, in the simplest case, coincide with the initial positions of each fluid particle, so that the coordinate of the particle at time t will be

$$\mathbf{r} = \mathbf{r}(\mathbf{a}, t)\,. \tag{4.18}$$

More simply, and most frequently, the vector $\mathbf{a}$ is related to the origin of the particle coordinates:

$$\mathbf{a} = \mathbf{r}(\mathbf{a}, 0)\,.$$

Hence it becomes clear that originally there are three independent Lagrangian invariants,

$$\mathbf{a} = \mathbf{a}(\mathbf{r}, t)\,,$$

that are the inverse map to (4.18). All other Lagrangian invariants are functions of $\mathbf{a}$. If we now assign the equations

$$\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} \equiv \frac{\partial\mathbf{a}}{\partial t} + (\mathbf{v}\nabla)\mathbf{a} = 0$$

for $\mathbf{a}$ as constraints† in the Lagrangian for the fluid we immediately come to three new pairs of the canonical variables (λ_l, a_l), $l = 1, 2, 3$ with the velocity in the form

$$\mathbf{v} = u_l \nabla a_l\,. \tag{4.19}$$

Here $u_l = \lambda_l/\rho$ and the density ρ is expressed through $\mathbf{a}$ by means of

$$\rho = \frac{\rho_0(\mathbf{a})}{J}\,,$$

where $\rho_0(\mathbf{a})$ is the original density, $J = \det \hat{J}_{ij}$ is the Jacobian of the mapping (4.18), and $\hat{J}_{ij} = \partial x_i/\partial a_j$ is the Jacobi matrix (for more details, see Sections 5, 6). The vector $\mathbf{u}$ in this formula is expressed in terms of the velocity components v_i by

$$\mathbf{u} = \hat{J}^{\mathrm{T}}\mathbf{v}\,,$$

where the superscript T means transpose.

Representation (4.19) is the most general. In particular, all the changes of variables presented above follow from this formula. It can be simplified although remaining general.

Let us consider reversible smooth changes of variables:

$$\mathbf{a} = \mathbf{a}(\tilde{\mathbf{a}})\,.$$

Under these changes representation (4.19) remains invariant,

$$\mathbf{v} = \tilde{u}_l \nabla\tilde{a}_l\,,$$

but the vector $\mathbf{u}$ transforms as

$$\tilde{u}_l = u_k\,\frac{\partial a_k}{\partial\tilde{a}_l}\,.$$

If we now require that one of the components, say u_3, is equal to 1, representation (4.19) becomes (compare with Ref. [49])

$$\mathbf{v} = \nabla\phi + \frac{\lambda_1}{\rho}\,\nabla\mu_1 + \frac{\lambda_2}{\rho}\,\nabla\mu_2\,. \tag{4.20}$$

If now in this equation we put the entropy S for μ_2, then we come back to the transformation (4.17). Note that such a reduction is possible if the family of surfaces of constant entropy, $S(\mathbf{r}, t) = \mathrm{const}$, are homotopic, say, to the family of surfaces $a_1(\mathbf{r}, t) = \mathrm{const}$. Hence, in particular, it follows that

† These constraints are often called Lin's constraints [51].

in the barotropic case it is enough to take two pairs of the Clebsch variables in order to describe any fluid flow. One pair of Clebsch variables, as we will see later, describes a partial type of flows. Nevertheless, locally any flow can be parameterized by one pair of Clebsch variables [17].

5. Non-canonical Poisson brackets

Now let us consider how one introduces a Hamiltonian structure into hydrodynamics in terms of the natural physical variables. To do so, it is sufficient to construct Poisson brackets that satisfy all the necessary requirements. The simplest way of constructing such brackets is to convert the Poisson brackets expressed in terms of canonical variables to an expression in terms of the natural variables. Note that in this case the arising symplectic operator appears to be local in those variables. As an example we carry out a conversion of the formula for barotropic flows of an ideal fluid. The calculations for other models can be done in exactly the same way.

According to (4.16), the Poisson brackets are given by the expression:

$$\{F, G\} = \int \left[\left(\frac{\delta F}{\delta \rho} \frac{\delta G}{\delta \varphi} - \frac{\delta F}{\delta \varphi} \frac{\delta G}{\delta \rho} \right) + \left(\frac{\delta F}{\delta \lambda} \frac{\delta G}{\delta \mu} - \frac{\delta F}{\delta \mu} \frac{\delta G}{\delta \lambda} \right) \right] \mathrm{d}\mathbf{r}\,. \tag{5.1}$$

Here the velocity is expressed in terms of λ, μ and ρ, φ by the formula

$$\mathbf{v} = \frac{\lambda}{\rho} \nabla \mu + \nabla \varphi\,,$$

by means of which one can calculate the variational derivatives of F with respect to ρ, φ, λ and μ:

$$\begin{aligned} &\left.\frac{\delta F}{\delta \rho}\right|_{\lambda} = \left.\frac{\delta F}{\delta \rho}\right|_{v} - \frac{\lambda \nabla \mu}{\rho} \frac{\delta F}{\delta \mathbf{v}}\,, \qquad \frac{\delta F}{\delta \varphi} = -\mathrm{div}\, \frac{\delta F}{\delta \mathbf{v}}\,, \\ &\frac{\delta F}{\delta \lambda} = \frac{\nabla \mu}{\rho} \frac{\delta F}{\delta \mathbf{v}}\,, \qquad \frac{\delta F}{\delta \mu} = -\mathrm{div} \left(\frac{\lambda}{\rho} \frac{\delta F}{\delta \mathbf{v}} \right). \end{aligned} \tag{5.2}$$

In these formulae the variational derivatives on the left-hand sides are taken with fixed λ, μ, ρ, φ, and those on the right-hand sides for constant ρ and $\mathbf{v}$. Substitution of these relations into Eqn (5.1) leads us to the brackets [15]

$$\begin{aligned} \{F, G\} = &\int \left[\left(\nabla \frac{\delta F}{\delta \rho}, \frac{\delta G}{\delta \mathbf{v}} \right) - \left(\nabla \frac{\delta G}{\delta \rho}, \frac{\delta F}{\delta \mathbf{v}} \right) \right] \mathrm{d}\mathbf{r} \\ &+ \int \left(\frac{\mathrm{rot}\, \mathbf{v}}{\rho}, \left[\frac{\delta F}{\delta \mathbf{v}} \times \frac{\delta G}{\delta \mathbf{v}} \right] \right) \mathrm{d}\mathbf{r}\,, \end{aligned} \tag{5.3}$$

the Jacobi identity (2.9) being satisfied automatically.

In terms of these brackets, the continuity and Euler equations have the form

$$\begin{aligned} &\frac{\partial \rho}{\partial t} = -\mathrm{div}\, (\rho \mathbf{v}) = \{\rho, H\}\,, \\ &\frac{\partial \mathbf{v}}{\partial t} = -(\mathbf{v}, \nabla)\mathbf{v} - \nabla w(\rho) = \{\mathbf{v}, H\}\,, \end{aligned}$$

where $H = \int \left[\rho \mathbf{v}^2/2 + \varepsilon(\rho) \right] \mathrm{d}\mathbf{r}$.

The brackets (5.3) have a more obvious meaning if we go over to the new variable $\mathbf{p} = \rho \mathbf{v}$, the momentum density. In these variables these brackets are changed to the BKK brackets [14]:

$$\begin{aligned} \{F, G\} = &\int \rho \left[\left(\nabla \frac{\delta F}{\delta \rho}, \frac{\delta G}{\delta \mathbf{p}} \right) - \left(\nabla \frac{\delta G}{\delta \rho}, \frac{\delta F}{\delta \mathbf{p}} \right) \right] \mathrm{d}\mathbf{r} \\ &+ \int \left(\mathbf{p}, \left[\left(\frac{\delta G}{\delta \mathbf{p}} \nabla \right) \frac{\delta F}{\delta \mathbf{p}} - \left(\frac{\delta F}{\delta \mathbf{p}} \nabla \right) \frac{\delta G}{\delta \mathbf{p}} \right] \right) \mathrm{d}\mathbf{r}\,. \end{aligned} \tag{5.4}$$

Using Eqn (5.4) to calculate brackets between components of $\mathbf{p}$ and ρ, we find that

$$\begin{aligned} &\{p_i(\mathbf{r}), p_j(\mathbf{r}')\} = \left[p_j(\mathbf{r}') \nabla_i' - p_i(\mathbf{r}) \nabla_j \right] \delta(\mathbf{r} - \mathbf{r}')\,, \\ &\{p_i(\mathbf{r}), \rho(\mathbf{r}')\} = \rho \nabla_i\, \delta(\mathbf{r} - \mathbf{r}')\,. \end{aligned} \tag{5.5}$$

In accordance with Eqn (2.12), these relations give a Lie algebra, which coincides with the algebra of vector fields [58, 14] in this case.

The brackets (5.4) and (5.5) can also be obtained in other ways. The simplest method is to regard the Poisson brackets as the classical limit of the corresponding quantum commutators, which were first calculated for hydrodynamics by L D Landau [12]. Another method for calculating the Poisson brackets for hydrodynamic models, proposed by G E Volovik and I E Dzyaloshinskii [13], is based on the fact that $\mathbf{p}$ and ρ are the densities of the generators of translations and gauge transformations.

For the sake of completeness we give the expressions for the Poisson brackets for the hydrodynamic equations of ideal fluids for an arbitrary dependence of the pressure on both the density and the entropy [15]:

$$\begin{aligned} \{F, G\} = &\int \left[\left(\nabla \frac{\delta F}{\delta \rho}, \frac{\delta G}{\delta \mathbf{v}} \right) - \left(\nabla \frac{\delta G}{\delta \rho}, \frac{\delta F}{\delta \mathbf{v}} \right) \right] \mathrm{d}\mathbf{r} \\ &+ \int \left(\frac{\mathrm{rot}\, \mathbf{v}}{\rho}, \left[\frac{\delta F}{\delta \mathbf{v}} \times \frac{\delta G}{\delta \mathbf{v}} \right] \right) \mathrm{d}\mathbf{r} \\ &+ \int \left(\frac{\nabla S}{\rho}, \left[\frac{\delta F}{\delta \mathbf{v}} \frac{\delta G}{\delta S} - \frac{\delta G}{\delta \mathbf{v}} \frac{\delta F}{\delta S} \right] \right) \mathrm{d}\mathbf{r}\,. \end{aligned} \tag{5.6}$$

We want to repeat once more that the introduction of the Poisson brackets to a system means that such systems possess a Hamiltonian structure in the weakest sense. For example, for the above equations of ideal hydrodynamics it is reflected in the fact that the brackets expressed in terms of natural variables are degenerate, i.e, there exist annulators (Casimirs) of these Poisson brackets which, as we will see in the next sections, are connected with a specific gauge symmetry of the hydrodynamic equations, providing, in particular, the conservation of fluid velocity circulation. Besides, it means that a direct conversion, i.e., passing from Eqn (5.3) or (5.6) to the canonical basis is impossible in general. For this case at first we need to resolve all our constraints (Casimirs). A typical example just consists in introducing Clebsch variables. This is all the more interesting as, so far, we have not explicitly known what these Casimirs look like.

Of particular interest is the introduction of a Hamiltonian structure for an incompressible fluid. In this case ρ is no longer an independent variable, and can be eliminated by using the relation $\mathrm{div}\, \mathbf{v} = 0$. Thus in the limit of the incompressible fluid there is only one pair of canonical variables λ and μ, and the Poisson brackets in this case take the form

$$\{F, G\} = \int \left(\frac{\delta F}{\delta \lambda} \frac{\delta G}{\delta \mu} - \frac{\delta F}{\delta \mu} \frac{\delta G}{\delta \lambda} \right) \mathrm{d}\mathbf{r}\,.$$

By means of relations analogous to (5.2), one can derive

$$\frac{\delta F}{\delta \lambda} = \left(\frac{\nabla \mu}{\rho}, \frac{\delta F}{\delta \mathbf{v}} - \nabla \frac{1}{\Delta} \operatorname{div} \frac{\delta F}{\delta \mathbf{v}}\right),$$

$$\frac{\delta F}{\delta \mu} = -\operatorname{div} \frac{\lambda}{\rho} \left(\frac{\delta F}{\delta \mathbf{v}} - \nabla \frac{1}{\Delta} \operatorname{div} \frac{\delta F}{\delta \mathbf{v}}\right).$$

As a result, we arrive at the equation

$$\{F, G\} = \int \left(\operatorname{rot} \mathbf{v}, \left[\left(\frac{\delta F}{\delta \mathbf{v}} - \nabla \frac{1}{\Delta} \operatorname{div} \frac{\delta F}{\delta \mathbf{v}}\right) \times \left(\frac{\delta G}{\delta \mathbf{v}} - \nabla \frac{1}{\Delta} \operatorname{div} \frac{\delta G}{\delta \mathbf{v}}\right)\right]\right) \mathrm{d}\mathbf{r}. \quad (5.7)$$

(Here we put $\rho = 1$.) This expression shows that the manifold G coincides with the algebra of vector fields $\mathbf{A}(\mathbf{r})$ for which $\operatorname{div} \mathbf{A} = 0$. These brackets are expressed in a more compact form using $\mathbf{\Omega} = \operatorname{rot} \mathbf{v}$ [18], to read

$$\{F, G\} = \int \left(\mathbf{\Omega}, \left[\operatorname{rot} \frac{\delta F}{\delta \mathbf{\Omega}} \times \operatorname{rot} \frac{\delta G}{\delta \mathbf{\Omega}}\right]\right). \quad (5.8)$$

As a result, the Euler equation for $\mathbf{\Omega}$,

$$\frac{\partial \mathbf{\Omega}}{\partial t} = \operatorname{rot}[\mathbf{v} \times \mathbf{\Omega}], \quad (5.9)$$

becomes the Hamiltonian equation [9, 18]

$$\frac{\partial \mathbf{\Omega}}{\partial t} = \{\mathbf{\Omega}, H\},$$

where

$$H = \int \frac{\mathbf{v}^2}{2} \, \mathrm{d}\mathbf{r}.$$

The brackets (5.8) also give a Hamiltonian structure for two-dimensional hydrodynamics. In this case $\mathbf{\Omega}$ has a single component, which is conveniently expressed in terms of the stream function ψ:

$$\Omega = -\Delta \psi, \qquad \left(v_x = \frac{\partial \psi}{\partial y}, \, v_y = -\frac{\partial \psi}{\partial x}\right).$$

In the two-dimensional case the equation of motion (5.9) and the Poisson brackets (5.8) have the following form:

$$\frac{\partial \Omega}{\partial t} = \{\Omega, H\} = -\frac{\partial \Omega}{\partial x}\frac{\partial \psi}{\partial y} + \frac{\partial \Omega}{\partial y}\frac{\partial \psi}{\partial x} \equiv -\frac{\partial(\Omega, \psi)}{\partial(x, y)}, \quad (5.10)$$

$$\{F, G\} = \int \Omega \, \frac{\partial(\delta F/\delta \Omega, \delta G/\delta \Omega)}{\partial(x, y)} \, \mathrm{d}x \, \mathrm{d}y, \quad (5.11)$$

$$H = \frac{1}{2}\int (\nabla \psi)^2 \, \mathrm{d}x \, \mathrm{d}y.$$

A Hamiltonian structure is introduced analogously into the Rossby equation, which differs from (5.11) in having the additional term $\beta(\partial\psi/\partial x)$ entering [1]:

$$\frac{\partial}{\partial t} \Delta \psi + \beta \frac{\partial \psi}{\partial x} = -\frac{\partial(\Delta \psi, \psi)}{\partial(x, y)}. \quad (5.12)$$

It is then easy to see that the change $\Omega \to \Omega - \beta y$ reduces this equation to (5.11). Thus, the Poisson brackets for Eqn (5.12) are given analogously by [1]

$$\{F, G\} = \int (\Omega + \beta y) \frac{\partial(\delta F/\delta \Omega, \delta G/\delta \Omega)}{\partial(x, y)} \, \mathrm{d}x \, \mathrm{d}y, \quad (5.13)$$

while the Hamiltonian H is still defined by the earlier expression

$$H = \frac{1}{2}\int (\nabla \psi)^2 \, \mathrm{d}x \, \mathrm{d}y.$$

One should add that the Poisson brackets (5.11) and (5.13) for flows with closed stream lines can be reduced to the Gardner – Zakharov – Faddeev brackets used in the theory of integrable equations [40]. Details of such a consideration can be found in the original papers [38, 39].

Thus, introducing non-canonical Poisson brackets on the basis of canonical ones represents the most simple way to find them. Moreover the Hamiltonian structure given by means of these brackets is the weakest Hamiltonian formulation of the equations. In this formulation, in particular, it is impossible to write the variational principle explicitly. On the other hand, as will be shown later, the representation of the hydrodynamic type equations by means of the non-canonical Poisson brackets can be written for arbitrary flows. However, the arbitrariness is paid for by the brackets degeneracy, i.e., by existence of Casimirs annulling non-canonical brackets.

6. Ertel's theorem

In this section and those after we show, mainly by following results expounded in Refs [41, 43], that for perfect fluids with arbitrary dependence of pressure on the fluid density and entropy, Ertel's theorem as well as Kelvin's theorem on the conservation of velocity circulation are a consequence of the specific gauge symmetry connected with the relabeling of fluid particles. We also discuss the role played in the Hamiltonian structures by this symmetry.

Ertel's theorem [42] for a perfect fluid says that the quantity

$$I_{\mathrm{L}} = \frac{(\mathbf{\Omega} \nabla S)}{\rho} \quad (6.1)$$

is a Lagrangian invariant. Here $\mathbf{\Omega} = \operatorname{rot} \mathbf{v}$ is the vorticity, $\mathbf{v}$ is the fluid velocity which satisfies the Euler equation,

$$\frac{\partial \mathbf{v}}{\partial t} - (\mathbf{v}\nabla)\mathbf{v} = -\frac{\nabla p}{\rho}, \quad (6.2)$$

and S the specific entropy advected by the fluid:

$$\frac{\partial S}{\partial t} + (\mathbf{v}\nabla) S = 0. \quad (6.3)$$

The density ρ is defined from the continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0. \quad (6.4)$$

We omit a proof of this theorem, the validity of which can be checked by direct calculations (see, for instance, Ref. [54]).

The invariance of I_{L} means that it depends only on the Lagrangian coordinates $\mathbf{a}$, and does not change in time moving together with a fluid particle.

