Алгебра и анализ
Том 6 (1994), вып. 3

Dedicated to
Ludwig Dmitrievich
Faddeev for the 60th birthday

ON SOME DEVELOPMENTS OF THE $\bar{\partial}$-DRESSING METHOD

L. V. Bogdanov, V. E. Zakharov

Abstract. Some developments of the $\bar{\partial}$-dressing method concerning an algebraic scheme of constructing integrable equations and construction of solutions with special properties are considered. It is demonstrated how the matrix KP equation appears from the scalar dressing and, more generally, how to construct the integrable system corresponding to an arbitrary triad of polynomials. Using the nonlocal $\bar{\partial}$-problem approach in $(2+1)$ dimensions, it is shown that the $\bar{\partial}$-problem with a shift and (for decreasing solutions) the Riemann problem with a shift naturally arise in $(1+1)$ dimensions. The Boussinesq equation and the first order relativistically-invariant systems are investigated. The developed approach allows one also to investigate the structure of the continuous spectrum and the inverse scattering problem for an arbitrary order ordinary differential operator on the infinite line.

§1. Introduction

This work may be considered as a development of paper [1]. Some approaches outlined in that paper are developed here in more detail. We also present new results concerning relativistically-invariant systems of equations, the inverse problem for an operator on the infinite line and a simple interesting result for an algebraic scheme of constructing equations integrable via $\bar{\partial}$-dressing.

The starting point of our work is the dressing method based on the nonlocal $\bar{\partial}$-problem [1–5]. This is a powerful method of constructing $(2+1)$-dimensional integrable equations together with a broad class of their solutions.

The dressing method establishes the correspondance between a triad of rational functions of one complex variable and an integrable system of equations. Using different normalizations of the nonlocal $\bar{\partial}$-problem [5], we show how to construct this system of equations effectively for an arbitrary triad of polynomials.

Deleting the dependence on one variable, we go to a one-dimensional case. In terms of the $\bar{\partial}$-problem this leads us to a problem with a special kind of nonlocality—the $\bar{\partial}$-problem with a shift and to the Riemann problem with a shift. It appears that these *scalar*

Key words and phrases. Inverse scattering problem method, the dressing method, $\bar{\partial}$-problem, the Riemann problem.

nonlocal problems are a general and natural technical tool in the $(1+1)$-dimensional case.

The main objects of our investigation are the KP equation and the system of equations of the N-wave type [5]. Slightly modifying the KP equation and descending to $(1+1)$ dimensions, one obtains the Boussinesq equation. The developed approach enables us to construct small nonsingular decreasing solutions of this equation and to investigate the continuous spectrum of its L-operator. The geometry of this spectrum is rather interesting, the spectral data are localized on a hyperbola in the complex plane and on a segment of the real axis and decreasing solutions are given by the Riemann problem with a shift on this curve (see another approach in [6]).

Using N-wave type equations in $(2+1)$ dimensions, we obtain relativistically-invariant systems in $(1+1)$ dimensions. These systems were integrated first by Zakharov and Mikhailov [7], using the *matrix local* Riemann problem. In our work we use the *scalar nonlocal* problem and obtain some new spectral information for these equations.

Finally we use the developed technique to study the inverse scattering problem for the differential operators of arbitrary order on a line. A special case of this problem was considered in detail in [8]. Our investigation is not so detailed. We use the ideology of the dressing method and do not treat the direct scattering problem at all. The dressing method yields the structure of the continuous spectrum, which is quite nontrivial. The spectral data are localized on certain algebraic curves in the complex plane, and the inverse problem is reduced to the Riemann problem with a shift on these curves.

We would like to emphasize that the calculations in this work are based on the $\bar{\partial}$-dressing technique and they are mainly formal. A rigorous investigation of the underlying analysis is beyond the scope of this paper. The integral operators are supposed to be "small" in some sense, so that the integral equations are uniquely solvable. But we hope that this work produces a relationship between the formal technique and spectral theory.

§2. The $\bar{\partial}$-dressing: the basic technique

We briefly outline the technique of the $\bar{\partial}$-dressing method in this paper, restricting ourselves to the scalar case as the simplest. We would like to emphasize that most of the content of this part is not original and is in the main described in the papers mentioned above (excluding the derivation of the matrix KP equation through the scalar dressing and construction of the U–V pair corresponding to an arbitrary triad of polynomials).

