Dynamics of Free Surface of an Ideal Fluid Without Gravity and Surface Tension

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Abstract

Using a combination of the canonical formalism for free-surface hydrodynamics and conformal mapping to a horizontal strip we obtain a simple system of pseudo-differential equations for the surface shape and hydrodynamic velocity potential. The system is well-suited for numerical simulation. It can be effectively studied in the case when the Jacobian of the conformal mapping takes very high values in the vicinity of some point on the surface. At first order in an expansion in inverse powers of the Jacobian one can reduce the whole system of equations to a single equation which coincides with the well-known Laplacian Growth Equation (LGE). In the framework of this model one can construct remarkable special solutions of the system describing such physical phenomena as formation of finger-type configurations or changing of the surface topology – generation of separate droplets.

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1 Introduction

Two-dimensional irrotational motion of an ideal fluid in a domain confined between a free surface and a flat bottom is one of the classical subjects of investigation in hydrodynamics. The method of conformal mapping is the traditional approach to its study. The first important result in this area dates from the middle of the last century and belongs to Stokes [1]. Since the classic works of Nekrasov [2] and Levi-Civita [3] performed in the 1920s, many publications have been devoted to this subject. (See, for instance, the beautiful book of Stoker [4] and references therein.) The mathematical aspects of these works gave a powerful impulse to the development of certain branches in the theory of integral equations and in functional analysis.

For the nonstationary surface phenomena studied in the 1960s and later, the Lagrangian description was more common [5, 6]. Some authors (see [7] and the review [8] and references therein) tried to perform an analytical continuation with respect to Lagrangian coordinates. However, these coordinates do not allow a proper conformal mapping, since their analytical continuation has singularities in both half-planes. Recently Tanveer [9, 10] suggested applying the conformal mapping to the nonstationary problem directly in the Euler description. He applied for the periodic deep water case the mapping of the fluid region into the interior of the unit circle. The equations obtained are quite complicated and therefore are difficult for both analytical and numerical analysis.

A convenient approach to the description of the potential flow of a fluid with a free surface in any dimension may be obtained by using of the Canonical Formalism known since 1968 (see [11]). For $2-D$ geometry a combination of the Canonical Formalism and the conformal mapping appears to be the most natural approach to the problem. This approach was implemented for the deep fluid in the recent paper [12]. Both gravity and surface tension were taken into consideration. The equations obtained in the paper [12] can be written in two different forms - implicit (not resolved with respect to time derivatives of surface shape and surface potential) and explicit (resolved with respect to time derivatives). The implicit equations are simple and symmetric. In the absence of surface tension they contain only quadratic nonlinearity. The explicit equations, though not as simple as the implicit ones, are perfectly suited for numerical simulation.

The most interesting unresolved problems in free-surface hydrodynamics are associated with formation of singularities (wave breaking) and essential modification of the surface geometry – wave generation, sprays, plumes, and droplets. Only in very special cases [13] they can be solved by using a traditional perturbation technique against the background of the flat surface. In these cases the Jacobian J of the conformal mapping remains close to unity. However, in many typical cases the Jacobian takes very large values in some piece of the surface. In this situation one can treat the inverse Jacobian $1/J$ as a small parameter and expand the solution in its powers. The first step in this direction was done in our paper [12]. Due to the essential nonlocality of the basic equation, the whole procedure of expansion in powers of $1/J$ is tricky, but a first approximation can be found easily. It is interesting that in this case the system of two equations for surface shape and surface potential reduces to a single equation, coinciding for some mysterious reason with the well-known Lagrangian Growth Equation (LGE). This equation is completely integrable, it has an infinite set of special solutions expressed in elementary functions. Among them are the solutions describing formation of finger-type configurations (quite similar to the Saffman-Taylor fingers [14]) as well as the solutions describing formation of droplets.

In the present paper we shall present a more detailed description of the results, briefly announced in the short letter [12]. We also extend most of our previous results to the case of a fluid of a finite depth.

The paper is organized as follows. In the second section we introduce the Lagrangian description of a free-surface fluid of a finite depth, combining Canonical Formalism and conformal mapping, and derive the implicit equation as the corresponding Euler-Lagrange equation. In section 3 we find the explicit equation. In section 4 we consider stationary waves and calculate the dependence of their dispersion relation on the wave amplitude. The first approximation in the high-Jacobian expansion is introduced in the section 5. In sections 6 and 7 we study finger-type and droplet-type solutions, respectively.

Checking analyticity violation is the most sensitive tool for studying that set of collapses. Loss of analyticity of vortex sheets at the nonlinear stage of the Kelvin-Helmholz instability [23] is such an example. Various aspects of the singularity formation for vortex sheet motion have so far been studied in a number of papers, both numerically and analytically [23, 24, 25, 26]. The recent paper [26] should be mentioned in particular, which provides a considerable numerical evidence of arising of the infinite surface curvature in a finite time. The root (in space) character of the arising singularity has been checked in [26] too. As for analytical consideration, though showing the existence of singularities, it is still lacking, in our opinion, demonstration of explicit collapsing solutions. The question also remains open, whether root singularities are generic for the Cauchy problem in this system.

In this paper we will consider how the singularities appear as a result of the analyticity breaking on the interface between two ideal liquids in the absence of both gravity and surface tension. This question is very important, also, for understanding the evolution of the boundary between two fluids while studying sea surface waves and the nonlinear stage of the Rayleigh-Taylor instability resulting in the finger structure (see, for instance, [27] and references therein). We present the analytical solution of the problem based both on the perturbation approach, assuming small angles of the interface variations, and on using the Hamiltonian formalism for the description of the interface motion. For the case of liquid with a free surface, the problem was formulated [28] by one of the authors (V.Z.) of the present paper. It is supposed that the singularity formation on a free surface of an ideal fluid or in more general case, for the boundary between two ideal fluids, is mainly connected with inertial forces, other factors give minor correction. This means that if one considers, for instance, motion of the ideal liquid drop (without both gravity and surface tension) then on the surface of the drop there will appear a singularity of the wedge type. This idea was later confirmed by direct numerical integration of the Euler equation for the case of the deep water [29].

Adopting only the small slope approximation, we give the solution of the Cauchy problem for the motion of the boundary between two liquids.

The main conjecture of this paper is as follows. The formation of singularities on the interface for small angle approximation can be considered as the process of the wave breaking in the complex plane where the solution can be extended to. This results in the motion of both branch points of the analytical continuation of the velocity potential and singular points of the analytical extension of the surface elevation. When for the first time the most "rapid" singular point will reach the real axis it will be just the singularity appearance. Respectively three kinds of singularities are possible. For the first kind at the touching moment the tangent velocity on the interface has the infinite first derivative and simultaneously the second space derivative of the interface coordinate $z = \eta(x,t)$, i.e. η_{xx}, also turns into infinity. These are weak singularities of the root character ($\eta_{xx} \sim \mid x \mid^{-1/2}$) which can be assumed to serve as an origin of more powerful singularities, observed in the numerical experiments [29], or to represent the separate type of singularities. This kind of singularities turns out to be consistent with an assumption about small surface angles. It is shown that the interaction of two movable branch points of the tangent velocity can lead under some definite conditions to the formation of the second type of singularities - wedges on the surface shape . Close to the collapse time the self-similar solution for such singularities occurs to be compatible with the complete system of equations describing arbitrary angle values. The third type is caused by the initial analytical properties of $\eta_0(x)$ resulting in the formation of strong singular interface profile.