As was mentioned before, the choice of the Lagrangian variables is arbitrary: they label each fluid particle. Therefore these coordinates are often called the Lagrangian markers. Usually the Lagrangian coordinates are chosen to coincide with the initial positions of the fluid particles, $\mathbf{r}|_{t=0} = \mathbf{a}$. Thus, a transition from one (Euler) description to another (Lagrangian) one is accomplished by means of a change of variables,

$$\mathbf{r} = \mathbf{r}(\mathbf{a}, t)\,, \tag{6.5}$$

with $\mathbf{a}$ being the label of each fluid particle. The velocity of a particle at point $\mathbf{r}$ is given by the usual formula

$$\mathbf{v}(\mathbf{r}, t) = \dot{\mathbf{r}}\Big|_{\mathbf{a}}\,, \tag{6.6}$$

where the dot means a derivative with respect to time t. In terms of the Lagrangian variables, the solution to the equations (6.4) and (6.3) can be written as

$$\rho(\mathbf{r}, t) = \frac{\rho(\mathbf{a})}{J}\,, \qquad S(\mathbf{r}, t) = S_0(\mathbf{a})\,, \tag{6.7}$$

where $J = \det \hat{J}_{ij}$ is a Jacobian and

$$\hat{J}_{i\alpha} = \frac{\partial x_i}{\partial a_\alpha}$$

is a Jacobi matrix of the mapping (6.5), which is assumed to be one-to-one. Further we will suppose $J \neq 0$ everywhere, that guarantees the existence of the mapping inverse to (6.5).

The Jacobi matrix plays the basic role. Knowing this allows the determination of not only the main flow parameters but also its geometrical characteristics, in particular the metric tensor. The equation of motion for the Jacobi matrix follows directly from the definition of the velocity (6.6). Consider the vector $\delta\mathbf{r}$ connecting two adjacent fluid particles:

$$\delta\mathbf{r} = \mathbf{r}(\mathbf{a} + \delta\mathbf{a}, t) - \mathbf{r}(\mathbf{a}, t)\,.$$

Using definition (6.6) it is easy to get the equation for this quantity:

$$\frac{\mathrm{d}\delta\mathbf{r}}{\mathrm{d}t} = (\delta\mathbf{r}, \boldsymbol{\nabla})\mathbf{v}\,. \tag{6.8}$$

Expanding then $\delta\mathbf{r}$ relative to the small vector $\delta\mathbf{a}$,

$$\delta x_i = \hat{J}_{ik}\delta a_k\,, \tag{6.9}$$

we arrive at the equation of motion for the Jacobi matrix,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{J} = U\hat{J}\,, \tag{6.10}$$

containing the matrix elements

$$U_{ij} = \frac{\partial v_i}{\partial x_j}\,.$$

The symmetric part of U,

$$B = \frac{1}{2}(U + U^{\mathrm{T}})\,,$$

is a stress tensor, and its antisymmetric part corresponds to the vorticity,

$$\Omega = \frac{1}{2}(U - U^{\mathrm{T}})\,.$$

Hence the equation for the matrix inverse to $\hat{J}$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{J}^{-1} = -\hat{J}^{-1}U\,, \tag{6.11}$$

that in the component notation has the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial a_\alpha}{\partial x_i} = -\frac{\partial a_\alpha}{\partial x_j}\frac{\partial v_j}{\partial x_i}\,. \tag{6.12}$$

The metric tensor is defined by means of the distances between two adjacent Lagrangian particles,

$$(\delta x_i)^2 = g_{ik}\delta a_i \delta a_k\,,$$

and equal to

$$g_{ik} = \hat{J}_{li}\hat{J}_{lk}\,.$$

The invariant I_{L} is local in Lagrangian variables. Therefore if one takes its convolution with an arbitrary function $f(\mathbf{a})$, then one can get the infinite family of conservation laws in the integral form:

$$I_i = \int I_{\mathrm{L}}(\mathbf{a})f(\mathbf{a})\,\mathrm{d}\mathbf{a}\,. \tag{6.13}$$

To begin with, we show that for barotropic fluids (when pressure p depends only on the density ρ) Kelvin's theorem follows from this relation. Notice that in this case there is one additional freedom: the entropy S has no link with the pressure and therefore instead of S in Eqns (6.1) and (6.7) we can take an arbitrary function of Lagrangian markers $\mathbf{a}$. Also one should note that in the first equation of (6.7), without any loss of generality, one can set $\rho_0(\mathbf{a}) = 1$ †, so that

$$\rho(\mathbf{r}, t) = \frac{1}{J}\,. \tag{6.14}$$

Substitute Eqn (6.1) into (6.13) and integrate once by parts. Accounting for Eqn (6.7) and $J\,\mathrm{d}\mathbf{a} = \mathrm{d}\mathbf{r}$, we get

$$I_i = \int \big(\mathbf{v}, [\boldsymbol{\nabla} f \times \boldsymbol{\nabla} S]\big)\,\mathrm{d}\mathbf{r}\,. \tag{6.15}$$

Here the gradient is taken with respect to $\mathbf{r}$, but the functions f and S are functions of $\mathbf{a} = \mathbf{a}(\mathbf{x}, t)$. Therefore we come back again to the integration with respect to $\mathbf{a}$. As a result of simple algebra we arrive at the expression

$$I_i = \int \dot{x}_i J\epsilon_{ijk}\frac{\partial a_\alpha}{\partial x_j}\frac{\partial a_\beta}{\partial x_k}\frac{\partial f(a)}{\partial a_\alpha}\frac{\partial S_0(a)}{\partial a_\beta}\,\mathrm{d}\mathbf{a}\,.$$

Taking then into account the identity

$$J\epsilon_{ijk}\frac{\partial a_\alpha}{\partial x_j}\frac{\partial a_\beta}{\partial x_k} = \epsilon_{\alpha\beta\gamma}\frac{\partial x_i}{\partial a_\gamma}\,, \tag{6.16}$$

the integral is transformed into

$$I_i = \int A_j(\mathbf{a})\dot{x}_i\frac{\partial x_i}{\partial a_j}\,\mathrm{d}\mathbf{a}\,. \tag{6.17}$$

Here the vector function $\mathbf{A}(\mathbf{a})$ reads:

$$\mathbf{A}(\mathbf{a}) = [\boldsymbol{\nabla} f \times \boldsymbol{\nabla} S_0]\,. \tag{6.18}$$

It has zero divergence:

$$\mathrm{div}_a\,\mathbf{A}(\mathbf{a}) = 0\,. \tag{6.19}$$

† Corresponding to a change of variables $\mathbf{b} = \mathbf{b}(\mathbf{a})$ which eliminates ρ_0: $J_{ab} = \rho_0$.

Note that till now we have never used the fact that the fluid is barotropic, i.e., equation (6.17) is applicable for any equation of state including the general dependence of the pressure on both the density and the entropy. For the barotropic case the entropy S_0 can be considered as an arbitrary function of $\mathbf{a}$. Therefore $\mathbf{A}(\mathbf{a})$ can also be considered as arbitrary with Eqn (6.19) the only constraint.

Let this vector function $\mathbf{A}(\mathbf{a})$ be concentrated on some closed curve: it is equal to zero everywhere outside this curve. We will parameterize the curve by the arc length s,

$$\mathbf{a} = \mathbf{a}(s), \qquad \mathbf{a}(s + l) = \mathbf{a}(s)\,, \tag{6.20}$$

where l is the curve length.

It is then easy to check that the function

$$\mathbf{A} = \int_0^l \frac{\mathrm{d}\mathbf{a}(s)}{\mathrm{d}s}\, \delta\big[\mathbf{a} - \mathbf{a}(s)\big]\, \mathrm{d}s$$

satisfies all the necessary conditions: it concentrates on the curve $\mathbf{a} = \mathbf{a}(s)$ and has zero divergence. Substituting this formula into integral (6.17), after simple integration, we come to Kelvin's theorem for a barotropic fluid:

$$I_{\mathrm{K}} = \int_C \big(\mathbf{v}(\mathbf{r}, t),\, \mathrm{d}\mathbf{l}\big)\,. \tag{6.21}$$

Here the contour C, moving together with the fluid, is the image of the closed curve (6.20). Thus, we have shown that Kelvin's theorem is a direct consequence of Ertel's theorem applied to the case of barotropic fluids.

Kelvin's theorem is also valid for an arbitrary dependence $p(\rho, S)$. This property is not widely known in the literature, for instance, it is absent in the Landau – Lifshits course. Curiously, the answer in this case will have the same form as (6.21). The only difference will be connected with the choice of contour. For the barotropic case, as we saw before, the only restriction was connected with condition (6.19) which provides the closure of the contour. For the general dependence $p = p(\rho, S)$, in addition to Eqn (6.19), one needs to satisfy the condition (6.18). According to the latter the lines of the vector field $\mathbf{A}(\mathbf{a})$ must lie on the surfaces of constant entropy $S_0(\mathbf{a})$. Therefore if we choose the closed contour lying on this (fluid!) surface we immediately arrive at Kelvin's theorem (6.21). Thus, Kelvin's theorem in the general case says that the velocity circulation is conserved in time if the fluid contour lies on the surface $S\big(\mathbf{a}(\mathbf{r}, t)\big) = \mathrm{const}$ advected by the fluid.

At the end of this section we examine an interesting interpretation of Kelvin's theorem. According to Ref. [11] conservation of the velocity circulation can be considered as a consequence of the conservation of the relative Poincare invariant

$$\oint \mathbf{p}\, \mathrm{d}\mathbf{q}\,. \tag{6.22}$$

For the barotropic flows one can relate to each fluid particle the Hamiltonian

$$h = \frac{p^2}{2} + w(\rho)\,,$$

where $\mathbf{p} = \dot{\mathbf{r}}$, and the enthalpy w plays the role of its potential energy.

If, instead of the contour in Eqn (6.22), one now takes the fluid contour, then it can be seen that the Poincare invariant will coincide with the velocity circulation

$$\oint \mathbf{v}\, \mathrm{d}\mathbf{r}\,,$$

and, thus, Kelvin's theorem becomes a direct consequence of the conservation of the relative Poincare invariant.

This concept has been very useful for other hydrodynamic systems, in particular, for some problems in plasma physics [52, 51], when the motion of a fluid particle can be reduced to the Hamiltonian equation for a charged particle in a magnetic field in the presence of a self-consistent potential. In such cases the analog of Kelvin's theorem is simply a consequence of the conservation of the relative Poincare invariant.

7. Gauge symmetry — relabeling group

In this section we consider how the conservation of the Ertel invariants follows from the variational principle.

To begin, we make two remarks.

Firstly, let $\mathbf{I}_\mathrm{l} = (I_1, \ldots, I_n)$ be a set of Lagrangian invariants, each moving with the fluid:

$$\frac{\mathrm{d}I_k}{\mathrm{d}t} = \frac{\partial I_k}{\partial t} + \mathbf{v}\nabla I_k = 0\,.$$

Then any function of $\mathbf{I}_\mathrm{l}$ will also be a Lagrangian invariant. To construct an Eulerian conservative density from the given Lagrangian one it is enough to be convinced that the quantity $I_{\mathrm{eu}} = \rho I_k$ obeys the continuity equation

$$\frac{\partial I_{\mathrm{eu}}}{\partial t} + \mathrm{div}\,(I_{\mathrm{eu}}\mathbf{v}) = 0\,.$$

The equations of ideal hydrodynamics, as we saw above, have two Lagrangian invariants, i.e., the Ertel invariant I_{L} given by Eqn (6.1) and s†. Both these integrals generate the conservation law

$$I_i = \int \rho f(I_{\mathrm{L}}, s)\, \mathrm{d}\mathbf{r}\,, \tag{7.1}$$

with $f(I_{\mathrm{L}}, s)$ being an arbitrary function of its arguments.

Secondly, the Euler equation (6.2) in terms of the Lagrangian variables is nothing other than the Newton equation for a fluid particle,

$$\ddot{x}_i = -\frac{\nabla_i p}{\rho}\,. \tag{7.2}$$

Multiplying this equation by the Jacobi matrix $\hat{J}$ we get

$$\frac{\partial x_i}{\partial a_k}\, \ddot{x}_i = -\frac{1}{\rho}\, \frac{\partial p(\rho, s)}{\partial a_k}\,. \tag{7.3}$$

This equation in the form (7.2) or (7.3) is closed by means of Eqns (6.7) and (6.14).

The action in terms of the Lagrangian (material) variables is written in the same form as in classical mechanics [17],

$$S = \int \mathrm{d}t\, L = \int \mathrm{d}t\, \mathrm{d}\mathbf{r} \left[\rho\, \frac{\dot{x}_i^2}{2} - \varepsilon(\rho, s)\right], \tag{7.4}$$

† To avoid confusion in this section only we denote the entropy as s, elsewhere the entropy retains the previous notation S.

where ε is the internal energy density connected with the enthalpy w by means of the thermodynamic relation

$$d\varepsilon = \rho T\, ds + w\, d\rho\,, \tag{7.5}$$

with T denoting temperature.

Let us now check that varying the action, $\delta S = 0$, is equivalent to the equation of motion (7.3).

At first let us pass from integration over $\mathbf{r}$ to $\mathbf{a}$ in Eqn (7.4). As a result, the action can then be transformed as follows:

$$S = \int dt\, d\mathbf{a} \left[\frac{\dot{x}_i^2}{2} - \tilde{\varepsilon}(\rho, s) \right]. \tag{7.6}$$

Here the time derivative of x is taken for fixed $\mathbf{a}$, $\tilde{\varepsilon} = \varepsilon/\rho$ is the function of ρ and s which are defined with the help of relations (6.7) and (6.14). Because only ρ in the internal energy $\tilde{\varepsilon}$ contains the dependence on x through the Jacobian (6.14), the main difficulty with a variation will be connected with the second term in Eqn (7.6).

Using both the identity (6.16) and the formula

$$J = \frac{1}{6} \epsilon_{ijk} \epsilon_{\alpha\beta\gamma} \frac{\partial x_i}{\partial a_\alpha} \frac{\partial x_j}{\partial a_\beta} \frac{\partial x_k}{\partial a_\gamma}\,,$$

one can get

$$\begin{aligned} \delta S &= \int dt\, d\mathbf{a} \left(-\ddot{x}_i \delta x_i + \rho^2 \frac{\partial \tilde{\varepsilon}}{\partial \rho}\, \delta J \right) \\ &= \int dt\, d\mathbf{a} \left[-\ddot{x}_i - \frac{1}{2} \frac{\partial}{\partial a_\alpha} \left(\rho^2 \frac{\partial \tilde{\varepsilon}}{\partial \rho} \right) \epsilon_{\alpha\beta\gamma} \epsilon_{ijk} \frac{\partial x_j}{\partial a_\beta} \frac{\partial x_k}{\partial a_\gamma} \right] \delta x_i \\ &= \int dt\, d\mathbf{a} \left[-\ddot{x}_i - \frac{1}{\rho} \frac{\partial}{\partial a_\alpha} \left(\rho^2 \frac{\partial \tilde{\varepsilon}}{\partial \rho} \right) \frac{\partial a_\alpha}{\partial x_i} \right] \delta x_i = 0\,, \end{aligned} \tag{7.7}$$

or

$$\ddot{x}_i = -\frac{1}{\rho} \frac{\partial}{\partial a_\alpha} \left(\rho^2 \frac{\partial \tilde{\varepsilon}}{\partial \rho} \right) \frac{\partial a_\alpha}{\partial x_i}\,.$$

Hence it is seen that the resulting equation coincides with the equation of motion (7.3) if one puts

$$p(\rho, s) = \rho^2 \frac{\partial \tilde{\varepsilon}}{\partial \rho}\,.$$

[The last equality is a direct consequence of the thermodynamic relation (7.5).]

Thus, we have proved that the equations of motion of an ideal fluid in the Lagrangian form follow directly from the variational principle.

The simplest conservation laws, i.e., the conservation of momentum

$$\mathbf{P} = \int \dot{\mathbf{x}}\, d\mathbf{a} = \int \rho \mathbf{v}(\mathbf{r}, t)\, d\mathbf{r}\,,$$

and the conservation of energy,

$$E = \int \left[\frac{\dot{\mathbf{x}}^2}{2} + \tilde{\varepsilon}(\rho, s) \right] d\mathbf{a} = \int \left[\frac{\rho v^2}{2} + \varepsilon(\rho, s) \right] d\mathbf{r}\,,$$

follow as a result of the invariance of the action relative to two independent symmetries, translations in space and time.

The equations of hydrodynamics, as was first shown in Ref. [41], have an additional non-trivial symmetry connected with the arbitrariness in the possible choice of the Lagrangian markers. Nothing should depend on this choice: the fluid dynamics as well as the equations of motion remain the same. From all possible relabeling transformations, the action invariance requirement restrains some certain class. In the case of barotropic fluids the action appears to be invariant if the transformations $\mathbf{b} = \mathbf{b}(\mathbf{a})$ are incompressible, i.e., for which the Jacobian is equal to 1:

$$J = \det \frac{\partial b_i}{\partial a_j} = 1\,. \tag{7.8}$$

All these transformations form the group of diffeomorphisms preserving the volume. (It is interesting to note that the same group governs the motion of an incompressible fluid.) This symmetry, in accordance with the Noether theorem, generates new conservation laws. To find them it is enough to consider infinitesimal transformations. In the given case those are defined by

$$\mathbf{b} = \mathbf{a} + \delta\mathbf{a}\,,$$

where the function $\delta\mathbf{a} = \boldsymbol{\alpha}$ satisfies the condition

$$\frac{\partial \alpha_i(\mathbf{a})}{\partial a_i} = 0\,, \tag{7.9}$$

which is a direct consequence of Eqn (7.8).

For the general equation of state $p = p(\rho, s)$ the invariance of the action implies that the transformations should preserve the surfaces $s_0(\mathbf{a}) = \text{const}$, being simultaneously incompressible. As a result, we have one additional constraint on the function $\boldsymbol{\alpha}(\mathbf{a})$:

$$[\nabla s \times \boldsymbol{\alpha}] = 0\,. \tag{7.10}$$

If in the first case Eqn (7.9) can be resolved by introducing the vector potential

$$\boldsymbol{\alpha} = \operatorname{rot} \boldsymbol{\zeta}\,,$$

for example, with the Coulomb gauge $\operatorname{div} \boldsymbol{\zeta} = 0$, then in the general case both equations (7.9) and (7.10) are satisfied if one puts

$$\boldsymbol{\alpha} = [\nabla s \times \nabla \psi]\,.$$

Here ψ is a scalar function and the gradient is taken with respect to $\mathbf{a}$.