2 Lagrangian description of a finite depth fluid

Let an incompressible fluid fill a domain on the (x, y)-plane bounded by the free surface,

$$y = \eta(x,t), \qquad -\infty < x < \infty, \tag{2.1}$$

and the bottom,

$$y = -\bar{h}\,.$$

The fluid flow in $-\bar{h} < y < \eta$ is potential,

$$V = \nabla\Phi\,, \quad \Delta\Phi = 0. \tag{2.2}$$

The Laplace equation (2.2) must be considered together with the following boundary conditions,

$$\frac{\partial \eta}{\partial t} + \frac{\partial \Phi}{\partial x}\frac{\partial \eta}{\partial x} = \frac{\partial \Phi}{\partial y}\bigg|_{y=\eta}, \tag{2.3}$$

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}\Big(\frac{\partial \Phi}{\partial x}\Big)^2 + \frac{1}{2}\Big(\frac{\partial \Phi}{\partial x}\Big)^2\bigg|_{y=\eta} + g\eta = 0\,, \tag{2.4}$$

$$\frac{\partial \Phi}{\partial y}\bigg|_{y=-\bar{h}} = 0\,. \tag{2.5}$$

In (2.4) g is gravity acceleration. Let us introduce the quantity

$$\Psi(x,t) = \Phi(x,y,t)|_{y=\eta} = \Phi(x,\eta(x,t),t)\,. \tag{2.6}$$

As found in [11], $\eta(x,t)$ and $\Psi(x,t)$ are canonically conjugate variables,

$$\begin{aligned} \frac{\partial \eta}{\partial t} &= \frac{\delta H}{\delta \Psi}\,, \\ \frac{\partial \Psi}{\partial t} &= -\frac{\delta H}{\delta \eta}\,, \end{aligned} \tag{2.7}$$

where the Hamiltonian H is the total energy of the fluid,

$$\begin{aligned} H &= T + U\,, \\ T &= \frac{1}{2}\int_{-\infty}^{\infty} dx \int_{-\eta}^{\eta} (\nabla\Phi)^2 dy\,, \end{aligned} \tag{2.8}$$

$$U = \frac{g}{2}\int \eta^2 dx\,. \tag{2.9}$$

Equations (2.7) and (2.8) extremize the action,

$$\delta S = 0\,, \qquad S = \int L dt\,, \tag{2.10}$$

$$L = \int \Psi \frac{\partial \eta}{\partial t} dx - H\,. \tag{2.11}$$

Let us apply the conformal mapping of the domain on the plane $z = x + iy$,

$$-\infty < x < \infty, \quad -\bar{h} \le y \le \eta(x,t)\,, \tag{2.12}$$

to the strip,

$$-\infty < u < \infty, \quad -h \le v \le 0\,, \tag{2.13}$$

on the plane $\omega = u + iv$[1]. After this transformation, the shape of the surface is given parametrically by

$$y = y(u,t), \quad x = u + \tilde{x}(u,t)\,. \tag{2.14}$$

[1] If $\eta \to 0$ at $|x| \to \infty$, then $\bar{h} = h$. In the periodical case $\bar{h}$ and h are different.

Functions y and $\tilde{x}$ are connected by the relation,

$$y = \hat{R}\tilde{x}\,. \tag{2.15}$$

Here $\hat{R}$ is the operator,

$$\hat{R}f(u) = \frac{1}{2h} P.V. \int_{-\infty}^{\infty} \frac{f(u')}{\sinh \pi/2h(u'-u)} du'\,. \tag{2.16}$$

Taking Fourier transforms gives

$$y_k = R_k x_k \qquad R_k = i \tanh kh\,. \tag{2.17}$$

In the limiting case of infinitely deep water $h \to \infty$ and $\hat{R}$ goes over to the Hilbert transformation,

$$\lim_{h\to\infty} \hat{R} = \hat{H}\,, \tag{2.18}$$

$$\hat{H}f = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(u')du'}{u'-u}\,. \tag{2.19}$$

One can introduce also the inverse operator $\hat{T}$,

$$\tilde{x} = \hat{T}y, \quad \text{with} \quad \hat{R}\hat{T} = \hat{T}\hat{R} = 1\,, \tag{2.20}$$

in which

$$\hat{T}f = \frac{1}{h} P.V. \int_{-\infty}^{\infty} \frac{f(u')}{1 - e^{-\pi/h(u-u')}} du'\,. \tag{2.21}$$

Asymptotically as $h \to \infty$,

$$\hat{T} \to -\hat{R}\,, \quad \text{but} \quad \hat{T}^{-1} \neq -\hat{R}\,.$$

Both operators $\hat{R}, \hat{T}$ are anti-self-adjoint,

$$\hat{R}^{+} = -\hat{R} \quad \text{and} \quad \hat{T}^{+} = -\hat{T}\,, \tag{2.22}$$

with Fourier transforms,

$$\tilde{x}_k = T_k y_k \quad \text{and} \quad T_k = -i \coth kh\,. \tag{2.23}$$

With the help of the operator $\hat{R}$ for any real smooth function $\phi(w)$ vanishing at the infinity, $u \to \infty$, one can construct a complex function θ, given at the real axis $v = 0$,

$$\theta = \phi + i\hat{R}\phi, \tag{2.24}$$

which can be analytically extended into the strip $0 \leq v \leq -h$. The real part of this function automatically will be satisfied to the boundary condition

$$\left.\frac{\partial \phi}{\partial v}\right|_{v=-h} = 0.$$

After the conformal mapping, the velocity potential $\Phi = \Phi(u, v)$ remains harmonic, i.e., it obeys the Laplace equation,

$$\frac{\partial^2 \Phi}{\partial u^2} + \frac{\partial^2 \Phi}{\partial v^2} = 0\,, \tag{2.25}$$

with boundary conditions,

$$\left.\frac{\partial \Phi}{\partial v}\right|_{v=-h} = 0 \quad \text{and} \quad \Phi|_{v=0} = \Psi(u, t)\,. \tag{2.26}$$

Applying the procedure (2.24) to the function Ψ we arrive at the complex potential

$$\Theta = \Phi + i\Xi,$$

where Φ and stream function Ξ are harmonic functions in the strip and satisfy at the bottom to the corresponding boundary conditions. On the real axis $\Theta = \Psi + i\hat{R}\Psi$ and therefore

$$\left.\frac{\partial \Phi}{\partial v}\right|_{v=0} = -\frac{\partial}{\partial u}\hat{R}\Psi = -\hat{R}\Psi_u\,, \tag{2.27}$$

so that the kinetic energy is

$$T = -\frac{1}{2}\int_{-\infty}^{\infty} \Psi\hat{R}\Psi_u du\,. \tag{2.28}$$

Then,

$$dx = x_u du\,,$$
$$\eta_t dx = (y_t x_u - x_t y_u)du\,, \tag{2.29}$$

and the Lagrangian can be expressed as follows,

$$L = \int_{-\infty}^{\infty} \Psi(y_t x_u - x_t y_u)du + \frac{1}{2}\int_{-\infty}^{\infty} \Psi\hat{R}\Psi_u du - \frac{g}{2}\int_{-\infty}^{\infty} y^2 x_u du + \int_{-\infty}^{\infty} f(y - \hat{R}\tilde{x})du\,. \tag{2.30}$$

Here f is the Lagrange multiplier which imposes the relation (2.20). Hamilton's principle,

$$\frac{\delta S}{\delta \Psi} = 0\,,$$

gives the following equation,

$$y_t x_u - x_t y_u = -\hat{R}\Psi_u\,, \tag{2.31}$$

or

$$(1 + \tilde{x}_u)y_t - \tilde{x}_t y_u = -\hat{R}\Psi_u\,.$$

The mean level of fluid is constant, so

$$\int_{-\infty}^{\infty}(y_t x_u - x_t y_u)du = 0\,. \tag{2.32}$$

From (2.31) one finds,

$$\Psi = -\hat{T}\partial_u^{-1}(y_t x_u - x_t y_u) + C(t)\,. \tag{2.33}$$

Substituting (2.33) to (2.30) and using the identity (2.32) implies,

$$L = T - U + \int_{-\infty}^{\infty} f(y - \hat{R}\tilde{x})du\,. \tag{2.34}$$

If $y = y(u,t)$ is chosen as the set of coordinates, one can drop the last term in (2.34). Consequently,

$$L = T - U\,. \tag{2.35}$$

Consider now the conditions,

$$\frac{\delta S}{\delta y} = 0 \quad \text{and} \quad \frac{\delta S}{\delta x} = 0\,.$$

These conditions imply the equations,

$$\Psi_t x_u - \Psi_u x_t + gyx_u = f\,, \tag{2.36}$$

$$\Psi_t y_u - \Psi_u x_t + gyy_u = -\hat{R}f\,, \tag{2.37}$$

which are equivalent to the equation,

$$\Psi_t y_u - \Psi_u y_t + gyy_u + \hat{R}(\Psi_t x_u - \Psi_u x_t + gyx_u) = 0\,. \tag{2.38}$$

Equations (2.31) and (2.38) constitute the complete system of equations describing the potential flow of a free-surface fluid. They are not resolved with respect to time-derivatives, rather they are written in implicit form.

3 Explicit form of the motion equations

Now we show that the system (2.31), (2.36) and (2.37) can be resolved with respect to time derivatives of Ψ and z given at $v = 0$.

Since $z = x + iy$, we have $x = \frac{1}{2}(z + z^*)$ and $y = \frac{1}{2i}(z - z^*)$. So equation (2.34) can be rewritten as

$$z_t z_u^* - z_t^* z_u = -2i\hat{R}\Psi_u\,,$$

or,

$$\operatorname{Im}\left[\frac{z_t}{z_u}\right] = \left.\frac{\hat{R}\Psi_u}{|z_u|^2}\right|_{v=0}. \tag{3.1}$$

Hence the function $\frac{z_t}{z_u}$, being analytical in the strip, can be restored completely by means of (2.24):

$$z_t = -z_u(\hat{T} + i)\frac{\hat{R}\Psi_u}{|z_u|^2}\,, \tag{3.2}$$

or,

$$y_t = -(y_u\hat{T} + x_u)\frac{\hat{R}\Psi_u}{|z_u|^2}\,, \tag{3.3}$$

$$x_t = (y_u - x_u\hat{T})\frac{\hat{R}\Psi}{|z_u|^2}\,. \tag{3.4}$$

Subtracting (2.35) multiplied by x_u from (2.36) multiplied by y_u and using (2.34) yields

$$-\Psi_u\hat{R}\Psi_u = y_u f + x_u\hat{H}f\,. \tag{3.5}$$

Both sides of this equation can be represented as imaginary parts of analytical functions in the strip:

$$-\frac{1}{2}\mathrm{Im}(\Psi_u + i\hat{R}\Psi_u)^2 = \mathrm{Im}[z_u(f + i\hat{R}f)].$$

Hence due to the analyticity we conclude that analytical functions coincide, i.e.,

$$-\frac{1}{2}(\Psi_u + i\hat{R}\Psi_u)^2 = z_u(f + i\hat{R}f)$$

or

$$f + i\hat{R}f = -\frac{1}{2z_u}(\Psi_u + i\hat{R}\Psi_u)^2. \tag{3.6}$$

We now subtract equation (2.35) multiplied by x_t from equation (2.36) multiplied by y_t to find

$$(\Psi_t + gy)(y_t x_u - x_t y_u) = y_t f + x_t\hat{R}f = \mathrm{Im}[z_t(f + i\hat{R}f)]\,. \tag{3.7}$$

Time-derivatives x_t, y_t, z_t can be excluded by using (2.34), (3.2). Hence, after simple algebra we finally get

$$\Psi_t + gy = -\frac{(\Psi_u)^2 - (\hat{R}\Psi_u)^2}{2J} - \Psi_u\hat{T}(\frac{\hat{R}\Psi_u}{J}) \tag{3.8}$$

where $J = |z_u|^2$. Using the identity

$$(\Psi_u)^2 - (\hat{R}\Psi_u)^2 = 2\hat{T}(\Psi_u\hat{R}\Psi_u)$$

which follows from the analyticity of the function $(\Psi_u + i\hat{R}\Psi_u)^2$ the last equation can be rewritten as

$$\Psi_t + gy = -\frac{\hat{T}(\Psi_u\hat{R}\Psi_u)}{J} - \Psi_u\hat{T}(\frac{\hat{R}\Psi_u}{J})\,. \tag{3.9}$$

Thus, we have coupled equations (3.2) and (3.8) or (3.9) for z and Ψ resolved relative their time derivatives. It is easy to show that the kinematic condition (2.3) after the conformal mapping transforms at first into the form (2.34) and then into Eq. (3.2). Respectively, the dynamic condition (2.4) on the free surface transforms with the help of Eq.(3.2) into the equation (3.8) or into its equivalent form (3.9).

In the case of the deep water ($h \to \infty$) Eqs. (3.2), (3.8) and (3.9) have the form,

$$z_t = z_u(\hat{H} - i)\frac{\hat{H}\Psi_u}{|z_u|^2}\,, \tag{3.10}$$

$$\Psi_t + gy = -\frac{(\Psi_u)^2 - (\hat{H}\Psi_u)^2}{2J} + \Psi_u\hat{H}(\frac{\hat{H}\Psi_u}{J})\,, \tag{3.11}$$

$$\Psi_t + gy = \frac{\hat{H}(\Psi_u\hat{H}\Psi_u)}{J} + \Psi_u\hat{H}(\frac{\hat{H}\Psi_u}{J})\,. \tag{3.12}$$

4 Stationary waves

We rewrite the implicit equations for surface shape and hydrodynamic potential as

$$y_t(1+\tilde{x}_u) - x_t y_u = -\hat{R}\Psi_u\,, \tag{4.1}$$

$$\Psi_t y_u - \Psi_u y_t + gyy_u + \hat{R}\left(\Psi_t(1+\tilde{x}_u) - x_t\Psi_u + gy(1+\tilde{x}_u)\right) = 0\,. \tag{4.2}$$

The last equation has a particular solution,

$$\begin{aligned} y &= y(u-ct), \tilde{x} = \tilde{x}(u-ct)\,, \\ \Psi &= \Psi(u-ct) - gb_0 t\,, \end{aligned} \tag{4.3}$$

which describes stationary waves propagating with a constant velocity c.

Plugging (4.3) into (4.1) yields

$$cy_u = \hat{R}\Psi_u\,. \tag{4.4}$$

Substituting (4.3) into (4.2) and using the relation (4.4) leads to

$$-(c^2 + 2gb_0)y_u + gy_u + g\hat{R}(y(1+\tilde{x}_u)) = 0\,. \tag{4.5}$$

Hereafter, we assume that all functions are periodic with period $L = \frac{2\pi}{k}$. We assume further that the total amount of fluid is conserved,

$$< y(1+\hat{x}_u) >= 0\,. \tag{4.6}$$

One can then find a solution of equation (4.5) in the form

$$y = a_0 + \sum_{n=1}^{\infty} a_n \cos nku\,. \tag{4.7}$$

It is convenient to introduce the operator

$$\hat{S} = \hat{R}\partial^{-1}\,,$$

satisfying

$$\hat{S}^{-1}y = \sum_{n=1}^{\infty} \frac{a_n}{S_n} \cos nku\,,$$

where

$$S_n = \frac{\tanh knh}{kn}\,.$$

From (4.6) one can define a_0 as

$$a_0 = -\frac{1}{2}\sum_{n=1}^{\infty} \frac{a_n^2}{S_n} = \bar{h} - h\,. \tag{4.8}$$

For b_0 there is the relation following from (3.8),

$$b_0 = a_0 - \frac{c^2}{g} < 1 - \frac{1}{|z_u|^2} > \tag{4.9}$$

The other coefficients a_n satisfy the following system of equations,

$$(\tilde{c}^2 - gS_1)a_1 = \frac{g}{2}\sum_{m=1}^{\infty}\left(1 + \frac{S_1(S_m + S_{m+1})}{S_m S_{m+1}}\right) a_m a_{m+1}\,, \tag{4.10}$$

$$\begin{aligned}(\tilde{c}^2 - gS_n)a_n &= \frac{g}{2}\sum_{m=1}^{\infty}\left(1 + \frac{S_n(S_m + S_{n+m})}{S_m S_{n+m}}\right) a_m a_{n+m} \\ &\quad + \frac{g}{4}\sum_{m=1}^{n-1}\left(1 + \frac{S_n(S_m + S_{n-m})}{S_m S_{n-m}}\right) a_m a_{n-m}\,.\end{aligned} \tag{4.11}$$