Omitting all the intermediate derivation of the conservation law (it is a standard procedure, for reference see, for instance, Ref. [80]) we present only the final answers:

(i) For a **barotropic** fluid the conservation law has the form

$$\frac{d}{dt} [\nabla_a \dot{x}_i \times \nabla_a x_i] = 0\,,$$

or it gives the whole conserved vector

$$\mathbf{I}_{\mathrm{L}} = [\nabla_a \dot{x}_i \times \nabla_a x_i]\,. \tag{7.11}$$

This integral has been known since the last century: it was found by Cauchy [17] (see also Refs [44, 45]).

The matrix notation of equation (7.11) has the form

$$\hat{J}_t^{\mathrm{T}} \hat{J} - \hat{J}^{\mathrm{T}} \hat{J}_t = \Omega^{(0)} , \tag{7.12}$$

where the index T means transposition, and the matrix $\Omega^{(0)}$ is expressed through the vector invariant $\mathbf{I}_{\mathrm{L}}$ with the help of the formula

$$\Omega_{ij}^{(0)} = \epsilon_{ijk} I_{\mathrm{L}_k} .$$

Recently this matrix representation of equation (7.12) was used by the authors of paper [62] to construct a set of exact three-dimensional solutions to the Euler equation for incompressible fluids.

Returning to the Euler description and using identity (6.16) this vector integral can be transformed into the form

$$\mathbf{I}_{\mathrm{L}} = J(\boldsymbol{\Omega}, \boldsymbol{\nabla})\mathbf{a} \equiv \frac{\rho_0(\mathbf{a})}{\rho} (\boldsymbol{\Omega}, \boldsymbol{\nabla})\mathbf{a} . \tag{7.13}$$

Here $\mathbf{a}$ is considered as a function of $\mathbf{r}$ and t. If $\mathbf{a}$ are the initial coordinates of fluid particles, then the vector (7.13) can be expressed through the initial distributions $\boldsymbol{\Omega}_0(\mathbf{a})$ and $\rho_0(\mathbf{a})$ as follows:

$$\mathbf{I}_{\mathrm{L}} = \boldsymbol{\Omega}_0(\mathbf{a}) .$$

From (7.13) it follows immediately for the vector $\mathbf{B} = \boldsymbol{\Omega}/\rho$ that

$$\mathbf{B}(\mathbf{r}, t) = \hat{J}\mathbf{B}_0(\mathbf{a}) .$$

Thus, the Jacobi matrix becomes the evolution operator for the vector $\boldsymbol{\Omega}/\rho$.

The invariants (7.13), indeed, are well-known in hydrodynamics but in a slightly different form. Let us write down the equation of motion for the fraction $\boldsymbol{\Omega}/\rho$ which directly follows from Eqns (6.2) and (6.4):

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{B} = (\mathbf{B}, \boldsymbol{\nabla})\mathbf{v} . \tag{7.14}$$

Here $\mathrm{d}/\mathrm{d}t \equiv \partial/\partial t + v\boldsymbol{\nabla}$. Comparing this equation with equation (6.8) for $\delta\mathbf{r}$ one can see that both the quantities $\mathbf{B}$ and $\delta\mathbf{r}$ obey the same equation. This means that the vorticity is frozen into a fluid, a well-known statement in hydrodynamics. Sometimes this property is called as the frozenness of the vorticity into a fluid. Then, multiplying Eqn (7.14) from the right by $\hat{J}^{-1}$ and Eqn (6.11) from the left by $\boldsymbol{\Omega}/\rho$, after summation of the obtained results we arrive at the conservation of the vector invariant (7.13). These integrals are the mathematical formulation of the frozenness of the vorticity into a fluid. The corresponding equation for the vector field $\mathbf{B}$ is called the frozenness equation.

(ii) In the **general** case (for an arbitrary dependence of pressure on both density and entropy) the only scalar that survives from this vector invariant is a projection of $\mathbf{I}_{\mathrm{L}}$ onto the vector $\boldsymbol{\nabla}s$:

$$I_{\mathrm{L}} = \left(\boldsymbol{\nabla}_a s_0, [\boldsymbol{\nabla}_a \dot{x}_i \times \boldsymbol{\nabla}_a x_i]\right) .$$

Here all derivatives are taken with respect to $\mathbf{a}$. Passing to the Eulerian variables and using the identity

$$\epsilon_{\alpha\beta\gamma} \frac{\partial x_i}{\partial a_\alpha} \frac{\partial x_j}{\partial a_\beta} \frac{\partial x_k}{\partial a_\gamma} = \epsilon_{ijk} J ,$$

one can get

$$I_{\mathrm{L}} = \frac{(\boldsymbol{\Omega}\boldsymbol{\nabla}s)}{\rho} .$$

This integral is the Ertel invariant (6.1). Thus, the conservation of the Ertel invariant as well as Kelvin's theorem about the conservation of the velocity circulation are a consequence of a specific gauge symmetry — the relabeling group.

It is interesting to follow how all the above formulae transform in two dimensions. In this case the Ertel invariant is identically equal to zero, due to the orthogonality of the vectors $\boldsymbol{\Omega}$ and $\boldsymbol{\nabla}s$. Therefore non-trivial answers appear only for a barotropic fluid.

Applying the identity

$$\epsilon_{\alpha\beta} \frac{\partial x_i}{\partial a_\alpha} \frac{\partial x_j}{\partial a_\beta} = \epsilon_{ij} J$$

to Eqn (7.11), it is easy to get that the Cauchy invariant transforms into the well-known Lagrangian invariant:

$$\frac{\Omega}{\rho} = \mathrm{const}(a) .$$

It is important to note that, unlike the three-dimensional case, this relation does not contain the Jacobi matrix.

Let us turn to incompressible fluids. In this case the obtained formulae are simplified. For example, relation (7.13) in three dimensions is written in the form

$$\mathbf{I}_{\mathrm{L}} = (\boldsymbol{\Omega}, \boldsymbol{\nabla})\mathbf{a} . \tag{7.15}$$

In formula (7.15) $\mathbf{I}_{\mathrm{L}}$ coincides with

$$\boldsymbol{\Omega}_0(\mathbf{a}) = \mathrm{rot}_a \mathbf{u} ,$$

where the vector $\mathbf{u}$ is defined by means of Eqn (4.19). This, in particular, means that the transverse part of the vector $\mathbf{u}$ is conserved (being the Lagrangian invariant), and its temporal variation is due to its longitudinal part. Moreover, as pointed out in the fourth section, the choice of this vector is arbitrary due to the arbitrariness in the choice of Lagrangian markers. The same applies to the vector $\boldsymbol{\Omega}_0(\mathbf{a})$. If one performs the contact transformations $\mathbf{b} = \mathbf{b}(\mathbf{a})$ under the condition $\partial(b_1 b_2 b_3)/\partial(a_1 a_2 a_3) = 1$, then the vector $\boldsymbol{\Omega}_0(\mathbf{a})$ will be transformed as

$$\tilde{\Omega}_{0i}(\mathbf{b}) = \frac{\partial b_i}{\partial a_j} \Omega_{0j}(\mathbf{a}) . \tag{7.16}$$

This is a transformation of the gauge type, being the generalization [45] of the gauge transformations for the Clebsch variables (4.15)†.

Let, as a result of these transformations, the vector $\tilde{\boldsymbol{\Omega}}_0(\mathbf{b})$ have one nonzero component, say, a z-component, equal to 1:

$$\tilde{\Omega}_{01} = (\boldsymbol{\Omega}_0 \boldsymbol{\nabla}_a) b_1 = 0 , \tag{7.17}$$

$$\tilde{\Omega}_{02} = (\boldsymbol{\Omega}_0 \boldsymbol{\nabla}_a) b_2 = 0 , \tag{7.18}$$

$$\tilde{\Omega}_{03} = (\boldsymbol{\Omega}_0 \boldsymbol{\nabla}_a) b_3 = 1 . \tag{7.19}$$

† Another approach to gauge transformations in hydrodynamics was developed in Ref. [81].

These relations within the given 'vorticity' $\mathbf{\Omega}_0(\mathbf{a})$ represent the equations to determine the dependence $\mathbf{b}(\mathbf{a})$. These are the linear differential equations of the first order, which allow the application of the method of characteristics. The equations for characteristics are the same here for all three equations of the system (7.17)–(7.19),

$$\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}s} = \mathbf{\Omega}_0(\mathbf{a}),$$

that define the 'vortex' line for $\mathbf{\Omega}_0(\mathbf{a})$. (Here s may be understood as the arc length of the 'vortex' line.) Equations on the characteristics (for the components of $\mathbf{b}$) are then given by

$$\frac{\mathrm{d}b_1}{\mathrm{d}s} = 0, \tag{7.20}$$

$$\frac{\mathrm{d}b_2}{\mathrm{d}s} = 0, \tag{7.21}$$

$$\frac{\mathrm{d}b_3}{\mathrm{d}s} = 1. \tag{7.22}$$

The first two components b_1 and b_2 are constant along the characteristics. Therefore b_1 and b_2 can be chosen as two independent integrals c_1 and c_2 of the system for the characteristics, and the third component is a linear function of the arc length s. It is important to notice that a solution to the system (7.20)–(7.22) can always be found, at least locally, in the vicinity of some nonsingular surface supplied with a coordinate system given, say, by the invariants c_1 and c_2. Rigorously speaking this is not a global solution as it is usual when one uses the method of characteristics.

Hence, by using the equation $\mathrm{rot}_b\,\tilde{\mathbf{u}} = \tilde{\mathbf{\Omega}}_0(b)$, one can reconstruct the velocity $\tilde{\mathbf{u}}$:

$$\tilde{u}_1 = \frac{\partial\phi}{\partial b_1}, \tag{7.23}$$

$$\tilde{u}_2 = \frac{\partial\phi}{\partial b_2} + b_1, \tag{7.24}$$

$$\tilde{u}_3 = \frac{\partial\phi}{\partial b_3}. \tag{7.25}$$

After substitution of these expressions into equation (4.19) we come back to the Clebsch representation with one pair of canonical variables (for more details, see Ref. [44]) which yields

$$\mathbf{v} = b_1\nabla b_2 + \nabla\phi.$$

So, the vorticity $\mathbf{\Omega}(\mathbf{r}, t)$ takes the form

$$\mathbf{\Omega}(\mathbf{r}, t) = [\nabla b_1 \times \nabla b_2] = \frac{\partial\mathbf{r}}{\partial b_3}(\mathbf{b}, t). \tag{7.26}$$

The last equality is a direct consequence of the fact that transformation $\mathbf{b} = \mathbf{b}(\mathbf{r},t)$ is a diffeomorphism preserving the volume. It is easy to check that expression (7.26) with $\mathbf{r}$ replaced by $\mathbf{b}$ also satisfies the system (7.17)–(7.19). In this case the first equation of the system becomes the equation $\partial(b_1b_2b_3)/\partial(a_1a_2a_3) = 1$.

Thus, locally any flow of incompressible fluid can be parameterized by one pair of the Clebsch variables. In the general situation one needs two pairs of such variables.

8. The Hopf invariant and the degeneracy of the Poisson brackets

So far we have not discussed the question of which classes of flows are described by the canonical variables introduced in the preceding sections.

To begin with, we consider this question for the example of an ideal incompressible fluid.

Let a flow be parameterized in terms of Clebsch variables in a simply-connected domain:

$$\mathbf{v} = \lambda\nabla\mu + \nabla\varphi.$$

Take some point inside this domain and draw through this point some closed curve. Starting from this point and constructing continuously Clebsch variables on each piece of this curve we come back to the original point. Generally speaking, the Clebsch variables will take different values. Thus, the Clebsch variables will be multi-valued functions of space coordinates. One partial case of fluid flows with multi-valued Clebsch variables allows the following geometrical interpretation.

Consider a compact oriented two-dimensional manifold M^2 and suppose that λ and μ are local coordinates on this manifold.

The gauge transformations associated with the non-uniqueness of the choice of Clebsch variables lead to the appearance of a whole family of gauge-equivalent manifolds obtainable from one another by continuous deformations preserving the surface element:

$$\mathrm{d}\lambda\,\mathrm{d}\mu = \mathrm{d}\lambda'\,\mathrm{d}\mu'.$$

It is therefore sufficient to select one representative from each such family. For example, among the surfaces of genus zero having the same area, it is natural to select the sphere S^2.

It is easy to understand that the inverse image of any point of M^2 in R^3 is a closed curve coinciding with a vortex line. This follows directly from the expression for the curl of the velocity:

$$\mathbf{\Omega} = \mathrm{rot}\,\mathbf{v} = \left[\nabla\lambda \times \nabla\mu\right]. \tag{8.1}$$

The vortex line is the intersection of the two surfaces $\lambda(\mathbf{r}) = \mathrm{const}$, $\mu(\mathbf{r}) = \mathrm{const}$. If the variables λ and μ are single-valued functions, then the manifold M^2 cannot be a closed surface of genus g. Then the flows given by such variables have no knots. This fact can also be proved differently.

It is known [56, 57] that the degree of knottiness of a flow is characterized in ideal hydrodynamics by the conserved quantity

$$I = \int(\mathbf{v}, \mathrm{rot}\,\mathbf{v})\,\mathrm{d}\mathbf{r}. \tag{8.2}$$

The conservation of this integral follows immediately from Kelvin's theorem. In order to illustrate this statement, following [57] we consider two closed vortex lines

$$\mathbf{\Omega} = \int \kappa_1\mathbf{n}_1\delta\left[\mathbf{r} - \mathbf{l}_1(s_1)\right]\mathrm{d}s_1 + \int \kappa_2\mathbf{n}_2\delta\left[\mathbf{r} - \mathbf{l}_2(s_2)\right]\mathrm{d}s_2,$$

where $\mathbf{n}_{1,2}$ are the tangents and $\mathrm{d}s_{1,2}$ the arc elements of these curves.

Calculating the velocity circulation around the contours $\mathbf{r} = \mathbf{l}_1(s_1)$ and $\mathbf{r} = \mathbf{l}_2(s_2)$, we find

$$\oint (\mathbf{v},\, \mathrm{d}\mathbf{l}_1) = m\kappa_2 \,, \qquad \oint (\mathbf{v}, \mathrm{d}\mathbf{l}_2) = m\kappa_1 \,,$$

where m is the linking number of these two curves. Multiplying the first equation by κ_2 and the second by κ_1, and adding the results, we get the integral I:

$$\int (\mathbf{v}, \kappa_1\, \mathrm{d}\mathbf{l}_1 + \kappa_2\, \mathrm{d}\mathbf{l}_2) = \int (\mathbf{v}, \operatorname{rot}\mathbf{v})\, \mathrm{d}\mathbf{r} = 2m\kappa_1\kappa_2 \,.$$

This formula is generalized without difficulty to a vortex, and then to a continuous distribution. The conservation law (8.2) is valid not only for an infinite region but for a finite one when the vorticity lines are tangent to the boundary.

This integral is thus identically equal to zero for a flow with trivial topology, in particular, for flows parameterized in terms of single-valued Clebsch variables.

We shall show that the Clebsch variables in the formulation (8.1) describe knotted flows, and illustrate their topological meaning.

Suppose that the variables λ and μ are local coordinates on S^2. In this case λ and μ are expressed in terms of the polar and azimuthal angles, θ and φ, so that

$$\mathbf{\Omega} = 2A\left[\nabla \cos\theta \times \nabla\varphi\right] ,$$

where A is a dimensional constant. Now the Clebsch variables are no longer single-valued functions, and on a contour enclosing the z axis the angle φ acquires an addition 2π. It is also convenient to go over, in the expression for the vector field Ω, from the angles θ and φ to the $\mathbf{n}$-field ($\mathbf{n}^2 = 1$) [58]:

$$\Omega_\alpha = \varepsilon_{\alpha\beta\gamma}\left(\mathbf{n}, [\partial_\beta\mathbf{n} \times \partial_\gamma\mathbf{n}]\right) . \tag{8.3}$$

We shall limit our considerations to the flows for which $\mathbf{n}$ tends sufficiently rapidly at infinity to a constant vector $\mathbf{n}_0$. For this class of flows R^3 is isomorphic to the three-dimensional sphere S^3. Thus the classification of the flows is a problem of classification of smooth mappings $S^3 \to S^2$. Such mappings are characterized by the homotopy group $\pi_3(S^2) = \mathrm{Z}$, i.e., any class of flows is characterized by the linking number that coincides with the winding number of any two lines $\mathbf{n}(\mathbf{r}) = \mathbf{n}_1$ and $\mathbf{n}(\mathbf{r}) = \mathbf{n}_2$ ($\mathbf{n}_{1,2} = \text{const}$). The index N for smooth mappings is called the Hopf invariant [59]. One can show that the Hopf invariant coincides with the integral I up to a constant factor [60]:

$$I = \int (\mathbf{v}, \mathbf{\Omega})\, \mathrm{d}\mathbf{r} = 64\pi^2 N A^2 \,.$$

The derivation of this relation is based on the well-known formula of Gauss for the linking number of two curves.

It should be mentioned that in the quantum case, according to Ref. [60], $A = \hbar/2m$. The remaining manifolds are of secondary interest from the point of view of topology. Say, a manifold M^2, which is a surface with boundary, is homotopic to a bouquet of circles. Therefore its homotopic group π_3 is trivial. The groups π_3 are also trivial for closed surfaces of genus $g \geqslant 1$. Topologically non-trivial situations occur only for surfaces with zero genus.

We now give an example of a non-trivial mapping with $N = 1$ (the Hopf mapping):

$$(\mathbf{n}, \sigma) = q^+ \sigma_3 q \,, \qquad q = (1 - \mathrm{i}\mathbf{r}\sigma)(1 + \mathrm{i}\mathbf{r}\sigma)^{-1} \,, \tag{8.4}$$

where σ are the Pauli matrices.