Here $\tilde{c}^2 = c^2 - 2g(a_0 - b_0)$. In these equations the amplitude of the first harmonic, a_1, is arbitrary. It must be treated as small. In a zero approximation,

$$a_0 = 0, b_0 = 0 \quad \text{and} \quad c^2 = \tilde{c}^2 = \frac{g}{k}\tanh kh\,, \tag{4.12}$$

in accordance with the linear theory. In the first nontrivial approximation,

$$\begin{aligned} a_2 &= a_1^2 k \frac{3 + \tanh^2 kh}{4\tanh^3 kh}\,, \\ c^2 &= \frac{g}{k}\tanh kh\left(1 + \left[\frac{9 - 6\tanh^2 kh + 5\tanh^4 kh}{8\tanh^4 kh}\right](ka_1)^2\right)\,. \end{aligned} \tag{4.13}$$

If we get back to the physical depth $\bar{h}$ the dispersion relation (4.13) exactly coincides with that of in [15],

$$c^2 = \frac{g}{k}\tanh k\bar{h}\left(1 + \left[\frac{9 - 10\tanh^2 k\bar{h} + 9\tanh^4 k\bar{h}}{8\tanh^4 k\bar{h}}\right](ka_1)^2\right)\,. \tag{4.14}$$

In the limiting case of deep water one gets

$$c^2 = \frac{g}{k}(1 + (ka_1)^2)\,. \tag{4.15}$$

This result was first obtained by Stokes [1]. For shallow water,

$$\hat{S} \simeq h(1 - \frac{h^2}{3}\frac{\partial^2}{\partial x^2})\,,$$

and equation (4.5) goes over to a differential equation describing KdV-type solitons.

5 Small-angle approximation

In this section we study the free surface dynamics in the approximation of small surface angles for fluids of a finite depth in the case when the gravity is absent. As we mentioned in the Introduction for the deep water case the system can be effectively analyzed [13]. The trick which was used in these papers was connected with a possibility of analytical continuation of solution into the lower (physical) half-plane of z, where the the complex velocity potential obeys the differential equation in time and z. Moreover, in this approximation the equation for Θ occurs to be splitted from that for the free surface shape.

As we show in this section, the construction is the same as for the deep water case works for a finite depth also. The difference is connected only with a change of the analyticity region. In the given case this is the strip $0 \leq x \leq -h$. Therefore we will drop many details which one-to-one can be rewritten from the deep water case to this one considering only the main points.

First of all, make a reduction to small surface angles in the equations of motion (3.8) (with $g = 0$) and (2.31). Small angles means that the Jacobian J is close to unity, $J \approx 1 + 2x_u$, and therefore in the leading approximation Eq. (3.8) reads

$$\Psi_t = \frac{1}{2}(\hat{R}\Psi_u)^2 - 3(\Psi_u)^2). \tag{5.1}$$

Let us return in this equation from conformal variables to the physical ones. As a result in the leading order one can get

$$\Psi_t = \frac{1}{2}(\hat{R}\Psi_x)^2 - (\Psi_x)^2). \tag{5.2}$$

The equation for η taking account quadratic nonlinearity transforms as follows

$$\frac{\partial \eta}{\partial t} = \hat{k}\Psi - \left[\hat{k}(\eta\hat{k}\Psi) + \nabla(\eta\nabla\Psi)\right], \tag{5.3}$$

$$\frac{\partial \Psi}{\partial t} = \frac{1}{2}\left[(\hat{k}\Psi)^2 - (\nabla\Psi)^2\right] \tag{5.4}$$

The remarkable property of these equations is the splitting off equation (5.4), which involves only variable Ψ, from that of (5.3), which governs the behavior of elevation η. Such a separation is a peculiarity of the used perturbation order and is being lost in next orders, when η appears in equation (5.4) as well. Since we assume $|\nabla\eta| \ll 1$, it is possible to omit the second term in the r.h.s. of equation (5.3):

$$\frac{\partial\eta}{\partial t} = \hat{k}\Psi. \tag{5.5}$$

$$H = \frac{1}{2}\int \Psi\hat{k}\Psi d\vec{r}_\perp + \frac{1}{2}\int \left[(\nabla\Psi)^2 - (\hat{k}\Psi)^2\right]\eta d\vec{r}_\perp, \tag{5.6}$$

To study the dynamics of this system and for the sake of simplicity we will consider the one-dimensional case when functions Ψ and η depend only on x (and t) and the operator $\hat{k}$ may be presented in the form

$$\hat{k} = -\frac{\partial}{\partial x}\hat{H},$$

where

$$(\hat{H}f)(x) = \frac{1}{\pi}V.P.\int_{-\infty}^{+\infty}\frac{f(x')}{x'-x}\,dx'.$$

is the Hilbert transform. By introducing a new function $v = \frac{\partial\Psi}{\partial x}$, which has a meaning of the tangent velocity on the interface, equations (5.4), (5.5) can be rewritten as

$$\frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x}\left[(\hat{H}v)^2 - v^2\right], \tag{5.7}$$

$$\frac{\partial\eta}{\partial t} = -\,\hat{H}v. \tag{5.8}$$

We exploit further that property of the Hilbert transform $\hat{H}$, that two operators $\hat{P}^{\pm} = \frac{1}{2}(1 \mp i\hat{H})$, are the projection operators. Namely, they decompose a function into the sum of two ones, $v = v^{(+)} + v^{(-)}$, with $v^{(\pm)} = \hat{P}^{\pm}v$ being a function, analytically continued into the upper (lower) complex half-plane. Then, the Hilbert transform acts as follows,

$$\hat{H}v = i(v^{(+)} - v^{(-)}). \tag{5.9}$$

Relation (5.9) should be substituted into both equations (5.8) for η and (5.7) for v. As a result, the latter decomposes into separate equations for the upper ($v^{(+)}$) and lower ($v^{(-)}$) analytical parts of v:

$$\frac{\partial v^{(\pm)}}{\partial t} + 2v^{(\pm)}\frac{\partial v^{(\pm)}}{\partial x} = 0. \tag{5.10}$$

Equations (5.10) look like those for motion of a free particle and can be solved by the standard method of characteristics :

$$v^{(\pm)} = F^{(\pm)}(x_0), \tag{5.11}$$

$$x = x_0 + 2F^{(\pm)}(x_0)t, \tag{5.12}$$

where functions $F^{(\pm)}$ are defined from initial conditions. On the real axis functions $v^{(\pm)}$ are complex conjugate, so it is enough to find a solution only for $v^{(+)}$, for example.

2. General Solution. Let in (5.10) $F^{(+)}(x_0)$ be some analytical function in the upper half-plane of complex x_0 with its singularities in lower half-plane. To find the solution of equations (5.10) one needs to resolve at first equation (5.12) with respect to x_0 . The mapping: $x \to x_0$ becomes ambiguous in the points, where

$$\frac{\partial x}{\partial x_0} = 1 + 2F^{(+)\prime}(x_0)t = 0. \tag{5.13}$$

Solution of (5.13) gives some trajectory on the complex plane x_0: $x_0 = x_0(t)$. The roots of (5.13) together with (5.11) define the corresponding movable branch points of the function $v^{(+)}(x,t)$

$$z_{br}(t) = x_0(t) + 2F^{(+)}(x_0(t))t. \tag{5.14}$$

These points should be connected with a set of cuts, providing for the uniqueness of the function $v^{(+)}(x,t)$. The choice of these cuts has to be made in such a way that at the moment $t = 0$ $v^{(+)}(x,t)$ would have the initial singularities. These movable branch points originate from the singularities of the function $F^{(+)}(z_0)$. At the moment when the most 'rapid' branch touches the real axis, the analyticity of $V^{(+)}(x,t)$ breaks, and, respectively, a singularity appears in the solution of system (5.10).

At first define the touching time t_0 from the requirement z_{br} to be real, $z_{br} = x_{br}$. Assuming $\tau = t_0 - t \ll t_0$, and considering a small vicinity of $z = x_{br}$, expansion of (5.12) up to the leading orders gives

$$F''t_0(\delta x_0)^2 - 2F'\tau\delta x_0 - 2F_0\tau - x' = 0. \tag{5.15}$$

where $F'' = F''(x_0(t_0))$, $\delta x_0 = x_0 - x_0(t_0)$, $x' = x - x_{br}$, $F_0 = F^{(+)}(z_0(t_0))$.

From this equation we find

$$x_0 = x_0(t_0) + \frac{F'\tau}{F''t_0} + \sqrt{(\frac{F'\tau}{F''t_0})^2 + \frac{2F_0\tau + x'}{F''t_0}}. \tag{5.16}$$

If $F_0 \neq 0$ the leading term under square root is the linear one with respect to τ. Therefore with the needed accuracy

$$x_0 = x_0(t_0) + C(x' + 2F_0\tau)^{1/2}. \tag{5.17}$$

where $C = [F^{(+)\prime\prime}(x_0(t_0))]^{-1/2}$.

At the vicinity of $x = x_{br}$ and $t = t_0$, such a general form of x_0 provides the self-similar singular dependences for $\partial v/\partial x$ and η_{xx} which follow after substitution

(5.17) into (5.11) and forthcoming integration of equation (5.8). The first step gives for the tangent velocity with the same accuracy as for (5.17)

$$v = 2\mathrm{Re}[F_0 - \frac{1}{t_0}C(x' + 2F_0\tau)^{1/2}]. \tag{5.18}$$

Hence we get for the first derivative of v

$$\frac{\partial v}{\partial x} = -\frac{1}{t_0}\mathrm{Re}\left[\frac{C}{\sqrt{x' + 2F_0\tau}}\right]. \tag{5.19}$$

So, close to the touching time t_0 v_x behaves in a self-similar way, $x' \sim \tau$, increasing as $\tau^{-1/2}$. In the limit $\xi = x'/\tau \to \infty$ this function does not depend on τ,

$$\frac{\partial v}{\partial x} \sim |x'|^{-1/2}. \tag{5.20}$$

It means this profile is formed at first at the periphery and then propagates to the center ($x' = 0$, resulting in a singularity at $\tau = 0$.

The curvature η_{xx} demonstrates the same self-similar behavior. In fact, the elevation η^+, governed by equation (5.8), can be presented in the following form:

$$\eta^{(+)} = -i[tF(x_0) - \int_x^{x_0(x,t)} \frac{(x - x_0)}{2F^{(+)}(x_0)} F'^{(+)}(x_0)dx_0],$$

where the dependence $x_0(x,t)$ is defined by means of (5.12). Thereafter, differentiation $\eta^{(+)}$ with respect to x yields the explicit expression for η_x:

$$\eta_x = \mathrm{Im}\log\frac{F^{(+)}(x)}{F^{(+)}(x_0)}. \tag{5.21}$$

This formula together with (5.17) leads to the same answer which we have got for v_x: η_{xx} becomes infinite while approaching the singularity,

$$\eta_{xx} = \frac{1}{\tau^{1/2}}h(\frac{x'}{\tau}), \tag{5.22}$$

where

$$h(\xi) = -\sqrt{2}\left[\frac{1 + \sqrt{1 + 2\xi^2}}{1 + 2\xi^2}\right]^{1/2}$$

At the critical moment of $\tau = 0$, η_{xx} looks like

$$\eta_{xx} \sim -|x|^{-1/2} \tag{5.23}$$

that gives after integration the following behavior of $\eta \sim \frac{4}{3}|x|^{3/2}$+regular terms. In so doing both functions η and η_x remain finite at the singular point. The singularities, thus obtained, are the general ones for system (5.7) and (5.8).

Now, let us show how the general formulas work for a simple example when $F^{(+)}(x_0)$ is a rational function with one simple pole in the lower half-plane,

$$F^{(+)}(x_0) = \frac{A}{x_0 + ia},$$

where Re$a > 0$. Then the dependence $x_0 = x_0(x,t)$ can be readily found by means of (5.12),

$$x_0 + ia = \frac{1}{2}(x + ia) + \sqrt{\frac{1}{4}(x + ia)^2 - 2At}. \tag{5.24}$$

Thus, instead of the initial pole at the point $x = -ia$ there appears a cut, connecting two moving branch points $x_{1,2} = -ia \pm 2\sqrt{2At}$.

The points $x_{1,2}(t)$ move (except of positive A) under some angle to the real axis. If, for instance, $A = -1/8$ and $a = 1$, the cut spreads in the vertical direction axis and reaches the real axis at the break moment of time $t = t_0 = 1$ at the point $x_{br} = 0$. At the vicinity of $\tau = 0$ and $x = 0$ expressions for $\frac{\partial v}{\partial x}$ and η_{xx} can be represented in the form:

$$2\eta_{xx} = \frac{\partial v}{\partial x} \approx -\frac{1}{\sqrt{4x^2 + \tau^2}} \left[\frac{1}{2}\left(\tau + \sqrt{4x^2 + \tau^2}\right)\right]^{1/2}. \tag{5.25}$$

Thus, at the critical moment of $\tau = 0$ velocity derivative looks like

$$\frac{\partial v}{\partial x} \approx -\frac{1}{2}|x|^{-1/2}. \tag{5.26}$$

Evidently, formulas (5.25), (5.26) are in a full correspondence with general ones, (5.19), (5.22).

3. Wedges. Let us show that the system (5.7), (5.8) has a special solution which describes an another type of singularity. This solution arises if $F_0 = 0$. For this particular case formulae (5.16) transforms into

$$x_0 = z_0(t_0) + \frac{F'\tau}{F''t_0} + \sqrt{\left(\frac{F'\tau}{F''t_0}\right)^2 + \frac{x'}{F''t_0}}$$

and, as a sequence, v can be approximately written in the form

$$v \approx (x_0 - z_0(t_0))F'. \tag{5.27}$$

Such dependence gives a new kind of self-similar behavior, $x \sim \tau^2$, that provides the surface singularity of the wedge type. Indeed, when substituting (5.21) into (5.27) and considering the asymptotics of η_x for $x'/\tau^2 \to \infty$, one gets

$$\eta_x \to -\frac{\pi}{4}\mathrm{sign}(x'),$$

that corresponds to the wedge surface profile with the angle $\alpha = 2\arctan\frac{4}{\pi} \approx 103,7^\circ$. This angle is far from π and our assumption about small surface angles breaks close to the singularity. However, the solution obtained above appears to be meaningful, because, first, the angle α is close to that calculated by Stokes for the critical stationary gravity surface wave on a deep water and, second, the self-similarity of the type $x \sim \tau^2$ is retained even by the complete system of equations (2.7). It is worth noting that $F_0 = 0$ can be got from the initial conditions with two poles:

$$F^{(+)}(z) = i\mu\left[\frac{a}{z+ia} - \frac{a^*}{z+ia^*}\right]$$

where Re$a < 0$, Im$\mu = 0$.

The dynamics of the branch points generated by these two poles is also interesting: at the initial moment of time the poles produce two pairs of branch points, two of which move towards imaginary axis and collide; after collision points move along imaginary axis in opposite directions; the touching of the real axis by one of them produces the singularity appearance.