In toroidal coordinates, one has

$$x + \mathrm{i}y = \frac{\sinh U}{\cosh U + \cos\beta} \exp(\mathrm{i}\alpha) \,, \qquad z = \frac{\sin\alpha}{\cosh U + \cos\beta}$$
$$(0 \leqslant U < \infty \,, \qquad 0 < \alpha \,, \qquad \beta < 2\pi) \,,$$

and Eqn (8.4) reads

$$\arctan\frac{n_y}{n_x} = \alpha - \beta \,, \qquad n_z = 1 - 2\tanh^2 U \,.$$

These formulae show that the flow looks as follows: the whole space is sliced up by the tori $U = \text{const}$, while any vortex line coils up on a torus, making one loop. Thus any vortex line links once. The expressions for $\mathbf{\Omega}$ and $\mathbf{v}$, calculated from Eqn (8.4) are not a solution of the stationary Euler equations, and can therefore be used as initial conditions for (5.9). It is obvious that the evolution of such a distribution does not take the solution out of the given class with Hopf invariant $N = 1$. The evolution of the vector field $\mathbf{n}$ is determined from the equation

$$\mathbf{n}_t + (\mathbf{v}\nabla)\mathbf{n} = 0 \,, \tag{8.5}$$

which is equivalent to the evolution equation for the variables λ and μ. Equations (8.5) are also Hamiltonian,

$$\mathbf{n}_t = 2A\left[\mathbf{n} \times \frac{\delta H}{\delta \mathbf{n}}\right],$$

and differ from the familiar Landau – Lifshits equations only by the choice of the Hamiltonian H.

The Poisson brackets in this case coincide with the BKK brackets (2.8), (2.11):

$$\{F, G\} = 2A \int \left(\mathbf{n},\, \left[\frac{\delta F}{\delta \mathbf{n}} \times \frac{\delta H}{\delta \mathbf{n}}\right]\right) \mathrm{d}\mathbf{r} \,.$$

When we go over in these brackets from the $\mathbf{n}$-field to $\mathbf{\Omega}$ according to formula (8.3) we get the Poisson brackets (5.8). It is important to note that brackets (5.8) are degenerate with respect to the invariant I: $\{I, \ldots\} = 0$, which again shows its origin. On one side, it is connected with its topology, on the other, with Kelvin's theorem. One should recall that the latter is a sequence of the gauge symmetry of the Lagrangian markers.

As we shall see below, the question about the degeneracy of the Poisson brackets for an arbitrary equation of state is directly connected with the gauge symmetry.

Let us discuss in more details this question for the hydrodynamic Poisson brackets. For this aim, we consider the most general form of the brackets for ideal hydrodynamics, namely, for non-barotropic fluids. The brackets in this case have the form of Eqn (5.6):

$$\begin{aligned}\{F, G\} = &\int \left[\left(\nabla \frac{\delta F}{\delta \rho}, \frac{\delta G}{\delta \mathbf{v}}\right) - \left(\nabla \frac{\delta G}{\delta \rho}, \frac{\delta F}{\delta \mathbf{v}}\right)\right] \mathrm{d}\mathbf{r} \\ &+ \int \left(\frac{\operatorname{rot}\mathbf{v}}{\rho}, \left[\frac{\delta F}{\delta \mathbf{v}} \times \frac{\delta G}{\delta \mathbf{v}}\right]\right) \mathrm{d}\mathbf{r} \\ &+ \int \left(\frac{\nabla S}{\rho}, \left[\frac{\delta F}{\delta \mathbf{v}}\frac{\delta G}{\delta S} - \frac{\delta G}{\delta \mathbf{v}}\frac{\delta F}{\delta S}\right]\right) \mathrm{d}\mathbf{r}.\end{aligned} \tag{8.6}$$

By substituting integral (7.1), $I_i = \int \rho f(I_{\mathrm{L}}, S)\,\mathrm{d}\mathbf{r}$, into this expression one can verify that this integral commutes with any functional:

$$\{I_i, .\} = 0\,.$$

In accordance with the definition of Section 2, this integral represents a Casimir of the brackets (8.6).

One should recall that the conservation of the integral (7.1) is a consequence of the special gauge symmetry of the ideal hydrodynamics equations, which, as we see, is responsible also for the degeneracy of the Poisson brackets.

In order to transform from these brackets to the canonical brackets it is necessary to resolve integral (7.1) by introducing new coordinates. We have already found one answer to the question of how to do it. If we take expression (4.17) for the velocity and put the Ertel invariant I_{L} instead of μ then integral (7.1) transforms into the dynamical conservation law with respect to the canonical brackets

$$\{F, G\} = \int \left[\left(\frac{\delta F}{\delta \rho} \frac{\delta G}{\delta \varphi} - \frac{\delta F}{\delta \varphi} \frac{\delta G}{\delta \rho} \right) + \left(\frac{\delta F}{\delta \lambda} \frac{\delta G}{\delta I_{\mathrm{L}}} - \frac{\delta F}{\delta I_{\mathrm{L}}} \frac{\delta G}{\delta \lambda} \right) + \left(\frac{\delta F}{\delta \beta} \frac{\delta G}{\delta s} - \frac{\delta F}{\delta s} \frac{\delta G}{\delta \beta} \right) \right] \mathrm{d}\mathbf{r}$$

so that

$$\{I_i, H\} = 0\,.$$

We can also remark that, as was shown by van Saarlos [53], the transition from the Lagrangian description in terms of the action (7.6) to the canonical variables is determined through the change (9.11) or (4.20).

9. Inhomogeneous fluid and surface waves

In this section we introduce canonical coordinates for the description of nonlinear waves in an ideal fluid of variable density. Here one distinguishes two types of waves. The first type refers to the so-called internal waves, propagating in a continuous medium with a smooth inhomogeneity. The second type refers to the situation where the density gradient changes sharply over the size of the wave length, and in the limit represents simply a jump. In this limit we talk about surface waves. Canonical variables can be introduced in both cases within the framework of the scheme developed in the preceding sections.

Consider an ideal fluid of varying density in the presence of a constant gravitational field $\mathbf{g}$ anti-parallel to the z axis. The fluid is assumed to be locally incompressible. This means that the density is convected along the fluid and is therefore a Lagrange variable:

$$\frac{\partial \rho}{\partial t} + (\mathbf{v}\nabla)\rho = 0 \quad \text{for} \quad \operatorname{div} \mathbf{v} = 0\,.$$

These two equations therefore appear in the Lagrangian as constraints:

$$L = \int \left[\rho\,\frac{\mathbf{v}^2}{2} - U(\rho, \mathbf{r}) - \alpha \left(\frac{\partial \rho}{\partial t} + (\mathbf{v}\nabla)\rho \right) + \varphi \operatorname{div} \mathbf{v} \right] \mathrm{d}\mathbf{r}\,. \tag{9.1}$$

Here $U(\rho, \mathbf{r})$ is the density of potential energy in the presence of the field $\mathbf{g}$, given by the expression

$$U(\rho, \mathbf{r}) = g \left[\rho(\mathbf{r}_{\perp}, z)(z - z') - \int_{z'}^{z} \rho_0(z'')\,\mathrm{d}z'' \right]. \tag{9.2}$$

The first term in this expression corresponds to the work in lifting a fluid element to the point z from the equilibrium point z', determined from the condition for equality of the equilibrium density $\rho_0(z')$ and the density of fluid at the given point:

$$\rho_0(z') = \rho(\mathbf{r}_{\perp}, z)\,.$$

This relation gives z' as a function $z' = z'(\rho)$ of density. The second term in Eqn (9.2) corresponds to the potential of the Archimedean force.

Variations of the Lagrangian with respect to $\mathbf{v}$ and φ lead us to the equations [26]

$$\rho \mathbf{v} = \nabla \varphi + \alpha \nabla \rho \tag{9.3}$$

and

$$\operatorname{div} \left[\rho^{-1} (\nabla \varphi + \alpha \nabla \rho) \right] = 0\,,$$

giving the connection between the new and old variables. Varying with respect to variable ρ, we get an equation

$$\frac{\partial \alpha}{\partial t} + (\mathbf{v}\nabla)\alpha + \frac{\mathbf{v}^2}{2} - \frac{\partial U}{\partial \rho} = 0$$

for the potential α with $\partial U / \partial \rho = g(z - z')$.

Next, substituting Eqn (9.3) into the Euler equation (4.8) and using the equations of motion for α and ρ, we obtain an expression for the pressure p up to a constant, analogous to the Bernoulli integral:

$$p = -\rho\,\frac{\mathbf{v}^2}{2} - \rho g (z - z') + \left(\frac{\partial}{\partial t} + (\mathbf{v}\nabla) \right) \varphi + \text{const}\,.$$

The Hamiltonian is formed in the standard way and coincides with the total energy

$$H = \int \left[\rho\,\frac{\mathbf{v}^2}{2} + U(\rho, \mathbf{r}) \right] \mathrm{d}\mathbf{r}\,,$$

while the variables α and ρ happen to be canonically conjugate:

$$\frac{\partial \alpha}{\partial t} = \frac{\delta H}{\delta \rho}\,, \qquad \frac{\partial \rho}{\partial t} = -\frac{\delta H}{\delta \alpha}\,.$$

The parameterization, presented here, for the velocity in terms of the density ρ and α imposes strong restrictions on the form of the initial distribution. As we see from Eqn (9.3), the curl of the mass current at all times, including the initial time, is orthogonal to the density gradient. Such motions are the analog of potential motions in a homogeneous fluid. This scheme can be considerably improved if one includes ‘non-potential’ motions. As far as the non-canonical Poisson brackets are concerned, they were introduced in paper [63].

If one needs to consider weakly nonlinear oscillations in a stratified fluid one should expand the Hamiltonian in powers of α and $\delta\rho$. In particular, the well-known Boussinesq approximation is obtained if the density in the kinetic energy is replaced by some averaged constant quantity:

$$H_{\mathrm{B}} = \int \left[\frac{\rho_0 \mathbf{v}^2}{2} + U(\rho, z) \right] \mathrm{d}\mathbf{r}\,.$$

Now let us consider one important limiting case of a stratified fluid, when the stratification is only due to a free boundary.

First we look at potential motions. Here the Lagrangian has the same form as before, in which the density ρ should be regarded as constant throughout the volume of the fluid, i.e.,

$$\rho = \rho_0 \theta\big(z - \eta(\mathbf{r}_\perp, t)\big)\,.$$

Here $\theta(z)$ is the Heaviside function, and $\eta(\mathbf{r}_\perp, t)$ is the deviation of the free surface from the horizontal plane $z = 0$. The element $\mathrm{d}s_\mathrm{n} = \mathrm{d}\mathbf{r}_\perp\big[1 + (\boldsymbol{\nabla}\eta)^2\big]^{1/2}$ of free surface and the vector normal to it, $\mathbf{n} = (-\boldsymbol{\nabla}\eta)\big[1 + (\boldsymbol{\nabla}\eta)^2\big]^{-1/2}$, are expressed in terms of the function $\eta(\mathbf{r}_\perp)$ explicitly, as is the potential energy

$$U = \int \left\{ \frac{\rho_0 g \eta^2}{2} + \sigma\left[\sqrt{1 + (\boldsymbol{\nabla}\eta)^2} - 1\right]\right\} \mathrm{d}\mathbf{r}_\perp\,,$$

in which we have taken into account the surface tension with coefficient σ.

It is easy to see that the continuity equation in the present case becomes the kinematic condition

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = \left(\frac{\partial}{\partial t} + \mathbf{v}\boldsymbol{\nabla}\right)\eta = v_z\,. \tag{9.4}$$

In accordance with this, the Lagrangian expresses

$$L = \int \mathrm{d}\mathbf{r}_\perp \int_{-h}^{\eta} \mathrm{d}z \left(\frac{\rho_0 \mathbf{v}^2}{2} + \varphi\,\mathrm{div}\,\mathbf{v}\right) + \int \psi\left[\frac{\partial\eta}{\partial t} - v_\mathrm{n}\sqrt{1 + (\boldsymbol{\nabla}\eta)^2}\right) \mathrm{d}\mathbf{r}_\perp - U\,. \tag{9.5}$$

Here

$$v_\mathrm{n} = \frac{[v_z - \mathbf{v}\boldsymbol{\nabla}\eta]_{z=\eta}}{\sqrt{1 + (\boldsymbol{\nabla}\eta)^2}}$$

is the normal component of the velocity and $\psi = -\alpha\rho_0$ the Lagrange multiplier given on the free surface.

The variation of L with respect to $\mathbf{v}$ within the bulk leads to the potential equation $\rho_0\mathbf{v} = \boldsymbol{\nabla}\varphi$, where φ is determined from the solution of the Laplace equation $\Delta\varphi = 0$. The variation of L with respect to $\mathbf{v}$ on the boundary (for fixed η) gives the boundary conditions for the Laplace equation:

$$\varphi\Big|_{z=\eta} = \psi\,. \tag{9.6}$$

The variation of the Lagrangian with respect to η is non-trivial. For this it is convenient to rewrite all the terms in (9.5) containing $\mathbf{v}$ in the form of a volume integral which we designate as L_v. Then, taking Eqn (9.6) into account, we have

$$L_v = \int \mathrm{d}\mathbf{r}_\perp \int_{-h}^{\eta} \mathrm{d}z \left(\frac{\rho_0 \mathbf{v}^2}{2} - \mathbf{v}\boldsymbol{\nabla}\varphi\right).$$

The variation δL_v for a change in η is composed of two terms. The first is caused by the volume change:

$$\int \mathrm{d}\mathbf{r}_\perp \left(\frac{\rho_0 \mathbf{v}^2}{2} - \mathbf{v}\boldsymbol{\nabla}\varphi\right)\delta\eta\,.$$

The second arises when we consider variations of $\mathbf{v}$ and φ not caused by the change in shape of these functions, for example,

$$\delta\varphi = \varphi(z - \delta\eta) - \varphi(z) = -\frac{\partial\varphi}{\partial z}\,\delta\eta\,.$$

Therefore the contribution to δL_v of this variation has the form

$$\int \mathrm{d}\mathbf{r}_\perp\, v_\mathrm{n}\sqrt{1 + (\boldsymbol{\nabla}\eta)^2}\,\frac{\partial\varphi}{\partial z}\,\delta\eta\,.$$

Collecting all the terms together we finally get

$$\frac{\partial\psi}{\partial t} = -\rho_0 g\eta + \sigma\,\mathrm{div}\,\frac{\boldsymbol{\nabla}\eta}{\sqrt{1 + (\boldsymbol{\nabla}\eta)^2}} + \left[\frac{\rho_0\mathbf{v}^2}{2} - \mathbf{v}\boldsymbol{\nabla}\varphi + \frac{\partial\varphi}{\partial z}\,v_\mathrm{n}\sqrt{1 + (\boldsymbol{\nabla}\eta)^2}\,\right]_{z=\eta}. \tag{9.7}$$

The Hamiltonian H, as before, coincides with the total energy of the system,

$$H = \int \mathrm{d}\mathbf{r}_\perp \int_{-n}^{\eta} \mathrm{d}z\,\frac{\rho_0(\boldsymbol{\nabla}\varphi)^2}{2} + U\,,$$

while the Hamiltonian equations have the form [25]

$$\frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta\psi}\,, \qquad \frac{\partial\psi}{\partial t} = -\frac{\delta H}{\delta\eta}\,.$$

Let us now consider the expansion of the Hamiltonian H in powers of the canonical variables. In the coordinate representation each term in this series is a non-local functional of η and ψ; the reason for this is that at each step of the iteration we must solve the Laplace equation. After applying Fourier transformation with respect to the coordinates in the horizontal plane and successive approximations, one can get

$$H = \frac{1}{2}\int (g + \sigma k^2)|\eta_k|^2\,\mathrm{d}\mathbf{k} + \frac{1}{2}\int k\psi|\psi_k|^2 \tanh(kh)\,\mathrm{d}\mathbf{k} + \frac{1}{2\cdot 2\pi}\int L_{kk_1k_2}\psi_k\psi_{k_1}\eta_{k_2}\delta_{k+k_1+k_2}\,\mathrm{d}\mathbf{k}\,\mathrm{d}\mathbf{k}_1\,\mathrm{d}\mathbf{k}_2 + \ldots\,, \tag{9.8}$$

where

$$L_{kk_1k_2} = \frac{1}{2}(k^2 + k_1^2 - k_2^2) - kk_1\tanh(kh)\tanh(k_1 h) \qquad (\rho_0 = 1)\,.$$

The expansion in Eqn (9.8) is performed with respect to the parameter $k\eta$, having the meaning of a characteristic angle of inclination of the fluid surface.

In the limit $kh \to 0$ of shallow water the above expression reduces to

$$L_{kk_1k_2} \to -(\mathbf{k}\mathbf{k}_1)\,,$$

i.e., the cubic term of the expansion H_1 becomes local in the variables ψ and η:

$$H_1 = \frac{1}{2}\int \eta(\boldsymbol{\nabla}\psi)^2\,\mathrm{d}\mathbf{r}_1\,. \tag{9.9}$$

In particular, the transition to the known Boussinesq model (cf., for example, Ref. [66]) is accomplished if we take Eqn (9.9) for the interaction Hamiltonian, and include the terms proportional to h^3 in H_0:

$$\frac{\partial\eta}{\partial t} = -h\Delta\psi - \operatorname{div}(\eta\nabla\psi) + \frac{h^3}{3}\Delta^2\psi = \frac{\delta H}{\delta\psi},$$
$$\frac{\partial\psi}{\partial t} = -g\eta + \sigma\Delta\eta - \frac{(\nabla\psi)^2}{2} = -\frac{\delta H}{\delta\eta}.$$

Here

$$H = \frac{1}{2}\int\left[g\eta^2 + \sigma(\nabla\eta)^2 + h(\nabla\psi)^2 - h^3(\Delta\psi)^2 + \eta(\nabla\psi)^2\right]\mathrm{d}\mathbf{r}_\perp.$$

In the limit of deep water, $L_{kk_1k_2}$ behaves like

$$L_{kk_1k_2} \to -(\mathbf{k}\mathbf{k}_1) - kk_1.$$

The transition to normal variables is given by the formulae

$$\eta_k = \left[\frac{\omega_k}{2(g+\sigma k^2)}\right]^{1/2}(a_k + a^*_{-k}),$$
$$\psi_k = -\mathrm{i}\left(\frac{g+\sigma k^2}{2\omega_k}\right)^{1/2}(a_k - a^*_{-k}),$$

where $\omega_k = \left[k(g+\sigma k^2)\tanh(kh)\right]^{1/2}$ is the dispersion law for surface waves.

In the same spirit as this was done in Section 4, one can include the contribution from non-potential flows [1]. For this it is necessary to involve an additional constraint in the Lagrangian,

$$\frac{\partial\mu}{\partial t} + \mathbf{v}\nabla\mu = 0,$$

so that the Lagrangian takes the form

$$L = \int \mathrm{d}\mathbf{r}_\perp \int_{-h}^{\eta}\mathrm{d}z\left[\frac{\rho_0\mathbf{v}^2}{2} + \varphi\operatorname{div}\mathbf{v} - \lambda(\mu_t + \mathbf{v}\nabla\mu)\right] + \int\psi\left[\frac{\partial\eta}{\partial t} - v_{\mathrm{n}}\sqrt{1+(\nabla\eta)^2}\right]\mathrm{d}\mathbf{r}_\perp - U. \qquad (9.10)$$

With such a choice for the Lagrangian $\mathbf{v}$ is given, as in Section 4, in terms of the Clebsch variables λ and μ:

$$\rho_0\mathbf{v} = \lambda\nabla\mu + \nabla\varphi. \qquad (9.11)$$

Evolution of λ and μ is given by Eqns (4.9) and (4.13). The function ψ, as for a potential flow, has the same meaning:

$$\varphi\Big|_{z=\eta} = \psi. \qquad (9.12)$$

The equation for this value takes the form of (9.7) where the velocity $\mathbf{v}$ is replaced by the expression (9.11).

Another way to introduce 'surface' canonical variables is given in Ref. [15]. In a similar way we can introduce canonical variables into a stratified fluid, taking into account the 'non-potential' variables λ and μ.

Non-canonical Poisson brackets for the case of arbitrary flows bounded by a free surface were introduced in paper [64]. The bracket represents a combination of the Zakharov's bracket [24, 25] for potential flow and the bracket (5.7):

$$\{F, G\} = \int\left(\operatorname{rot}\mathbf{v}, \left[\frac{\delta F}{\delta\mathbf{v}}\times\frac{\delta G}{\delta\mathbf{v}}\right]\right)\mathrm{d}\mathbf{r} + \int_\Sigma\left(\frac{\delta F}{\delta\Sigma}\frac{\delta G}{\delta\psi} - \frac{\delta G}{\delta\Sigma}\frac{\delta F}{\delta\psi}\right)\mathrm{d}s. \quad (9.13)$$

Here F and G are functionals of the velocity $\mathbf{v}$ ($\operatorname{div}\mathbf{v} = 0$) and the free surface Σ; $\mathrm{d}s$ is a surface element. The variational derivatives $\delta F/\delta\mathbf{v}$ and $\delta G/\delta\mathbf{v}$ are divergence free. The potential part of the velocity is introduced by the unique velocity decomposition (for more details, see Ref. [65])

$$\mathbf{v} = \mathbf{w} + \nabla\Phi,$$

where $\mathbf{w}$ is divergence free and tangential to Σ. The potential Φ is determined by the equations

$$\Delta\Phi = 0, \qquad \frac{\partial\Phi}{\partial n} = v_{\mathrm{n}}.$$

In Eqn (9.13) ψ is the limit of Φ on the free surface Σ.

The equations of motion

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}, \nabla)\mathbf{v} = -\nabla p, \qquad \frac{\partial\Sigma}{\partial t} = v_{\mathrm{n}}$$

with two additional conditions

$$\operatorname{div}\mathbf{v} = 0, \qquad p\Big|_\Sigma = \sigma\kappa,$$

where κ is the mean curvature of the free surface, by means of the brackets (9.13) can be written in the form

$$\frac{\partial\mathbf{v}}{\partial t} = \{\mathbf{v}, H\}, \qquad \frac{\partial\Sigma}{\partial t} = \{\Sigma, H\}.$$

Note that it is also possible to arrive at the brackets (9.13) by recounting the brackets, expressed through the canonical variables λ and μ, ψ and η, using transformation (9.11).

Several other means of introducing 'surface' canonical variables were shown in Ref. [23].

Exactly as in Eqn (9.11), the canonical variables are introduced in a stratified liquid, considering the 'non-potential' variables λ and μ. There is also no difficulty in introducing canonical variables for the description of interacting internal and surface waves. For this case the Lagrangian is a combination of the Lagrangians (9.1) and (9.5).

We would like to mention the interesting paper [69] where the canonical Hamiltonian approach was developed for the description of the interaction of surface waves and vortex filaments. The canonical variables introduced in this paper can be extracted from the general non-canonical Poisson brackets (9.13) by the corresponding limit to the vortex filament.

The introduction of canonical variables for internal and surface waves is also possible for more complicated systems, for example, for a dielectric fluid in an external electric field or a ferro-fluid in a magneto-static field [68]. For these systems the Hamiltonian coincides with the free energy in the external electric (magnetic) field, and the canonical variables remain the same as in the absence of the field.

10. Hamiltonian formalism for plasma and magnetohydrodynamics

The simplest hydrodynamic models of a plasma are of the type of (4.1) and (4.2). Let us consider the hydrodynamics of electrons interacting with a potential electric field in a plasma without magnetic field:

$$\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho\mathbf{v})=0\,,$$

$$\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\nabla)\mathbf{v}=-\nabla\left[\frac{e}{m}\,\varphi+\frac{3T}{m\rho_0}\,\delta\rho\right],\tag{10.1}$$

$$\Delta\varphi=-4\pi e\,\frac{\delta\rho}{m}\,,\qquad \delta\rho=\rho-\rho_0\,.$$

Here e and m are the electron charge and mass, and T is the temperature.

The internal energy of such a system is composed of the electrostatic energy

$$\mathcal{E}_{\mathrm{es}}=\frac{1}{8\pi}\int(\nabla\varphi)^2\,\mathrm{d}\mathbf{r}=\frac{e^2}{2m^2}\int\frac{\delta\rho(\mathbf{r})\delta\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{r}'$$

and the gas-kinetic energy

$$\mathcal{E}_T=\frac{3}{2}\,\frac{T}{m\rho_0}\int\delta\rho^2\,\mathrm{d}\mathbf{r}\,.$$

It is obvious that

$$\frac{e}{m}\,\varphi=\frac{e^2}{m^2}\int\frac{\delta\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}\mathbf{r}=\frac{\delta\mathcal{E}_{\mathrm{es}}}{\delta\rho}\,,$$

$$\frac{3T}{m\rho_0}\,\delta\rho=\frac{\delta\mathcal{E}_T}{\delta\rho}\,.\tag{10.2}$$

Formula (10.2) shows that system (10.1) belongs to the type of (4.1) and (4.2) with $\mathcal{E}_{\mathrm{in}}$ in the general form (4.6). The diagonalizing transformation for H_0 in this case has the form (4.4), in which one should set $\omega_k^2=\omega_{\mathrm{p}}^2+3k^2T/m$ ($\omega_{\mathrm{p}}^2=4\pi_0e^2/m^2$) while the coefficients U and V are determined from formulae (4.5), in which we should take $g=0$.

Now let us consider the hydrodynamics of slow motion of a non-isothermal plasma, whose electron temperature T_{e} significantly exceeds the ion temperature. By slow motion we shall understand wave motion with phase velocities ω/k much smaller than the electron thermal velocity $v_{T\mathrm{e}}$, but large compared to the ion thermal velocity. In this case we can assume that the electrons are distributed according to Boltzmann's law, $\rho_{\mathrm{e}}=\rho\exp(e\phi/T_{\mathrm{e}})$, while the thermal ion motion can be neglected. Then

$$\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho\mathbf{v})=0\,,\qquad \frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\nabla)\mathbf{v}=-\frac{e}{M}\,\nabla\varphi\,,$$

$$\Delta\varphi=\frac{4\pi e}{M}\left(\rho-\rho_0\exp\frac{e\varphi}{T_{\mathrm{e}}}\right),\tag{10.3}$$

where M is the ion mass.

This system also conserves the energy

$$H=\int\frac{\rho\mathbf{v}^2}{2}\,\mathrm{d}\mathbf{r}+\mathcal{E}_{\mathrm{in}}\,.\tag{10.4}$$

Here $\mathcal{E}_{\mathrm{in}}$ is the internal energy, equal to the sum of the electrostatic energy $\mathcal{E}_{\mathrm{es}}=(1/8\pi)\int(\nabla\varphi)^2\,\mathrm{d}\mathbf{r}$ and the thermal energy of the electron gas

$$\mathcal{E}_T=\frac{T_{\mathrm{e}}}{M}\int\rho_0\left[\left(\frac{e\varphi}{T_{\mathrm{e}}}-1\right)\exp\frac{e\varphi}{T_{\mathrm{e}}}+1\right]\mathrm{d}\mathbf{r}\,.$$

Calculating the variational derivative of $\mathcal{E}_{\mathrm{in}}$ with respect to the ion density ρ, we get

$$\frac{\delta\mathcal{E}_{\mathrm{in}}}{\delta\rho}=\int\varphi(\mathbf{r}')\left\{-\frac{1}{4\pi}\,\Delta\,\frac{\delta\varphi(\mathbf{r}')}{\delta\rho(\mathbf{r})}+\frac{e^2\rho_0}{M}\exp\frac{e\varphi}{T_{\mathrm{e}}}\,\frac{\delta\varphi(\mathbf{r}')}{\delta\rho(\mathbf{r})}\right\}\mathrm{d}\mathbf{r}'\,.$$

On the other hand, by varying the Poisson equation we have

$$-\frac{1}{4\pi}\,\Delta\,\frac{\delta\varphi(\mathbf{r}')}{\delta\rho(\mathbf{r})}+\frac{e^2\rho_0}{M}\exp\frac{e\varphi}{T_{\mathrm{e}}}\,\frac{\delta\varphi(\mathbf{r}')}{\delta\rho(\mathbf{r})}=\frac{e}{M}\,\delta(\mathbf{r}-\mathbf{r}')\,.$$

Comparing the two expressions, we arrive at

$$\frac{\delta\mathcal{E}_{\mathrm{in}}}{\delta\rho}=\frac{e}{M}\,\varphi\,.$$

From this it follows that the system (10.3) also belongs to the type of (4.1) and (4.2).

We note that for long-wave (such, that $kr_{\mathrm{d}}\ll1$, where $r_{\mathrm{d}}=(T_{\mathrm{e}}/4\pi n_0e^2)^{1/2}$ is the Debye radius) oscillations of small amplitude, the system of Boussinesq equations follows from Eqn (10.3). It is easy to see that in this limit the potential is determined from the Poisson equation:

$$\frac{e\varphi}{T_{\mathrm{e}}}\approx\frac{\delta\rho}{\rho_0}-\frac{1}{2}\left(\frac{\delta\rho}{\rho_0}\right)^2+rd^2\Delta\,\frac{\delta\rho}{\rho_0}\,.$$

Substitution of this expression into Eqn (10.4) leads to the following form for the internal energy [cf. Eqn (4.6)]:

$$\mathcal{E}_{\mathrm{in}}=\int\frac{\rho_0c_{\mathrm{s}}^2}{2}\left[\left(\frac{\delta\rho}{\rho_0}\right)^2-\frac{1}{3}\left(\frac{\delta\rho}{\rho_0}\right)^3-rd^2\left(\nabla\,\frac{\delta\rho}{\rho_0}\right)^2\right]\mathrm{d}\mathbf{r}\,,$$

$$c_{\mathrm{s}}^2=\frac{T_{\mathrm{e}}}{M}\,.$$

We now go over to a consideration of the relativistic gas dynamics of electrons, interacting with an arbitrary nonpotential electromagnetic field:

$$\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho\mathbf{v})=0\,,$$

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\nabla\right)\mathbf{p}=e\mathbf{E}+\frac{e}{c}[\mathbf{v}\times\mathbf{H}]-3T\nabla\,\frac{\delta\rho}{\rho_0}\,,$$

$$\operatorname{rot}\mathbf{E}=-\frac{1}{c}\,\frac{\partial\mathbf{H}}{\partial t}\,,$$

$$\operatorname{rot}\mathbf{H}=\frac{4\pi}{c}\,\frac{e\rho}{m}\,\mathbf{v}+\frac{1}{c}\,\frac{\partial\mathbf{E}}{\partial t}\,,$$

$$\operatorname{div}\mathbf{E}=4\pi e\,\frac{\delta\rho}{m}\,.$$

For the electromagnetic field we introduce the scalar and vector potentials φ and $\mathbf{A}$, where we choose for $\mathbf{A}$ the Coulomb gauge, $\operatorname{div}\mathbf{A}=0$.

We know that in the Coulomb gauge the vector potential is a canonical variable, if we change from ordinary momentum to generalized momentum

$$\mathbf{p} = \mathbf{p}_1 - \frac{e}{c}\,\mathbf{A}$$

determined from the equation

$$\frac{\partial \mathbf{p}_1}{\partial t} + \nabla (m^2c^4 + p^2c^2)^{1/2} - [\mathbf{v} \times \operatorname{rot} \mathbf{p}_1] + e\nabla\varphi = -3T\nabla\,\frac{\delta\rho}{\rho_0}\,.$$

The canonical conjugate of $\mathbf{A}$ is the vector

$$\mathbf{B} = \frac{1}{4\pi c}\left(\nabla\varphi + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right) = -\frac{\mathbf{E}}{4\pi c}\,.$$

The other variables are introduced by analogy with the Clebsch variables:

$$\frac{\mathbf{p}_1}{m} = \frac{\lambda}{\rho}\,\nabla\mu + \nabla\varphi\,.$$

Here (λ, μ), (ρ, φ), and $(\mathbf{B}, \mathbf{A})$ are canonically conjugate quantities,

$$\frac{\partial\lambda}{\partial t} = \frac{\delta H}{\delta\mu}\,, \quad \frac{\partial\mu}{\partial t} = -\frac{\delta H}{\delta\lambda}\,, \quad \frac{\partial\rho}{\partial t} = \frac{\delta H}{\delta\varphi}\,, \quad \frac{\partial\varphi}{\partial t} = -\frac{\delta H}{\delta\rho}\,,$$

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{\delta H}{\delta \mathbf{B}}\,, \quad \frac{\partial \mathbf{B}}{\partial t} = -\frac{\delta H}{\delta \mathbf{A}}\,,$$

with the Hamiltonian

$$H = \int\left[\frac{\rho}{m}(p^2c^2 + m^2c^4)^{1/2} + \frac{3}{2}\,T\,\frac{\delta\rho^2}{m\rho_0} + \frac{1}{8\pi}(\operatorname{rot}\mathbf{A})^2\right]\mathrm{d}\mathbf{r}$$
$$+ \int\left[2\pi c^2B^2 - c(\mathbf{B}\nabla\varphi) + \frac{1}{4\pi}\,\varphi\Delta\varphi\right]\mathrm{d}\mathbf{r}\,, \tag{10.5}$$

coinciding with the total energy of the system if the Poisson equation is satisfied identically.

We note that canonical variables are introduced analogously for the two-fluid model of the plasma. A more detailed presentation of these results can be found in papers [34, 35].

Another widely used model in plasma physics is the set of magnetohydrodynamic (MHD) equations, describing low-frequency (hydrodynamic) motions of the plasma as a whole. These equations, in particular, can be obtained from the equations of the two-fluid model.

For barotropic flows, the equations of MHD have the form

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{v}) = 0\,,$$
$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\nabla\,\frac{\delta\varepsilon}{\delta\rho} + \frac{1}{4\pi\rho}[\operatorname{rot}\mathbf{H}\times\mathbf{H}]\,, \tag{10.6}$$
$$\frac{\partial H}{\partial t} = \operatorname{rot}[\mathbf{v}\times\mathbf{H}]\,.$$

For this system, just as for the equations of hydrodynamics, the transition to canonical variables is accomplished using the Lagrange approach. For this we shall start from the well-known expression for the Lagrangian of a fluid interacting with the electromagnetic field in the MHD approximation. This means that in the Lagrangian we drop small terms of order v/c. Thus, for example, in the MHD approximation we should neglect the contribution from the electric field $[E \sim (v/c)H]$ compared to the corresponding contribution from the magnetic field.

We also pay attention to one important consequence of Eqns (10.6) following which the magnetic field is frozen in a plasma [70]. It is that the vector $\mathbf{H}/\rho$ moves together with the fluid particles. In other words, each magnetic field line is displaced together with the particles that are on it. This fact allows one to regard the magnetic field $\mathbf{H}$ and the density ρ as generalized coordinates.

Thus the Lagrangian in the MHD approximation including the constraint has the following form:

$$L = \int\left[\frac{\rho\mathbf{v}^2}{2} - \varepsilon(\rho) - \frac{\mathbf{H}^2}{8\pi} + \mathbf{S}\left(\frac{\partial\mathbf{H}}{\partial t} - \operatorname{rot}[\mathbf{v}\times\mathbf{H}]\right)\right.$$
$$\left. + \varphi\left(\frac{\partial\rho}{\partial t} + \operatorname{div}\rho\mathbf{v}\right) + \psi\operatorname{div}\mathbf{H}\right]\mathrm{d}\mathbf{r}\,.$$

Varying L with respect to the variables $\mathbf{v}$, ρ and $\mathbf{H}$, we get

$$\rho\mathbf{v} = [\mathbf{H}\times\operatorname{rot}\mathbf{S}] + \rho\nabla\varphi\,, \tag{10.7}$$
$$\frac{\partial\varphi}{\partial t} + \mathbf{v}\nabla\varphi - \frac{\mathbf{v}^2}{2} + \omega(\rho) = 0\,, \tag{10.8}$$
$$\frac{\partial\mathbf{S}}{\partial t} + \frac{\mathbf{H}}{4\pi} - [\mathbf{v}\times\operatorname{rot}\mathbf{S}] + \nabla\psi = 0\,. \tag{10.9}$$

From this we see that the undetermined Lagrange multipliers enter as generalized momenta. The appropriate transition to these variables is accomplished using formula (10.7), and their evolution is determined from Eqns (10.8) and (10.9). The gauge function ψ that enters these equations is chosen for convenience. For the natural condition $\operatorname{div}\mathbf{S} = 0$ we have

$$\psi = \Delta^{-1}\operatorname{div}[\mathbf{v}\times\operatorname{rot}\mathbf{S}] + \psi_0\,,$$

where ψ_0 is an arbitrary solution of the Laplace equation $\Delta\psi_0 = 0$. In particular, for finite motions of the plasma in a magnetic field $\mathbf{H}_0$, it is convenient to choose the quantity $\mathbf{S}$ so that $\mathbf{S} \to 0$ for $r \to \infty$. It is then obvious that

$$\psi_0 = \frac{(\mathbf{H}_0\mathbf{r})}{4\pi}\,.$$

The equivalence of the system of equations obtained here and the MHD equations is verified by a direct substitution of the velocity in the equation of motion (10.6).