4.Floating Singularities The new type of singularities is associated with a possibility of exact integration of equation (5.3) taking into account the second term in its r.h.s.. For this aim let us separate from (5.3) the equation for $\eta^{(+)}(x,t)$:

$$\frac{\partial \eta^{(+)}}{\partial t} + 2\hat{P}^{(+)}(v^{(-)}\eta^{(+)})_x = -i\Psi_x^{(+)}, \tag{5.28}$$

Introducing instead of $\eta^{(+)}$ a new function $\xi^{(+)}$ by means of $\eta^{(+)} = \frac{\partial \xi^{(+)}}{\partial x}$ and integrating (5.28) once one can get

$$\hat{P}^{(+)}[\frac{\partial \xi}{\partial t} + 2v^{(-)}\frac{\partial \xi}{\partial x}] = -i\Psi^{(+)}. \tag{5.29}$$

Here ξ is a function, for which $\hat{P}^{(+)}\xi = \xi^{(+)}$. Omitting then in both sides of (5.29) the operator $\hat{P}^{(+)}$, we arrive to the equation for ξ

$$\frac{\partial \xi}{\partial t} + 2v^{(-)}\frac{\partial \xi}{\partial x} = -i\Psi^{(+)} + \Phi^{(-)}, \tag{5.30}$$

where $\Phi^{(-)}$ is some lower analytical function (for which $\hat{P}^{(+)}\Phi^{(-)} = 0$). This equation can be integrated along the characteristics, defined by (5.12). General solution to (5.30) consists of two parts, $\xi = \xi_1 + \xi_2$, where

$$\xi_1 = -i\int_0^t \psi^{(+)}(x(x_0,t'),t')dt' + \int_0^t \Phi^{(-)}(x(x_0,t'),t')dt' \tag{5.31}$$

is the solution of the inhomogeneous equation with zero initial condition, and $\xi_2 = f(x_0)$ is that of homogeneous one, presenting simply the initial shape of ξ. The effect of ξ_1 is defined by the analytical properties of the tangent velocity only, while that of ξ_2 results from the interference of the the tangent velocity effect and intrinsic peculiarities of the initial elevation $\eta_0(x)$.

Analyzing the first term, ξ_1, we first underline that the integration of the function $\Phi^{(-)}$ along characteristics (5.12) with forthcoming application of the operator $\hat{P}^{(+)}$ gives zero. It is enough, therefore, to integrate only $\Psi^{(+)}$ in (5.30). In fact, the situation is even simpler, because we are interested in the solution behavior only close to the moment of t_0. Omitting details, we note only that taking into account the convective term in (5.28), as compared with simplified equation (5.5), though giving rise to some additional motion, does not change, in fact, the character of singularity in elevation ($\eta_{xx} \sim |x'|^{-1/2}$).

It is very important that the singularities obtained belong to the weak ones (see (5.23)), which do not destroy our basic assumption about small values of angles, $|\nabla\eta| \ll 1$. Note also, that the self-similar asymptotics of the form (5.22) is admitted by the complete set of equations (2.7).

Of greater interest now is the homogeneous part of the solution $\xi_2 = f(x_0)$ (not considered in the previous sections at all). The corresponding upper analytical part of elevation $\eta^{(+)}$ is defined as,

$$\eta_2^{(+)}(x,t) = (\frac{\partial \xi_2}{\partial x})^{(+)} = \hat{P}^{+}(\frac{\partial x_0}{\partial x}\frac{df}{dx_0})$$

Since at the initial moment of $t = 0$, $x = x_0$, $\frac{\partial x_0}{\partial x} = 1$, the function $\frac{df}{dx_0}$) coincides with $\eta_0^{(+)}(x_0)$, where $\eta_0(x)$ is the initial form of the interface. The exact form of $\eta_2^{(+)}$ may be written down as follows

$$\eta_2^{(+)}(x,t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{dx'}{x' - x - i0} \quad \frac{\partial x_0(x',t)}{\partial x'} \quad \eta_0^{(+)}(x_0).$$

Passing to x_0 as a new variable of integration, this integral reduces to the form

$$\eta_2^{(+)}(x,t) = \frac{1}{2\pi i}\int_C \frac{dx_0}{x'(x_0,t) - x - i0}\eta_0^{(+)}(x_0) \tag{5.32}$$

with x' and contour C, both defined from (5.12). The contour C initially coincides with the real axis, then, during the time, it is being deformed so that it is going partially in the lower half-plane. The motion of contour C towards singular points of $\eta^{(+)}(x_0)$ will define obviously the behavior and the singularity formation of function $\eta(x,t)$ for real x. To clarify this situation let us assume that $\eta^{(+)}(x_0)$ has one pole in the lower half-plane,

$$\eta^{(+)}(x_0) = \frac{iB}{x_0 - b}$$

where B is real, and $\mathrm{Im} b < 0$. Then integral (5.32) is found explicitly,

$$\eta_2^{(+)}(x,t) = \frac{iB}{x - x'(b,t)} = \frac{iB}{x - b - 2F^{(-)}(b)t}.$$

It is clear from this expression that the pole of $\eta^{(+)}$ is movable with the "velocity" $2F^{(-)}(b)$, being some regular function. Therefore if $F_2 = \mathrm{Im}F^{(-)}(b) > 0$ then there exists such a moment of time t_c when $\eta_2(x,t)$ becomes infinite. Evidently that $t_c = -\frac{b_2}{2F_2(b)}$, where $b_2 = \mathrm{Im}b$. Close to this time $\eta(x,t)$ has the Lorenz form,

$$\eta(x,t) = -\frac{B(b_2 + 2F_2 t)}{(x - b_1 - 2F_1(b)t)^2 + (b_2 + 2F_2 t)^2},$$

which transforms at $t = t_c$ into the $\delta-$function:

$$\eta(x,t) = B\pi\delta(x - b_1 + b_2\frac{F_1}{F_2}).$$

Thus, the proper singularities of the analytical function $\eta^{(+)}$, not generated by the velocity field and existed initially, remain during the time and occur to be movable. This statement can be readily checked for an arbitrary case, not only for poles.) It gives a new type of singularities of the free surface, generally speaking, of arbitrary kind appearing due to the proper analytical properties of the initial profile of the elevation. What kinds of the singularities will appear first depends on the initial conditions. If, for instance, the initial elevation is equal to zero then we get the first kind of singularities of the root character. One should pay attention to the fact that for the second kind of singularities our assumption about small surface angles breaks. Close to the time $t = t_c$ one should use the complete system (2.7) rather than reduced equations (5.3), (5.4).

5.Conclusion. In this paper we did not touch such a question as the stability problem of the collapsing regimes. According to the analysis performed in Section 3, the first regime of the root character is obviously stable in the framework of truncated system (5.7), (5.8). For the complete system, however, this is an open question as well as for two other regimes. It should be emphasized again, that from the very beginning we assumed the angle of the surface ($|\nabla\eta|$) to be small, and therefore, we can not pretend to the full description of all types of possible singularities, as described by the complete system of equations (2.7). However, the solutions corresponding to the weak singularity regime turn out to be consistent with the applicability condition of the truncated equations (5.4), (5.5).

In our opinion, there exist two possibilities of what role may play the root singularities in the general dynamics; either the singularities serve as an origin of more powerful ones observed in numerical experiments or represent new type of singularities. One should note also that the self-similar asymptotics for the wedge type of singularities are allowed by the exact system of equations. We believe therefore that just this type of singularity was observed in numerical experiments [29] (see also [28]).

6 High-Jacobian approximations

Let us suppose that the function $z = z(\omega)$ has a singularity in the upper half-plane on the imaginary axis close to zero. In the vicinity of zero the quantity $J = J(u)$ is a very singular function. But it might happen that $1/J$ is in this region a smooth function close locally to zero. In this case one can develop a new type of an approximate theory. We will discuss only the case of infinite depth. We will seek a solution of equation (3.10) in the form

$$\Psi = -\frac{1}{2}\int \lambda^2 dt + \lambda(t)y + \tilde{\Psi}\,. \tag{6.1}$$

Here

$$\dot{\lambda} = -g \tag{6.2}$$

and $\lambda = \lambda(t) = \lambda_0 - gt$ is a linear function of time.