Now changing to the Hamiltonian description, we get [33]

$$\frac{\partial\rho}{\partial t} = \frac{\delta H}{\delta\varphi}\,, \quad \frac{\partial\varphi}{\partial t} = -\frac{\delta H}{\delta\rho}\,, \quad \frac{\partial\mathbf{H}}{\partial t} = \frac{\delta H}{\delta\mathbf{S}}\,, \quad \frac{\partial\mathbf{S}}{\partial t} = -\frac{\delta H}{\delta\mathbf{H}}\,,$$

where the Hamiltonian

$$H = \int\left[\frac{\rho\mathbf{v}^2}{2} + \varepsilon(\rho) + \frac{\mathbf{H}^2}{8\pi} - \psi\operatorname{div}\mathbf{H}\right]\mathrm{d}\mathbf{r}$$

has a value that also coincides in value with the total energy of the system.

Another way to introduce canonical variables in MHD was suggested in Ref. [71]. In this paper both the velocity and the magnetic field are parameterized in terms of the Clebsch-type potentials:

$$\mathbf{v} = \nabla\phi + \rho^{-1}(\mu\nabla\lambda + M\nabla\Lambda)\,,$$
$$\mathbf{H} = [\nabla\lambda \times \nabla\Lambda]\,.$$

So doing the quantities λ and μ, Λ and M, ρ and ϕ form pairs of canonically conjugated variables. It is possible to show that the given parameterization for $\mathbf{H}$ and $\mathbf{v}$ can be reduced by appropriate gauge choice to the change (10.7).

For an incompressible fluid the canonical variables are $\mathbf{H}$ and $\mathbf{S}$: the potential φ can be eliminated using the continuity equation,

$$\Delta\varphi = -\mathrm{div}\,\frac{1}{\rho_0}[\mathbf{H}\times\mathrm{rot}\,\mathbf{S}]\,,$$

while the Hamiltonian takes the form

$$H = \int\left[\frac{\rho_0\mathbf{v}^2}{2} + \frac{\mathbf{H}^2}{8\pi} + \psi\,\mathrm{div}\,\mathbf{H}\right]\mathrm{d}\mathbf{r}\,.$$

For barotropic flows the variables (ρ,ϕ) and $(\mathbf{H},\mathbf{S})$ determine the canonical Poisson brackets:

$$\{F,G\} = \int\left[\left(\frac{\delta F}{\delta\rho}\frac{\delta G}{\delta\varphi} - \frac{\delta F}{\delta\varphi}\frac{\delta G}{\delta\rho}\right) + \left(\frac{\delta F}{\delta\mathbf{H}}\frac{\delta G}{\delta\mathbf{S}} - \frac{\delta F}{\delta\mathbf{S}}\frac{\delta G}{\delta\mathbf{H}}\right)\right]\mathrm{d}\mathbf{r}\,. \tag{10.10}$$

These brackets, as in the hydrodynamic case, allow recalculation to the natural variables, i.e., to the velocity $\mathbf{v}$, the density ρ and the magnetic field $\mathbf{H}$. As a result, the non-canonical brackets become a combination of (5.6) and the additional term containing the variational derivatives with respect to the magnetic field [15]:

$$\begin{aligned}\{F,G\} = &\int\left[\left(\nabla\,\frac{\delta F}{\delta\rho},\frac{\delta G}{\delta\mathbf{v}}\right) - \left(\nabla\,\frac{\delta G}{\delta\rho},\frac{\delta F}{\delta\mathbf{v}}\right)\right]\mathrm{d}\mathbf{r}\\ &+\int\left(\frac{\mathrm{rot}\,\mathbf{v}}{\rho},\left[\frac{\delta F}{\delta\mathbf{v}}\times\frac{\delta G}{\delta\mathbf{v}}\right]\right)\mathrm{d}\mathbf{r}\\ &+\int\left(\frac{\mathbf{H}}{\rho},\left[\mathrm{rot}\,\frac{\delta F}{\delta\mathbf{H}}\times\frac{\delta G}{\delta\mathbf{v}}\right] - \left[\mathrm{rot}\,\frac{\delta G}{\delta\mathbf{H}}\times\frac{\delta F}{\delta\mathbf{v}}\right]\right)\mathrm{d}\mathbf{r}.\end{aligned} \tag{10.11}$$

Without the barotropic constraint these brackets acquire the additional term (compare with Ref. [15])

$$\int\left(\frac{\nabla S}{\rho},\left[\frac{\delta F}{\delta\mathbf{v}}\frac{\delta G}{\delta S} - \frac{\delta G}{\delta\mathbf{v}}\frac{\delta F}{\delta S}\right]\right)\mathrm{d}\mathbf{r}\,. \tag{10.12}$$

The brackets (10.11), (10.12) are, as in the pure hydrodynamic limit ($\mathbf{H}=0$), degenerate.

The simplest annulators of these brackets were probably found in the paper [73]:

$$C = \int\rho f\left(S,\frac{\mathbf{H}\nabla}{\rho}\,S,\left(\frac{\mathbf{H}\nabla}{\rho}\right)^2 S,\ldots\right)\mathrm{d}\mathbf{r}\,. \tag{10.13}$$

It can be verified by the direct calculations that the Lagrangian invariants which generate integral (10.13) are written in the form

$$I_n = \left(\frac{\mathbf{H}\nabla}{\rho}\right)^n S\,. \tag{10.14}$$

The integrals (10.13), however, are only one of the possible sets of the Eulerian integrals of motion. There exist recurrent formulae for the construction of such integrals which can be obtained from the Lagrangian invariants I, the frozen field $\mathbf{B}$, the density ρ and the field of the Lamb momentum $\mathbf{p}$ [74, 29]. These quantities are defined by the corresponding equations of motion:

$$\frac{\partial I}{\partial t} + (\mathbf{v}\nabla)I = 0\,, \tag{10.15}$$

$$\frac{\partial\mathbf{B}}{\partial t} + (\mathbf{v}\nabla)\mathbf{B} = (\mathbf{B}\nabla)\mathbf{v}\,, \tag{10.16}$$

$$\frac{\partial\mathbf{p}}{\partial t} + (\mathbf{v}\nabla)\mathbf{p} + (\mathbf{p}\nabla)\mathbf{v} + [\mathbf{p}\times\mathrm{rot}\,\mathbf{v}] = 0\,. \tag{10.17}$$

The recurrent procedure for the construction of Lagrange invariants consists of several steps.

At first, it is easy to verify that the definition of the fields $\mathbf{B}$ and $\mathbf{p}$ by means of Eqns (10.16) and (10.17) remains without changes if one first multiplies them by I:

$$\mathbf{B}' = I\mathbf{B}\,,\qquad \mathbf{p}' = I\mathbf{p}\,. \tag{10.18}$$

In other words the new fields $\mathbf{B}'$ and $\mathbf{p}'$ obey the same equations as those for $\mathbf{B}$ and $\mathbf{p}$.

At the next step for the given I, $\mathbf{p}$, $\mathbf{B}$ we construct a new set of quantities I', $\mathbf{p}'$, $\mathbf{B}'$, which possess the same properties:

$$\mathbf{p}' = \nabla I\,,\qquad \mathbf{B}' = \frac{1}{\rho}\,\mathrm{rot}\,\mathbf{p}'\,,\qquad I' = \frac{1}{\rho}\,\mathrm{div}\,(\rho\mathbf{B})\,. \tag{10.19}$$

At the third step, by substituting Eqn (10.18) into (10.19), we obtain

$$I' = (\mathbf{v}\mathbf{B})\,,\qquad \mathbf{B}' = \frac{1}{\rho}[\mathbf{p}\times\mathbf{p}']\,,\qquad \mathbf{B}' = \frac{1}{\rho}[\nabla I\times\nabla I']\,. \tag{10.20}$$

Further recursion gives the following relations:

$$\mathbf{p} = \rho[\mathbf{B}\times\mathbf{B}']\,,\qquad I = \frac{1}{\rho}\big(\mathbf{p},[\nabla I'\times\nabla I'']\big)\,,$$

$$I = \frac{1}{\rho}\big(\nabla I'[\nabla I''\times\nabla I''']\big)\,. \tag{10.21}$$

Because an arbitrary function of Lagrangian variables is again a Lagrangian invariant, this in combination with Eqns (10.18)–(10.21) sets the prescription of the Lagrangian invariant reproduction. For example, the Lagrangian invariants of the first generation, consisting of the quantities ρ, $\mathbf{p}$, $\mathbf{B}$ and the three Lagrangian invariants given initially plus a lack constructed by means of Eqns (10.18)–(10.21), can be represented in the following form [29]:

$$I'_0 = \mathbf{p}\times\mathbf{B}\,,\qquad I'_{ik} = \frac{1}{\rho}\big(\mathbf{p}[\nabla I_i\times\nabla I_k]\big)\,,$$

$$I'_k = (\mathbf{B}\nabla)I_k\,,\qquad I' = \frac{1}{\rho}\big(\nabla I_1[\nabla I_2\times\nabla I_3]\big)\,. \tag{10.22}$$

Using this procedure sequentially one can get the subsequent generations of Lagrangian invariants.

Of particular interest is a question about Lagrangian invariant construction in the two-dimensional case. Here to construct an infinite hierarchy of invariants it is enough to have two input invariants. As was shown in Ref. [82] in the 2D case the whole set of conserved quantities forms a highly complicated Lie algebra, which contains very interesting subalgebras, e.g., a loop algebra with a layer in the algebra of area preserving diffeomorphisms of a plane.

Let us apply this approach to the MHD equations.

In the MHD case with an arbitrary equation of state $p = p(\rho, S)$ one should take entropy S for I, $\mathbf{H}/\rho$ instead of the vector field $\mathbf{B}$, and the field $\mathbf{p}$ should be changed by the vector potential $\mathbf{A}$ of the magnetic field ($\mathbf{H} = \mathrm{rot}\,\mathbf{A}$), imposing the gauge [72]

$$\frac{\partial \mathbf{A}}{\partial t} = [\mathbf{v} \times \mathrm{rot}\,\mathbf{A}] - \boldsymbol{\nabla}(\mathbf{vA}) \equiv -(\mathbf{v}\boldsymbol{\nabla})\mathbf{A} - (\mathbf{A}\boldsymbol{\nabla})\mathbf{v} - [\mathrm{rot}\,\mathbf{v} \times \mathbf{A}]\,.$$

It is easy to see that for MHD the transformation (10.19) reads as follows:

$$\mathbf{A}' = \boldsymbol{\nabla} S\,, \qquad \mathbf{B}' = \frac{\mathbf{H}}{\rho}\,, \qquad I' = 0\,.$$

The first equation reflects the gauge freedom of the vector potential, the second formula in this case is the definition of the frozen field.

If now in the third formula of Eqn (10.19), instead of $\mathbf{B}$, one takes its transforming value from Eqn (10.18), then as a result one can get the second (after S) Lagrangian invariant:

$$I_2 = \frac{(\mathbf{H}\boldsymbol{\nabla})S}{\rho}\,.$$

Its structure is similar to the Ertel invariant for hydrodynamic flows. Multiple use of this transformation leads to the invariants (10.14):

$$I_n = \left(\frac{\mathbf{H}\boldsymbol{\nabla}}{\rho}\right)^n S\,.$$

Transformation (10.20) generates another Lagrangian invariant

$$I_3 = \frac{(\mathbf{AH})}{\rho}\,.$$

Integration of I_3 with the help of formula (6.1) results in the integral

$$I_k = \int (\mathbf{AH})\,\mathrm{d}\mathbf{r}$$

of motion, which characterizes the degree of knottiness of lines of the magnetic field $\mathbf{H}$ [57].

The representation of the three invariants $I_1 = S$, I_2 and I_3, and also of the magnetic field $\mathbf{H}$ and its vector potential $\mathbf{A}$ permits, by use of the formula (10.22), all the sets of Lagrangian invariants to be found together with the Eulerian integrals

$$C = \int \rho f(I_1, I_2, \ldots)\,\mathrm{d}\mathbf{r} \tag{10.23}$$

of motion.

In the case of barotropic flows the given recursion changes. At first, one should exclude the entropy S as a quantity not entering into the equations of motion. Therefore from the set I_i ($i = 1, 2, 3$) of Lagrangian invariants of the first generation, explicitly expressed in terms of $\mathbf{H}$ and ρ, only the invariant $I_2 = (\mathbf{AH})/\rho$ remains. With its help all the series of the integrals of motion is written as follows [48]:

$$C = \int \rho f\left(I_2, \frac{\mathbf{H}\boldsymbol{\nabla}}{\rho} I_2, \ldots\right)\mathrm{d}\mathbf{r}\,. \tag{10.24}$$

Furthermore, in the barotropic case a new integral should be added. This is the topological invariant

$$C_{\mathrm{t}} = \int (\mathbf{v}, \mathbf{H})\,\mathrm{d}\mathbf{r}\,,$$

characterizing the degree of cross knottiness of the magnetic field and velocity lines.

It is possible to show that all the integrals presented above are the Casimirs relative to the brackets (10.11) and (10.12).

Thus, we demonstrated how canonical variables are introduced for hydrodynamical models of plasma. These variables to some extent generalize the Clebsch variables for ideal hydrodynamics. They differ in that, firstly, the number of canonical variables increases, so that the electromagnetic field itself is an additional canonical variable, and, secondly, due to this fact the Hamiltonian structure of the equations changes (becomes more complicated), especially for MHD.

11. Hamiltonian formalism in kinetics

In this section we introduce a Hamiltonian structure into the self-consistent type collisionless kinetic equation. We consider the simplest example that has sufficient contents from the point of view of generalization: the Vlasov kinetic equation for the distribution function f describing potential (electric field $E = -\boldsymbol{\nabla}\varphi$) oscillations of electrons relative to a homogeneous background of ions with density n_0:

$$\frac{\partial f}{\partial t} + (\mathbf{v}\boldsymbol{\nabla})f - \boldsymbol{\nabla}\varphi\,\frac{\partial f}{\partial \mathbf{v}} = 0\,,$$

$$\Delta\varphi = -4\pi\left[\int f\,\mathrm{d}\mathbf{v} - n_0\right] \qquad (e = m = 1)\,. \tag{11.1}$$

Kinetic equations of this type should also be regarded as hydrodynamic type systems. In the phase space $(\mathbf{r}, \mathbf{v})$ Eqn (11.1) describes the motion of an incompressible 'fluid', whose density is convected together with the 'fluid'. The behavior of the system here is in many ways similar to the situation which holds in a stratified fluid. In order to transform to canonical coordinates, we introduce the Lagrange coordinate ξ, which we determine from the condition that the distribution function f be equal to the equilibrium distribution function $f_0(\xi)$, not necessarily Maxwellian:

$$f(\mathbf{r}, \mathbf{v}, t) = f_0(\xi) \qquad \text{or} \qquad \mathbf{v} = \mathbf{V}(\mathbf{r}, \xi, t)\,.$$

Such a representation can be expressed in integral form:

$$f(\mathbf{r}, \mathbf{v}, t) = \int F(\mathbf{r}, \xi, t)\,\delta\big[\mathbf{v} - \mathbf{V}(\mathbf{r}, \xi, t)\big]\,\mathrm{d}\xi\,. \tag{11.2}$$

Substitution of Eqn (11.2) in (11.1) leads to the following system of equations:

$$\frac{\partial F}{\partial t} + \operatorname{div}(F\mathbf{V}) = 0\,, \tag{11.3}$$

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\nabla)\mathbf{V} = -\nabla\varphi\,, \tag{11.4}$$

$$\Delta\varphi = -4\pi\left[\int F(\xi,\mathbf{r},t)\,\mathrm{d}\xi - n_0\right]. \tag{11.5}$$

The internal energy

$$\mathcal{E}_{\mathrm{in}} = \frac{1}{8\pi}\int(\nabla\varphi)^2\,\mathrm{d}\mathbf{r} = \frac{1}{2}\int\frac{\left[\int F(\xi,\mathbf{r})\,\mathrm{d}\xi - n_0\right]\left[\int F(\xi',\mathbf{r}')\,\mathrm{d}\xi' - n_0\right]}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{r}'$$

of this system is a functional of the 'density' F, and therefore, according to the classification of Section 4 belongs to the type of Eqns (4.1) and (4.2).

Canonical variables for Eqns (11.3)–(11.5) are introduced in the standard way. For potential 'flows' $\mathbf{V} = \nabla\Phi$ and the equations of motion are of the Hamiltonian form [1]

$$\frac{\partial\Phi}{\partial t} = -\frac{\delta H}{\delta F}\,, \qquad \frac{\partial F}{\partial t} = \frac{\delta H}{\delta\Phi}\,,$$

with

$$H = \int\frac{F\mathbf{v}^2}{2}\,\mathrm{d}\xi\,\mathrm{d}\mathbf{r} + \mathcal{E}_{\mathrm{in}}\,.$$

Thus, the Poisson brackets have the canonical form

$$\{S,T\} = \int\mathrm{d}\xi\,\mathrm{d}\mathbf{r}\left[\frac{\delta S}{\delta F}\frac{\delta T}{\delta\Phi} - \frac{\delta T}{\delta F}\frac{\delta S}{\delta\Phi}\right].$$

They can be expressed in terms of the distribution function f [1]. By using Eqn (11.2), together with simple transformations, one can get brackets which were first obtained in Ref. [16]:

$$\{S,T\} = \int f\left[\left(\frac{\partial}{\partial\mathbf{v}}\frac{\delta S}{\delta f}\right)\left(\frac{\partial}{\partial\mathbf{r}}\frac{\delta T}{\delta f}\right) - \left(\frac{\partial}{\partial\mathbf{r}}\frac{\delta S}{\delta f}\right)\left(\frac{\partial}{\partial\mathbf{v}}\frac{\delta T}{\delta f}\right)\right]\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{v}\,.$$

Canonical variables are introduced analogously in the Vlasov–Maxwell equations, where the canonical Poisson brackets may be transformed into the brackets of Ref. [16], which locally depend on the distribution function and the electromagnetic field.

In concluding this section we mention another similar important example, in which there is an analogous construction. This is the Benney equations, describing surface waves in the approximation of 'shallow' water, where the flow of the fluid is not assumed to be potential:

$$h_t + \operatorname{div}\int_0^h\mathbf{U}\,\mathrm{d}z = 0\,, \tag{11.6}$$

$$\mathbf{U}_t + (\mathbf{U}\nabla)\mathbf{U} + W\frac{\partial\mathbf{U}}{\partial z} + \nabla h = 0\,, \tag{11.7}$$

$$\frac{\partial W}{\partial z} + \operatorname{div}\mathbf{U} = 0\,. \tag{11.8}$$

Here $h = h(\mathbf{r},t)$ [$\mathbf{r} = (x,y)$, $0 < z < h$] is the boundary of the free surface of the fluid, $\mathbf{U} = \mathbf{U}(\mathbf{r},z)$ is the horizontal velocity, $W = W(\mathbf{r},z)$ is the vertical component of the velocity, and $g = 1$. We first show that the system (11.6)–(11.8) can be reduced to an infinite system of two-dimensional hydrodynamic equations.

We introduce a coordinate ξ $(0 < \xi < l)$ which enumerates each layer of the fluid in equilibrium along the direction z. Then the coordinate of each layer at time t will be given by the functions

$$z = z(\mathbf{r},\xi,t)\,, \qquad h(\mathbf{r},t) = z(\mathbf{r},l,t)\,.$$

It is clear that the equations expressed in terms of this function are similar to Eqn (9.4):

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial t} + (\mathbf{U}\nabla)z = W\,. \tag{11.9}$$

Setting $\xi = l$ we see that Eqn (11.6) follows from Eqn (11.9). Derivatives taken for constant ξ and z are linked by the following formulae:

$$\left(\frac{\partial}{\partial t}\right)_z = \left(\frac{\partial}{\partial t}\right)_\xi - \frac{z_t}{\eta}\frac{\partial}{\partial\xi}\,, \qquad (\nabla)_z = (\nabla)_\xi - \frac{\nabla z}{\eta}\frac{\partial}{\partial\xi}\,. \tag{11.10}$$

In addition we have

$$\frac{\partial}{\partial z} = \frac{1}{\eta}\frac{\partial}{\partial\xi}\,, \tag{11.11}$$

where

$$\eta(\mathbf{r},\xi,t) = \frac{\partial z}{\partial\xi}\,.$$

Differentiating relation (11.9) with respect to ξ and using formulae (11.10) and (11.11), we easily obtain the equation

$$\frac{\partial\eta}{\partial t} + \operatorname{div}(\eta\mathbf{U}) = 0\,. \tag{11.12}$$

(Here and everywhere below the derivatives are taken at constant ξ.)

Applying the same formulae to Eqn (11.7), we find, after transformations,

$$\frac{\partial\mathbf{U}}{\partial t} + (\mathbf{U}\nabla)\mathbf{U} + \nabla h = 0\,, \tag{11.13}$$

where h and η are connected by the relation

$$h = \int_0^l\eta(\mathbf{r},\xi,t)\,\mathrm{d}\xi\,. \tag{11.14}$$

The system (11.12)–(11.14) is similar to that considered above and differs from it only in the consistency condition (11.14). Therefore, the canonical variables for potential ($\mathbf{U} = \nabla\varphi$) flows (in the x, y plane) remain the same [30]:

$$\frac{\partial\varphi}{\partial t} = -\frac{\delta H}{\delta\eta}\,, \qquad \frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta\varphi}\,. \tag{11.15}$$

Here

$$H = \frac{1}{2}\int\mathrm{d}\xi\,\mathrm{d}\mathbf{r}\,\eta(\nabla\varphi)^2 + \frac{1}{2}\int\mathrm{d}\mathbf{r}\,h^2\,.$$

If the flow depends only on x, the Hamiltonian structure can be given in terms of the variables η and U:

$$\frac{\partial\eta}{\partial t} = -\frac{\partial}{\partial x}\frac{\delta H}{\delta U}\,, \qquad \frac{\partial U}{\partial t} = -\frac{\partial}{\partial x}\frac{\delta H}{\delta\eta}\,.$$

It should be added that for one-dimensional flows another method for introducing a Hamiltonian structure was developed in paper [32]. One can show that the Hamiltonian structure introduced in Ref. [32] is equivalent to the structure (11.15).

At the end of this section we would like to pay attention to one more paper [76] where, in fact, the same idea as for the Vlasov and Benney equations was used. In Ref. [76] Virasoro supposes to describe flows of stratified fluid by the use of a mixed, Lagrangian–Eulerian representation. For two-dimensional flows the horizontal coordinate x and the Lagrangian coordinate β, labeling the levels of density ρ, serve as independent variables. In the case of two-dimensional hydrodynamics, when the suggested scheme can also be applied, one of the coordinates (of the Lagrange type) labels the vorticity levels Ω and the other coordinate may, for example, be the Cartesian x. Virasoro, from the very beginning, comes from the variational principle in the Lagrange form (7.6), and then performs a transformation to new variables, by introducing the generating function of this transformation. This function in the Lagrangian plays the role of the generalized coordinate.

Approximately the same ideas occur in papers [38, 39] where for equation (5.12), describing the Rossby waves, the Gardner–Zakharov–Faddeev brackets are derived from non-canonical Poisson brackets.

12. Classical perturbation theory and the reduction of Hamiltonians

If in the previous sections we dealt with introducing the Hamiltonian structure, then further we will suppose that we were able in some way to introduce canonical variables together with the normal variables diagonalizing a quadratic part of Hamiltonian. In this section we turn to the classical perturbation theory for the wave Hamiltonian systems which is based on an assumption about the smallness of wave amplitudes. The difference of the wave systems from the finite-dimensional systems is that the application of the perturbation theory to the wave systems leads to the appearance of resonant denominators not at separate points, as for finite-dimensional equations, but on whole manifolds. By their classification, we arrive at the whole set of standard Hamiltonians and corresponding equations. In particular, many well-known equations such as the nonlinear Schrödinger equation, the KdV equation, the KP equation, etc. are among them.

Suppose that in a medium there is one type of waves with dispersion law $\omega(k)$ and amplitudes $a(k)$, whose evolution is determined by Eqn (3.7):

$$\frac{\partial a_k}{\partial t} = -\mathrm{i}\,\frac{\delta H}{\delta a_k^*}\,. \tag{12.1}$$

Here

$$H = H_0 + H_1 + \ldots,$$

$$H_0 = \int \omega_k |a_k|^2\,\mathrm{d}k\,, \tag{12.2}$$

$$H_1 = \int (V_{kk_1k_2} a_k^* a_{k_1} a_{k_2} + \text{c.c.})\,\delta_{k-k_1-k_2}\,\mathrm{d}\mathbf{k}\,\mathrm{d}\mathbf{k}_1\,\mathrm{d}\mathbf{k}_2 + \frac{1}{3}\int (U_{kk_1k_2} a_k^* a_{k_2}^* a_{k_2}^* + \text{c.c.})\,\delta_{k+k_1+k_2}\,\mathrm{d}\mathbf{k}\,\mathrm{d}\mathbf{k}_1\,\mathrm{d}\mathbf{k}_2\,. \tag{12.3}$$

Consider a transformation from the variables $a(k)$ to new variables $c(k)$ in the form of an integral power series:

$$a_k = c_k + \int L_{kk_1k_2} c_{k_1} c_{k_2} \delta_{k-k_1-k_2}\,\mathrm{d}\mathbf{k}_1\,\mathrm{d}\mathbf{k}_2 + \int M_{kk_1k_2} c_k c_{k_2}^* \delta_{k_2-k-k_1}\,\mathrm{d}\mathbf{k}_1\,\mathrm{d}\mathbf{k}_2 + \int N_{kk_1k_2} c_k^* c_{k_1}^* \delta_{k+k_1+k_2}\,\mathrm{d}\mathbf{k}_1\,\mathrm{d}\mathbf{k}_2 + \ldots \tag{12.4}$$

We require such a transformation to eliminate the third order terms from the Hamiltonian and to be canonical. The last item means that

$$\{c_k, c_{k'}^*\} = \delta_{k-k'}\,, \qquad \{c_k, c_{k'}\} = \{c_k^*, c_{k'}^*\} = 0\,.$$

From these two requirements, after simple algebra, we can find that

$$a_k = c_k - \int \frac{V_{kk_1k_2} c_{k_1} c_{k_2}}{\omega_k - \omega_{k_1} - \omega_{k_2}}\,\delta_{k-k_1-k_2}\,\mathrm{d}\mathbf{k}_1\,\mathrm{d}\mathbf{k}_2 + 2\int \frac{V_{k_2kk_1}^* c_k c_{k_2}^*}{\omega_{k_2} - \omega_k - \omega_{k_1}}\,\delta_{k_2-k-k_1}\,\mathrm{d}\mathbf{k}_1\,\mathrm{d}\mathbf{k}_2 - \int \frac{U_{kkk_2} c_k^* c_{k_1}^*}{\omega_k + \omega_{k_1} + \omega_{k_2}}\,\delta_{k+k_1+k_2}\,\mathrm{d}\mathbf{k}_1\,\mathrm{d}\mathbf{k}_2 + \ldots \tag{12.5}$$

Here the first two integral terms guarantee the cancellation in H_1 of the second two terms, while the last term gives the cancellation of the other two, proportional to $a^*a^*a^*$ and aaa. These two transformations (eliminating both pairs from H_1) are independent and can be carried out separately. This procedure for successive elimination of perturbation terms in the Hamiltonian expansion by means of canonical transformations is called classical perturbation theory. In constructing such a theory we quickly come up against the problem of 'small denominators', related in the present case to the appearance of non-integrable singularities near the manifolds

$$\omega_k + \ldots + \omega_{k_i} - \omega_{k_{i+1}} - \ldots - \omega_{k_n} = 0\,,$$

$$\mathbf{k} + \ldots + \mathbf{k}_i - \mathbf{k}_{i+1} - \ldots - \mathbf{k}_n = 0\,,$$

which give the condition for an nth order resonance. The simplest manifolds already appear in the elimination of the three-wave Hamiltonian (3.8), when [cf. Eqn (12.5)]

$$\omega_k + \omega_{k_1} + \omega_{k_2} = 0\,, \qquad \mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2 = 0 \tag{12.6}$$

and

$$\omega_k - \omega_{k_1} - \omega_{k_2} = 0\,, \qquad \mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2 = 0\,. \tag{12.7}$$

Satisfying the first condition is possible if waves with negative energy exist in the medium, and then one of the frequencies ω_k must be negative. Such a situation, as a rule, occurs in unstable media, for example, in a plasma with a current. If there are no waves with negative energy in the medium, then the terms proportional to $a^*a^*a^*$ and aaa can be eliminated from H_1 by a canonical transformation, and in this sense they are unimportant (non-resonant).

The possible existence of solutions of the system (12.7) depends on the form of the functions $\omega(\mathbf{k})$. For isotropic media, in which $\omega(k)$ depends only on $|\mathbf{k}|$, there is no solution if $\omega(0) = 0$ and $\omega''(k) < 0$. Such a situation is realized, for example, for surface gravitational waves. For capillary waves the resonance conditions (12.7) are satisfied.

If the conditions (12.6) and (12.7) have no solutions then the three-wave terms can be eliminated. Among the fourth order terms the important contribution to the Hamiltonian is of the form

$$H_3 = \int T_{k_1 k_2 k_3 k_4} a^*_{k_1} a^*_{k_2} a_{k_3} a_{k_4} \delta_{k_1+k_2-k_3-k_4} \prod \mathrm{d}\mathbf{k}_i \,, \tag{12.8}$$

for which the resonance condition

$$\omega_{k_1} + \omega_{k_2} - \omega_{k_3} - \omega_{k_4} = 0 \,,$$
$$\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4 = 0$$

may be satisfied for any form of $\omega(k)$. Here the three-wave interaction leads to a renormalization of the vertex $T_{kk_1k_2k_3}$ in Eqn (12.8) (see Ref. [83]):

$$\begin{aligned} T_{kk_1k_2k_3} = T^{(0)}_{kk_1k_2k_3} &- 2\,\frac{U_{-k_2-k_3,k_2k_3} U^*_{-k-k_1,kk_1}}{\omega_{k+k_1} + \omega_k + \omega_{k_1}} \\ &+ 2\,\frac{V_{k_2+k_3k_2k_3} V^*_{k+k_1kk_1}}{\omega_{k+k_1} - \omega_k - \omega_{k_1}} - 2\,\frac{V_{kk_2k-k_2} V^*_{k_3k_1k_3-k_1}}{\omega_{k_3-k_1} + \omega_{k_1} - \omega_{k_3}} \\ &- 2\,\frac{V_{k_1k_3k_1-k_3} V^*_{k_2kk_2-k}}{\omega_{k_2-k} + \omega_k - \omega_{k_2}} - 2\,\frac{V_{k_1k_2k_1-k_2} V^*_{k_3-kk_3-k}}{\omega_{k_3-k} + \omega_k - \omega_{k_3}} \\ &- 2\,\frac{V_{kk_3k-k_3} V^*_{k_2k_1k_2-k_1}}{\omega_{k_2-k} + \omega_{k_1} - \omega_{k_2}} \,. \end{aligned} \tag{12.9}$$

Thus, we arrive at a sequence of standard interaction Hamiltonians: the Hamiltonian

$$H_{\mathrm{d}} = \int (V_{kk_1k_2} a^*_k a_{k_1} a_{k_2} + \mathrm{c.c.}) \delta_{k-k_1-k_2} \, \mathrm{d}\mathbf{k} \, \mathrm{d}\mathbf{k}_1 \, \mathrm{d}\mathbf{k}_2 \,, \tag{12.10}$$

is responsible for the process of decay $1 \to 2$ and the inverse process of fusion $2 \to 1$; the Hamiltonian

$$H_{\mathrm{ex}} = \frac{1}{3} \int (U^*_{kk_1k_2} a^*_k a^*_{k_1} a^*_{k_2} + \mathrm{c.c.}) \delta_{k+k_1+k_2} \, \mathrm{d}\mathbf{k} \, \mathrm{d}\mathbf{k}_1 \, \mathrm{d}\mathbf{k}_2 \,, \tag{12.11}$$

describes the so-called explosive instability, in which three quanta of the wave field are created simultaneously from vacuum ($0 \to 3$), the Hamiltonian

$$H_{\mathrm{sc}} = \int T_{kk_1k_2k_3} a^*_k a^*_{k_1} a_{k_2} a_{k_3} \delta_{k+k_1-k_2-k_3} \prod \mathrm{d}\mathbf{k}_i \,, \tag{12.12}$$

is responsible for the process $2 \to 2$, etc.

If several types of waves exist in the medium, the list of standard Hamiltonians is greatly increased. We give one of them, responsible for the interaction of high-frequency and low-frequency waves:

$$H_{\mathrm{int}} = \int (V_{kk_1k_2} b_k a^*_{k_1} a_{k_2} + \mathrm{c.c.}) \delta_{k-k_1-k_2} \, \mathrm{d}\mathbf{k} \, \mathrm{d}\mathbf{k}_1 \, \mathrm{d}\mathbf{k}_2 \,. \tag{12.13}$$

A Hamiltonian of type (12.13) describes the interaction of light and sound in dielectrics, Langmuir and ion-acoustic waves in plasma, etc.

In describing a system of nonlinear waves by means of some standard interaction Hamiltonian, we are naturally assuming that the level of nonlinearity, characterized by the wave amplitude, is small. Despite these limitations, the resulting phenomena are quite rich. Many of them can already be understood starting from the simplest models that arise from the reduction of the standard Hamiltonians.

As a first example let us consider the interaction of three spectrally narrow wave packets with wave vectors lying near $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$. Such an interaction is resonant if, for instance,

$$\omega(\mathbf{k}_1) = \omega(\mathbf{k}_2) + \omega(\mathbf{k}_3) \,,$$
$$\mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3 \,.$$

For this case $a(\mathbf{k})$ may be represented in the form

$$a(\mathbf{k}) = a_1(\mathbf{k}) + a_2(\mathbf{k}) + a_3(\mathbf{k}) \,,$$

where a_1, a_2, a_3 are the amplitudes of the waves in the packets. The characteristic width κ_i of the each packet is assumed to be small compared with $|\mathbf{k}_i|$. For such an interaction, a canonical transformation reduces the Hamiltonian H_{d} given by Eqn (12.10) to

$$H_{\mathrm{int}} = 2 \int \left[V a^*_1(k_1) a_2(k_2) a_3(k_3) + \mathrm{c.c.} \right] \delta_{k_1-k_2-k_3} \prod \mathrm{d}\mathbf{k}_i \,.$$

Now using the narrowness of these packets, we set

$$\omega(\mathbf{k}_1 + \boldsymbol{\kappa}) = \omega(\mathbf{k}_i) + (\boldsymbol{\kappa}\mathbf{v}_i) \,, \qquad \mathbf{v}_i = \frac{\partial\omega}{\partial\mathbf{k}_i}$$

in H_0 and make the change of variables:

$$c_i(x) = a_i(\kappa) \exp\left[\mathrm{i}\omega(k_i) t \right] .$$

As a result,

$$H \to H - \sum \int \omega_i |c_i|^2 \, \mathrm{d}\boldsymbol{\kappa} \,.$$

Taking the inverse Fourier transform of this Hamiltonian, using the formula

$$\psi_i(x) = \frac{1}{(2\pi)^{3/2}} \int c_i(\boldsymbol{\kappa}) \exp(\mathrm{i}\boldsymbol{\kappa}\mathbf{r}) \, \mathrm{d}\boldsymbol{\kappa} \,,$$

we obtain the well-known equations for resonant interaction [77]:

$$\frac{\partial\psi_1}{\partial t} + (\mathbf{v}_1 \nabla)\psi_1 = -\frac{\mathrm{i}V}{(2\pi)^{3/2}} \, \psi_2 \psi_3 \,, \tag{12.14}$$

$$\frac{\partial\psi_2}{\partial t} + (\mathbf{v}_2 \nabla)\psi_2 = -\frac{\mathrm{i}V^*}{(2\pi)^{3/2}} \, \psi_1 \psi^*_3 \,, \tag{12.15}$$

$$\frac{\partial\psi_3}{\partial t} + (\mathbf{v}_3 \nabla)\psi_3 = -\frac{\mathrm{i}V^*}{(2\pi)^{3/2}} \, \psi_2 \psi^*_3 \,. \tag{12.16}$$

In a similar fashion one gets the system of equations for describing the explosive instability of three wave packets. In this case the interaction Hamiltonian for the packets arises as the result of reduction of the Hamiltonian (12.11):

$$\frac{\partial\psi_1}{\partial t} + (\mathbf{v}_1\boldsymbol{\nabla})\psi_1 = -\frac{\mathrm{i}U^*}{(2\pi)^{3/2}}\,\psi_2^*\psi_3^*\,, \tag{12.17}$$

$$\frac{\partial\psi_2}{\partial t} + (\mathbf{v}_2\boldsymbol{\nabla})\psi_2 = -\frac{\mathrm{i}U^*}{(2\pi)^{3/2}}\,\psi_1^*\psi_3^*\,, \tag{12.18}$$

$$\frac{\partial\psi_3}{\partial t} + (\mathbf{v}_3\boldsymbol{\nabla})\psi_3 = -\frac{\mathrm{i}U^*}{(2\pi)^{3/2}}\,\psi_2^*\psi_3^*\,. \tag{12.19}$$