The idea of separating $\tilde{\Psi}$ from Ψ is the following. Let the singularity in Ψ be posed at a distance $\delta \ll 1$ from the real axis. Then the functions $\Psi(u)$ and $y(u)$ are sharp, they change their value essentially in a region $u \simeq \delta$. Our central assumption is that in (6.1) $\tilde{\Psi}$ is a "smooth" function. It varies on a scale of order of unity.

To justify this assumption we must write the equation for $\tilde{\Psi}$. To obtain this equation we exploit the identities,

$$\begin{aligned} (1 - i\hat{H})(\tilde{x}_u + iy_u)^2 &= 0\,, & (6.3)\\ (1 + i\hat{H})(\frac{1}{z_u^*} - 1) &= 0\,. & (6.4)\end{aligned}$$

These imply

$$\hat{H}(y_u \hat{H} y_u) = \frac{1}{2}\left[(\tilde{x}_u)^2 - y_u^2\right] \tag{6.5}$$

and

$$\frac{y_u}{J} + \hat{H}(\frac{x_u}{J} - 1) = 0\,. \tag{6.6}$$

After a simple transformation one finds

$$\begin{aligned}\frac{\partial \tilde{\Psi}}{\partial t} &= \frac{\lambda^2}{2J} + \lambda\left(\frac{1}{J}\hat{H}\tilde{\Psi}_u + \tilde{\Psi}_u \hat{H}(\frac{1}{J} - 1)\right)\\ &+ \frac{1}{2J}\left((\hat{H}\tilde{\Psi}_u)^2 - \tilde{\Psi}_u^2\right) + \tilde{\Psi}_u \hat{H}\frac{\hat{H}\tilde{\Psi}_u}{J}\,. \end{aligned} \tag{6.7}$$

Equation (6.7) is an *exact* relation. All information about the shape of surface is hidden in the function $1/J$, which according to our assumption is smooth and "large scale". Therefore, we may consider $\tilde{\Psi}$ smooth and large scale as well.

In the first approximation, we put

$$\tilde{\Psi} = 0\,. \tag{6.8}$$

The corresponding equation for the surface shape looks very simple in its implicit form,

$$y_t(1+\tilde{x}_u) - \tilde{x}_t y_u = \lambda \tilde{x}_u \,. \tag{6.9}$$

Let us denote

$$\tilde{z} = \tilde{x} + iy \,.$$

In terms of $\tilde{z}$ equation (6.9) can be rewritten as follows,

$$\tilde{z}_t - \tilde{z}_t^* + \tilde{z}_t \tilde{z}_u^* - \tilde{z}_t^* \tilde{z}_u = i\lambda(\tilde{z}_u + \tilde{z}_u^*) \,. \tag{6.10}$$

By introducing a new function

$$z = \tilde{z} + u - i \int \lambda dt \,,$$

one transforms (6.10) to the form

$$\Im(z_t z_u^*) = -\lambda \,. \tag{6.11}$$

In the simplest case $\lambda = const$, this equation has been known in the literature since 1945 [16, 17]. It is usually called the Laplacian Growth Equation (LGE) and is used widely in the theory of interfaces and dendrite growth.

7 Finger-type solutions

Equation (6.10) is an integrable system. One can find a general solution of this equation starting from a very wide class of special solutions (N-finger solutions),

$$\tilde{z} = \sum_{n=1}^{N} q_n \log(u - a_n(t)) \,. \tag{7.1}$$

Here N is any positive integer (including $N = \infty$), q_n are complex constants and $\Im a_n > 0$. Strictly speaking, to satisfy the condition $\tilde{z} \to 0$ at $|u| \to \infty$, one must demand

$$\sum_{n=1}^{N} q_n = 0 \,. \tag{7.2}$$

However, the constraint (7.2) is not significant from the physical view-point. For arbitrary choice of q_n it can be satisfied by adding to (7.1) one more term,

$$-\left(\sum_{n=1}^{N} q_n\right) \log(u - a_{N+1}(t)), \qquad \Im a_{N+1} \to +\infty.$$

Substituting (7.1) to (6.10) and using an expansion in the sum of elementary fractions, one obtains a system of ODE for a_n,

$$\dot{a}_n + \sum_m q_m^* \frac{\dot{a}_n - \dot{a}_m^*}{a_n - a_m^*} = -i\lambda(t)\,. \tag{7.3}$$

Integration by t gives the following system of transcendental equations,

$$a_n + \sum_m q_m^* \log(a_n - a_m^*) = -\int \lambda(t)dt + C_n\,, \tag{7.4}$$

where C_n are arbitrary complex constants. The simplest possible solution (one-finger solution) of this type is

$$\tilde{z} = -i\log(u - ib(t)),\ \text{where } b \text{ is real}\,. \tag{7.5}$$

Now

$$b + \log b = -\int \lambda dt + \log r,\ \text{where } r \text{ is a real constant}\,. \tag{7.6}$$

If $\int \lambda dt \to +\infty$, one finds the asymptotic behavior of b,

$$b \simeq r\exp\,(-\int \lambda dt) \to 0, \qquad t \to \infty\,. \tag{7.7}$$

In our case, we find

$$\begin{aligned} y &= -\log\sqrt{u^2 + b^2(t)}\,, \\ \tilde{x} &= \arctan\frac{u}{b(t)}\,, \\ \frac{1}{J} &= \frac{u^2 + b^2(t)}{(1 + b(t))^2 + u^2}\,. \end{aligned} \tag{7.8}$$

Let $\int \lambda dt$ be positive and large. Then

$$\frac{1}{J} \to \frac{u^2}{1 + u^2}\,. \tag{7.9}$$

This expression is small for $u \simeq b$, and is indeed smooth and large scale. From (6.2) one gets

$$\int \lambda dt = -\frac{1}{2}gt^2 + Ct\,. \tag{7.10}$$

For positive g (stable case), $\int \lambda dt$ can be large only during a finite time (if C is positive and large). For $g \leq 0$ (neutrally stable or unstable cases) the approximation improves as $t \to \infty$.

In the one-finger solution,

$$y(0, t) = -2\log b(t) \simeq \int \lambda dt,\ \text{as } t \to \infty. \tag{7.11}$$

At the same time, the curvature tends to a constant,

$$1/R = \eta_{xx} \to -\text{const}, \text{ as } t \to \infty\,. \tag{7.12}$$

The one-finger solution was found by Saffman and Taylor [14]. N-finger solution were studied in the articles [18, 19]. A general solution of the equation (6.10) can be obtained from (7.1) by a kind of transition to the limit $N \to \infty$. We will discuss this procedure elsewhere.

8 Droplet-type solutions

Let us consider the simplest case of rational solution for LGE, namely

$$\tilde{z} = \frac{\alpha}{u - ib}\,. \tag{8.1}$$

Here α and b both are positive. Substituting (8.1) into (6.11) and using an expansion in the sum of elementary fractions leads to a system of ODE for α and b,

$$\begin{aligned} \dot{\alpha} b + \alpha \dot{b} + \lambda \alpha &= 0\,, \\ \dot{\alpha} b^3 + \alpha \dot{\alpha} b - \alpha b^2 \dot{b} - \alpha^2 \dot{b} - \lambda \alpha b^2 &= 0\,. \end{aligned} \tag{8.2}$$

This set of equations has a solution which is valid until some final time t_0. Asymptotically, when $t \to t_0$ the solution approaches

$$\begin{aligned} \alpha &\Rightarrow \delta b\,, \\ b &\Rightarrow \frac{\lambda}{2}(t_0 - t)\,, \end{aligned} \tag{8.3}$$

where δ is a constant. This solution describes formation of a droplet which separates from the fluid at $t = t_0$. Again, this solution is valid provided $\hat{H}\Psi_u$ is small compared to $\tilde{x}_u$. To study the validity of this assumption one must estimate the solution of (6.7) where J is calculated from (8.1) and (8.3). Here we are able to solve only the linear (with respect to $\tilde{\Psi}$) part of (6.7),

$$\tilde{\Phi}_t + iU\tilde{\Phi}_u = \frac{\lambda^2}{2}U\,. \tag{8.4}$$

This is a linear complex transport equation which can be solved by the method of characteristics. Here,

$$\begin{aligned} U(u,t) &= \frac{1}{J} + i\hat{H}\frac{1}{J}\,, \\ \tilde{\Phi}(u,t) &= \tilde{\Psi} + i\hat{H}\tilde{\Psi}\,. \end{aligned} \tag{8.5}$$

All functions in the equation (8.4) are analytic in the lower half-plane. The equation for the characteristic is

$$\dot{u} = iU(u,t)\,. \tag{8.6}$$

In the vicinity of t_0 the complex velocity U is given by

$$U \Rightarrow -\frac{i}{2}\frac{\delta u}{u^2 - \delta b}\,. \tag{8.7}$$

In this case the characteristic equation can be solved exactly, namely

$$\frac{u}{u^2 + \delta b} = C\,, \tag{8.8}$$

and the general solution of equation (8.4) is

$$\tilde{\Phi} = \tilde{\Phi}(\frac{u}{u^2 + \delta b}) - i\frac{u}{2}\,, \tag{8.9}$$

where $\tilde{\Phi}$ is arbitrary function.

Let us impose zero initial condition for $\tilde{\Phi}$ at $\delta b = 1$. Then the solution acquires the form,

$$\tilde{\Phi} = \frac{1}{4}\left[i\frac{u^2 + \delta b}{u} + \sqrt{4 - \left(\frac{u^2 + \delta b}{u}\right)^2} - i\frac{u}{2}\right]\,. \tag{8.10}$$

The behavior of $\hat{H}\tilde{\Psi}_u = \Im\tilde{\Phi}_u$ at $u = 0$ is proportional to C/b and is similar to that for $\tilde{x}_u$.

9 Self-Similar Solutions for Deep Water

In this section we shall consider the case of deep water without gravity. The equations (2.31) and (2.38) read now:

$$\begin{aligned} y_t x_u - x_t y_u &= -\hat{H}\Psi_u\,, && (9.1)\\ \Psi_t x_u - \Psi_u x_t &= \hat{H}(\Psi_t y_u - \Psi_u y_t)\,. && (9.2) \end{aligned}$$

The system (9.1), (9.2) has a rich family of self-similar solutions. First we mention that it is invariant with respect to the transformation:

$$u \rightarrow \frac{u}{a(t)} \tag{9.3}$$

$a(t)$ - is an arbitrary function of time. This invariance has aa very simple origin - it steams from a possibility to rescale u in arbitrary way at any moment of time.

Then, we can seek for solutions of (9.1), (9.2) in a form:

$$\begin{aligned} y &= f(t)y_0(u)\,, \\ x &= f(t)x_0(u)\,, \\ \Psi &= g(t)\Psi_0(u)\,. \end{aligned} \tag{9.4}$$

From (9.1) one gets

$$g(t) = ff'\,, \tag{9.5}$$

and from (9.2) one can obtain

$$(ff')' = qf'^2 \qquad q \text{ - is a constant.} \tag{9.6}$$

The equation (9.6) has a powerlike solution

$$f(t) = c(t_0 - t)^s, \quad s = \frac{1}{(2-q)}, \quad \text{if } q \neq 2\,, \tag{9.7}$$

and an exponential one

$$f(t) = c\rceil^{\lambda t} \quad \text{if } q = 2\,. \tag{9.8}$$

y_0, x_0 and Ψ_0 satisfy the following equations:

$$\begin{aligned} y_0x_0' - x_0y_0' &= -\hat{H}\Psi_0'\,, \\ q\Psi_0x_0' - x_0\Psi_0' &= \hat{H}(q\Psi_0'y_0\prime - \Psi_0'y_0)\,, \end{aligned} \tag{9.9}$$

In natural variables (9.4) is a self-similar solution:

$$\begin{aligned} \eta &= f(t)\eta_0(\frac{x}{f(t)}\,, \\ \phi &= f(t)f'(t)\phi_0(\frac{x}{f(t)}, \frac{y}{f(t)})\,. \end{aligned} \tag{9.10}$$

If $s > 0$ ($q < 2$), this solution describes formation of singularity in a finite time. This is a wedge-type singularity, similar to that found in Section 5. One can find a limitations imposed on s by conservation of energy. From (9.10) one gets for kinetic energy

$$T = f^2f'^2\int(\nabla\phi_0(\vec{\xi}))^2 d\vec{\xi}, \quad \vec{\xi} = (\frac{x}{f(t)}, \frac{y}{f(t)})\,.$$

The self-similar solution can loose energy to a neighbour "background" or preserve it. But the energy can not grow. As far as

$$T \simeq (t_0 - t)^{4s-2},$$

and $\infty > s \geq \frac{1}{2}$, q must be in the limits $2 \leq q \leq 0$. For $q = 2$, $s = \infty$, collapse is reached in infinite time.

The case $q = s = 1$ is very special. Now one can find the exact solution of (9.1), (9.2) in a following form:

$$\begin{aligned} x &= x_0 t + x_1 \,, \\ y &= y_0 t + y_1 \,, \\ \Psi &= \Psi_0 t + \Psi_1 \end{aligned} \tag{9.11}$$

In (9.11) x_0, y_0 and Ψ_0 present the self-similar solution, satisfying the equations

$$\begin{aligned} y_0 x_0' - x_0 y_0' &= -\hat{H}\Psi_0' \,, \\ \Psi_0 x_0' - x_0 \Psi_0' &= \hat{H}(\Psi_0' y_0\prime - \Psi_0' y_0) \,, \end{aligned} \tag{9.12}$$

while x_1, y_1 and Ψ_1 obey the linear equations

$$\begin{aligned} y_0 x_1' - x_0 y_1' &= -\hat{H}\Psi_1' \,, \\ \Psi_0 x_1' - x_0 \Psi_1' &= \hat{H}(\Psi_0' y_1\prime - \Psi_1' y_0) \,. \end{aligned} \tag{9.13}$$

Starting from the self-similar solution (9.11), one can construct a general solution of the system (9.1), (9.2). It has to be found in the form of the infinite series:

$$\begin{aligned} x &= x_0 t + x_1 + \sum_{k=1}^{\infty} \frac{x_{-k}}{t^k} \,, \\ y &= y_0 t + y_1 + \sum_{k=1}^{\infty} \frac{y_{-k}}{t^k} \,, \\ \Psi &= \Psi_0 t + \Psi_1 + \sum_{k=1}^{\infty} \frac{\Psi_{-k}}{t^k} \,. \end{aligned} \tag{9.14}$$

First two terms of the expansion (9.14) satisfy the equations (9.12), (9.13) and further:

$$\begin{aligned} y_0 x_{-n}' - y_{-n}' x_0 + \hat{H}\Psi_{-n}' &= F_n \,, \\ \Psi_0 x_{-n}' - \Psi_{-n}\prime x_0 + \hat{H}(y_0 \Psi_{-n} - \Psi_0 y_{-n}') &= G_n \,, \end{aligned} \tag{9.15}$$

F_n, G_n are some expressions depending only on x_{-k}, y_{-k}, Ψ_{-k} $(k < n)$. In particulary

$$\begin{aligned} F_2 &= y_{-1} x_1' - x_{-1} y_1' \,, \\ G_2 &= \Psi_{-1} x_1' - x_{-1} \Psi_1' - \hat{H}(\Psi_{-1} y_1' - y_{-1} \Psi_1') \,. \end{aligned}$$

Suppose that the equations (9.12), (9.13) are solved and, moreover, one can solve explicitly linear, time-independent <u>n</u>onuniform equations (9.15) at any given F_n, G_n. Then one can find recurrently all terms in the seria (9.14). In this sense the system (9.1), (9.2) is exactly solvable.

So far, solutions of (9.12), (9.13) as well as an algorythm for solution of (9.15) are unknown. At the moment we don't know anything about existense of any solution of the equations (9.9) and can't offer any recipe for definition of s. We just remind, that in the small-angle approximation $s = 2$.

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