The following example refers to the reduction of the Hamiltonian (12.12) for a single spectrally narrow wave packet. Suppose that the center of the packet is at $\mathbf{k}_0$. Then setting

$$a(\mathbf{k}) = c(\kappa)\exp(-\mathrm{i}\omega_{k_0}t)\,, \qquad \mathbf{k} = \mathbf{k}_0 + \kappa\,,$$

$$H \to H - \omega(k_0)\int |c(\kappa)|^2\,\mathrm{d}\kappa\,,$$

$$\omega(\mathbf{k}) = \omega(\mathbf{k}_0+\kappa) = \omega(\mathbf{k}_0) + \kappa\mathbf{v}_{\mathrm{g}} + \frac{1}{2}\frac{\partial^2\omega}{\partial k_\alpha \partial k_\beta}\,\kappa_\alpha\kappa_\beta\,,$$

we get the nonlinear Schrödinger equation (NLSE)

$$\mathrm{i}(\psi_t + \mathbf{v}_{\mathrm{g}}\boldsymbol{\nabla}\psi) + \frac{\omega_{\alpha\beta}}{2}\frac{\partial^2\psi}{\partial x_\alpha \partial x_\beta} + \frac{T}{(2\pi)^3}\,|\psi|^2\psi = 0 \tag{12.20}$$

for the envelope $\psi(\mathbf{r})$, where

$$\omega_{\alpha\beta} = \frac{\partial^2\omega}{\partial k_\alpha \partial k_\beta}\,.$$

Equation (12.20) describes the self-interaction of a spectrally narrow wave packet in a nonlinear medium. In an isotropic medium, when the tensor $\omega_{\alpha\beta}$ is of the form

$$\omega_{\alpha\beta} = \frac{v_{\mathrm{g}}}{2k_0}(\delta_{\alpha\beta} - n_\alpha n_\beta) + \omega'' n_\alpha n_\beta \qquad \left(\mathbf{n} = \frac{\mathbf{k}}{k}\right),$$

this equation simplifies to

$$\mathrm{i}(\psi_t + v_{\mathrm{g}}\psi_x) + \frac{v_{\mathrm{g}}}{2k_0}\,\Delta_\perp\psi + \frac{\omega''}{2}\,\psi_{xx} + \frac{T}{(2\pi)^3}\,|\psi|^2\psi = 0\,, \tag{12.21}$$

where the x axis coincides with the direction of the group velocity. In this equation the second term is responsible for the propagation of the wave packet as a whole with the group velocity v_{g} (this term can evidently be excluded by passing to the system of reference moving with $\mathbf{v}_{\mathrm{g}}$); the next term describes the diffraction of the packet in the plane transverse to $\mathbf{v}_{\mathrm{g}}$, the fourth term corresponds to the dispersion of the broadening along the x-direction, finally, the last term in Eqn (12.21) accounts for the nonlinearity.

After performing rescaling transformations in this equation, the NLSE can be written in the canonical form:

$$\mathrm{i}\psi_t + \Delta_\perp\psi + \sigma\psi_{xx} + \eta|\psi|^2\psi = 0\,. \tag{12.22}$$

Here $\sigma = \mathrm{sign}(\omega'' v_{\mathrm{g}})$ and $\eta = \mathrm{sign}(Tv_{\mathrm{g}})$. This equation can be considered as the Schrödinger equation for quantum particle motion in self-consistent potential $U = -\eta|\psi|^2$ with a positive transverse mass and a longitudinal mass, the sign of which coincides with σ. This means that the character of the interaction in the transverse and longitudinal directions are different and depend on the signs of η and σ. If $\eta > 0$, then in the transverse direction the attraction takes place and the packet has to be compressed due the nonlinear interaction. In the opposite case, ($\eta < 0$), the nonlinearity helps the diffraction broadening. The same situation arises for longitudinal motion. If $\sigma\eta = 1$, then the compression takes place along the group velocity direction and respectively the repulsion in the opposite case ($\sigma\eta = -1$). There exists the only variant $\sigma = \eta = 1$, when simultaneously the nonlinearity leads to packet compression in all directions. In this case wave collapse is possible (for a review see Ref. [84]).

Thus, depending on σ and η, there exist four canonical forms for the NLSE:

$$\mathrm{i}\psi_t + \Delta\psi + |\psi|^2\psi = 0\,, \tag{12.23}$$

$$\mathrm{i}\psi_t + \Delta_\perp\psi - \psi_{xx} + |\psi|^2\psi = 0\,, \tag{12.24}$$

$$\mathrm{i}\psi_t + \Delta\psi - |\psi|^2\psi = 0\,, \tag{12.25}$$

$$\mathrm{i}\psi_t + \Delta_\perp\psi - \psi_{xx} - |\psi|^2\psi = 0\,. \tag{12.26}$$

All these equations belong to the Hamiltonian type; they can be written as

$$\mathrm{i}\psi_t = \frac{\delta H}{\delta\psi^*}\,, \qquad H = \int\left\{|\boldsymbol{\nabla}_\perp\psi|^2 + \sigma|\psi_x|^2 - \frac{\eta}{2}\,|\psi|^4\right\}\mathrm{d}\mathbf{r}\,. \tag{12.27}$$

In deriving Eqn (12.20) we have assumed that the kernel $T_{k_1k_2k_3k_4}$ is a continuous function of its arguments [the vertex appearing in Eqn (12.9) is the value of this kernel at $\mathbf{k}_i = \mathbf{k}_0$]. However, this situation is not typical, in particular if $\omega(0) = 0$. At the same time, according to Goldstone's theorem (cf. Ref. [78]) the matrix element $V_{kk_1k_2}$ vanishes if one of the wave vectors $\mathbf{k}$, $\mathbf{k}_1$ or $\mathbf{k}_2$ is zero. Thus, in expression (12.9) for the matrix element T, there are indeterminacies when $\mathbf{k}_i = \mathbf{k}_0$. To remove them we must calculate a limit of the type

$$\lim_{k\to 0}\frac{|V_{k_0,k_0+k,-k}|^2}{\omega(k) - (\mathbf{k}\mathbf{v}_{\mathrm{g}})}\,.$$

For example, for surface waves of infinite depth

$$V_{kk_0k_0} \sim k^{3/4}\,, \qquad \omega(k) \sim k^{1/2}\,,$$

and all the indeterminacies vanish. For finite depth one has $V_{kk_0k_0} \sim k^{1/2}$, $\omega(k) \sim k$, so that this limit is finite in each direction, while the quantity $T_{k_0k_0k_0k_0}$ remains undetermined. Indeterminacy of this type is related to the excitation of forced motion of the medium as a whole. Such a situation occurs for all waves whose dispersion laws ω_k become linear as $k \to 0$. In addition to the surface waves considered above, such waves include ion-acoustic waves in a plasma, sound waves in a solid, etc.

In this situation one needs separate equations for describing induced low-frequency motions. This problem is a special case of a more general question: the interaction of a spectrally narrow high-frequency wave packet with a low-frequency acoustic type oscillation. The Hamiltonian for such an interaction can be constructed from general principles, based on the classical notion of an adiabatic invariant. [Of course, there is also a direct method of calculation, based on the reduction of the Hamiltonian (12.13).] We recall that for an oscillator with frequency ω there is the following remarkable relation between the energy E and the adiabatic invariant I:

$$\frac{E}{\omega} = I\,.$$

In the present case the adiabatic invariant is the quantity $\left|c(k)\right|^2$, so that

$$H_0 \approx \omega(k_0)\int\left|c(k)\right|^2 \mathrm{d}\mathbf{k} = \int \omega(k_0)|\psi|^2\,\mathrm{d}\mathbf{r}\,.$$

A nonlinear interaction with low-frequency motions does not destroy the adiabaticity, so

$$H_{\mathrm{int}} = \int \delta\omega|\psi|^2\,\mathrm{d}\mathbf{r}\,,$$

where $\delta\omega$ is the change in frequency due to variations of the local characteristics of the medium, namely, the density $\delta\rho$ and velocity $\mathbf{v}$:

$$\delta\omega = \frac{\partial\omega}{\partial\rho_0}\,\delta\rho + (\mathbf{k}_0\mathbf{v})\,.$$

(The second term corresponds to the Doppler effect.)

Setting $\mathbf{v} = \boldsymbol{\nabla}\varphi$ and remembering that $\delta\rho$ and φ are canonically conjugate functions for a compressible fluid, we get the equations [66, 79]

$$\mathrm{i}(\psi_t + \mathbf{v}_{\mathrm{g}}\boldsymbol{\nabla}\psi) + \frac{\omega_{\alpha\beta}}{2}\,\frac{\partial^2\psi}{\partial x_\alpha \partial x_\beta} + \left(\frac{\partial\omega}{\partial\rho_i}\,\delta\rho + \mathbf{k}_0\boldsymbol{\nabla}\varphi\right)\psi + \frac{T}{(2\pi)^3}\,|\psi|^2\psi = 0\,, \tag{12.28}$$

$$\frac{\partial}{\partial t}\,\delta\rho + \rho_0\Delta\varphi + (\mathbf{k}_0\boldsymbol{\nabla})|\psi|^2 = 0\,, \tag{12.29}$$

$$\rho_0\,\frac{\partial\varphi}{\partial t} + c_{\mathrm{s}}^2\delta\rho + \frac{\partial\omega}{\partial\rho_0}\,|\psi|^2 = 0\,, \tag{12.30}$$

where T is the regular part of the matrix element $\tilde{T}_{k_0k_0k_0k_0}$ without singularities.

The Hamiltonian of this system is a combination of the Hamiltonians for Eqns (12.20) and (4.3):

$$H_{\mathrm{c}} = \int\bigg[-\mathrm{i}\psi^*\mathbf{v}_{\mathrm{g}}\boldsymbol{\nabla}\psi + \frac{1}{2}\,\omega_{\alpha\beta}\,\frac{\partial\psi}{\partial x_\alpha}\,\frac{\partial\psi^*}{\partial x_\beta} + \left(\frac{\partial\omega}{\partial\rho_0}\,\delta\rho + \mathbf{k}_0\boldsymbol{\nabla}\varphi\right)|\psi|^2 + \frac{1}{2}\,\frac{T}{(2\pi)^3}\,|\psi|^4 + c_{\mathrm{s}}^2\,\frac{\delta\rho^2}{2\rho_0} + \rho_0\,\frac{(\boldsymbol{\nabla}\varphi)^2}{2}\bigg]\,\mathrm{d}\mathbf{r}\,. \tag{12.31}$$

Depending on the ratio between the group velocity v_{g} and the sound velocity c_{s}, Eqns (12.29) and (12.30) permit various simplifications. If $v_{\mathrm{g}} < c_{\mathrm{s}}$ and $v_{\mathrm{g}}\Delta k \gg T|\psi|^2$, where Δk is the width around k of the high-frequency packet, we can replace $\partial/\partial t$ by $\mathbf{v}_{\mathrm{g}}\boldsymbol{\nabla}$ in Eqns (12.29) and (12.30):

$$\begin{aligned} &-\mathbf{v}_{\mathrm{g}}\boldsymbol{\nabla}\delta\rho + \rho_0\Delta\varphi + (\mathbf{k}_0\boldsymbol{\nabla})|\psi|^2 = 0\,,\\ &-\rho_0(\mathbf{v}_{\mathrm{g}}\boldsymbol{\nabla})\varphi + c_{\mathrm{s}}^2\delta\rho + \frac{\partial\omega}{\partial\rho_0}\,|\psi|^2 = 0\,. \end{aligned} \tag{12.32}$$

For isotropic media the resulting system of equations described in a coordinate system moving with the group velocity goes over into the Davey – Stewartson equations

$$\mathrm{i}\psi_t + \frac{v_{\mathrm{g}}}{2k_0}\,\Delta_\perp\psi + \frac{\omega''}{2}\,\psi_{xx} + \left[\frac{\partial\omega}{\partial\rho_0} + \frac{k_0c_{\mathrm{s}}^2}{\rho_0 v_{\mathrm{g}}}\right]\delta\rho\psi + \left[\frac{T}{(2\pi)^3} + \frac{k_0}{\rho_0 v_{\mathrm{g}}}\,\frac{\partial\omega}{\partial\rho_0}\right]|\psi|^2\psi = 0\,, \tag{12.33}$$

$$\left(v_{\mathrm{g}}\,\frac{\partial}{\partial x}\right)^2\left(\delta\rho - \frac{k_0}{v_{\mathrm{g}}}\,|\psi|^2\right) = \Delta\left(c_{\mathrm{s}}^2\delta\rho + \frac{\partial\omega}{\partial\rho_0}\,|\psi|^2\right), \tag{12.34}$$

which were first obtained for gravitational waves on the surface of a fluid of finite depth [80].

In this system Eqn (12.32) or (12.34) represents a constraint for $\delta\rho$, φ and $|\psi|^2$, and the Hamiltonian for (12.33) is constructed taking these constraints into account. An explicit expression for it is easily obtained if we represent the constraint equations in the form

$$-(\mathbf{v}_{\mathrm{g}}\boldsymbol{\nabla})\delta\rho = \frac{\delta H}{\delta\varphi}\,, \qquad -(\mathbf{v}_{\mathrm{g}}\boldsymbol{\nabla})\varphi = \frac{\delta H}{\delta\rho}$$

with $H = H_{\mathrm{c}}$ given by Eqn (12.31). Then the Hamiltonian H_{DS} for the Davey – Stewartson equation is obtained from H_{c} by the rule

$$H_{\mathrm{DS}} = H_{\mathrm{c}} - \int \delta\rho(\mathbf{v}_{\mathrm{g}}\boldsymbol{\nabla})\varphi\,\mathrm{d}\mathbf{r} + \mathrm{i}\int \psi^*(\mathbf{v}_{\mathrm{g}}\boldsymbol{\nabla})\psi\,\mathrm{d}\mathbf{r}\,,$$

and the equations have the form

$$\mathrm{i}\psi_t = \frac{\delta H_{\mathrm{DS}}}{\delta\psi^*}\,.$$

If $v_{\mathrm{g}} > c_{\mathrm{s}}$, then in Eqns (12.29) and (12.30) we cannot replace $\partial/\partial t$ by the operator $(-\mathbf{v}_{\mathrm{g}}\boldsymbol{\nabla})$ no matter what the level of nonlinearity is. This is easily understood if we rewrite Eqns (12.29) and (12.30) in a Fourier representation. If we carry this out, we are confronted by a resonance denominator of the form

$$kc_{\mathrm{s}} = (\mathbf{k}\mathbf{v}_{\mathrm{g}})\,,$$

which corresponds to condition (12.7) for the decay of a high-frequency wave into high-frequency and sound waves. Under the condition $v_{\mathrm{g}} \gg c_{\mathrm{s}}$, corresponding, for example, to the interaction of light and sound in dielectrics, the contribution to $\delta\omega$ because of the Doppler effect is weak compared to the scattering by long-wave fluctuations $\delta\rho$ of the density (the relative parameter is $c_{\mathrm{s}}/v_{\mathrm{g}}$). In this case Eqns (12.28) – (12.30) simplify to the following form:

$$\mathrm{i}\psi_t + \frac{\omega''}{2}\,\psi_{xx} + \frac{v_{\mathrm{g}}}{2k_0}\,\Delta_\perp\psi + \left(\frac{\partial\omega}{\partial\rho}\,\delta\rho + \frac{T}{(2\pi)^3}\,|\psi|^2\right)\psi = 0\,,$$

$$\left[\left(\frac{\partial}{\partial t} - v_{\mathrm{g}}\,\frac{\partial}{\partial x}\right)^2 - c_{\mathrm{s}}^2\Delta_\perp\right]\delta\rho = \frac{\partial\omega}{\partial\rho_0}\,\Delta|\psi|^2\,.$$

Among the simplest reductions one should also include the reduction of the Boussinesq equation to the KdV equation. For the Boussinesq model the dispersion law is close to linear. This means that in the Hamiltonian H_1 with coefficients of the form (4.5) one should keep the terms proportional to a^*aa and aa^*a^*, and eliminate the other terms by canonical transformations, while in the quadratic Hamiltonian we can keep in $\omega(k)$ the term linear in the dispersion ν: $\omega(k) = kc_s[1 + (\nu\rho_0 k^2/2c_s^2)]$; then changing from the variables a_k to $u(x)$ according to the formulae

$$a_k = \frac{u_k}{\sqrt{k}}\,, \qquad u = \int_0^\infty \left[u_k \exp(\mathrm{i}kx) + u_k^* \exp(-\mathrm{i}kx)\right] \mathrm{d}k$$

we then arrive at the KdV equation

$$u_t + c_s u_x + \beta u u_x + c_s \gamma u_{xxx} = 0\,, \tag{12.35}$$

where

$$\gamma = -\frac{\nu\rho_0}{2c_s^2}\,, \qquad \beta = \frac{1}{2}\left(\frac{c_s}{\rho_0}\right)^{1/2}(1+g)\,.$$

The natural generalization of the KdV equation to many dimensions is the Kadomtsev–Petviashvili (KP) equation [85] which follows if one considers the reduction of the Hamiltonian (4.5) to the case of a narrow angular distribution of acoustic waves with a weak dispersion. Given that the packet mainly propagates along the x-axis the equation (12.35) will transform into the form

$$\frac{\partial}{\partial x}(u_t + c_s u_x + \beta u u_x + c_s \gamma u_{xxx}) = \frac{c_s}{2}\,\nabla_\perp^2 u\,, \tag{12.36}$$

where the term on the left-hand side of the equation describes the diffraction of acoustic waves in the transverse direction to x. It is necessary to emphasize that all terms in this equation are small compared to the second one, $c_s u_x$, responsible for the propagation along the x-axis of the packet with the velocity of sound. And in this sense the procedure for deriving the KP equation as well as the KdV equation represents one of the variants of the averaging methods when it is possible to distinguish two different temporal types of motion, rapid and slow.

The examples do not obviously exhaust all the possible reductions of Hamiltonians. We have only concentrated on the clearest ones, demonstrating their universality. A significant feature of this universality is that many of the models considered in this survey permit the application of the inverse scattering transform.

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