The scheme of the dressing method uses the nonlocal $\bar{\partial}$-problem with the special dependence of the kernel on additional (space and time) variables:

$$\bar{\partial}(\psi(\mathbf{x},\lambda)-\eta(\mathbf{x},\lambda)) = \hat{R}\psi(\mathbf{x},\lambda), \qquad (\psi(\mathbf{x},\lambda)-\eta(\mathbf{x},\lambda))_{|\lambda|\to\infty} \to 0, \tag{1}$$

$$\hat{R}\psi = \iint \psi(\lambda)R(\lambda,\mu)\exp(\varphi_i x_i)\,d\mu\wedge d\bar{\mu}, \quad \varphi_i = K_i(\mu)-K_i(\lambda), \ 1\leqslant i\leqslant 3, \tag{2}$$

where $\lambda\in\mathbb{C}$, $\bar{\partial}=\partial/\partial\bar{\lambda}$, $\eta(\mathbf{x},\lambda)$ is a rational function of λ (normalization), $K_i(\lambda)$ are rational functions, the choice of which determines the equation that can be solved using the problem (1). We suppose that the kernel $R(\lambda,\mu)$ equals zero in a neighborhood with respect to λ and to μ of the divisor of poles of the functions $K_i(\lambda)$, tends to zero as $\lambda,\mu\to\infty$, and that for the chosen kernel $R(\lambda,\mu)$ the problem (1) is uniquely solvable (at least for sufficiently small $\mathbf{x}$). The solution of the problem (1) normalized by η is the function

$$\psi(\mathbf{x},\lambda) = \eta(\mathbf{x},\lambda)+\varphi(\mathbf{x},\lambda),$$

where $\eta(\mathbf{x},\lambda)$ is a rational function of λ (normalization), $\varphi(\mathbf{x},\lambda)$ decreases as $\lambda\to\infty$ and is *analytic* in a neighborhood of the poles of $K_i(\lambda)$.

The problem (1) reduces to the integral equation for the function φ:

$$\varphi(\mathbf{x},\lambda)=\bar\partial^{-1}\hat R(\varphi(\mathbf{x},\lambda)+\eta(\mathbf{x},\lambda)), \tag{3}$$

here

$$(\bar\partial^{-1}\varphi)(\lambda)=(2\pi i)^{-1}\iint\frac{\varphi(\lambda')}{(\lambda'-\lambda)}d\lambda'\wedge d\bar\lambda'$$
$$=(2\pi i)^{-1}\lim_{\varepsilon\to 0}\iint\frac{\varphi(\lambda')(\bar\lambda'-\bar\lambda)}{(|\lambda'-\lambda|^2+\varepsilon)}d\lambda'\wedge d\bar\lambda',$$

which is assumed to be uniquely solvable for a given R. Solvability is guaranteed if the operator $\bar\partial^{-1}R$ is "small enough" (i.e., the norm of this operator is less then 1 for some properly chosen space of functions).

Let us introduce $\rho(\lambda,\bar\lambda)=\bar\partial\varphi$. Now

$$\psi(\mathbf{x},\lambda)=\eta+(2\pi i)^{-1}\iint\frac{\rho(\lambda')}{(\lambda'-\lambda)}d\lambda'\wedge d\bar\lambda'. \tag{4}$$

Substituting (4) in (3), we can get another form of the basic integral equation resolving the nonlocal $\bar\partial$-problem

$$\rho(\mathbf{x},\lambda)=\hat R\left(\eta+\bar\partial^{-1}\rho\right). \tag{5}$$

2.1. Special cases of the nonlocal $\bar\partial$-problem. In the most important cases the kernel $R(\lambda,\mu)$ is a singular function localized on some manifold in $\mathbb{C}^2$. This means that the kernel contains the δ-function localized on the corresponding manifold, or in other words that the measure of integration in the operator $\bar\partial^{-1}\hat R$ is localized on this manifold. The operator $\bar\partial^{-1}\hat R$ in this case is still well defined.

The $\bar\partial$-problem with a shift. In a typical situation this manifold is a covering of the complex λ-plane defined by the equation

$$f(\lambda,\bar\lambda,\mu,\bar\mu)=0, \tag{6}$$

where f is a function in $\mathbb{C}^2$. Equation (6) defines a multi-valued shift function $\mu=\mu_i(\lambda,\bar\lambda)$. The kernel of the problem (1) in this case reads

$$R=\sum_i R_i(\lambda,\bar\lambda)\delta(\mu-\mu_i(\lambda,\bar\lambda)).$$

We shall call this case the *$\bar\partial$-problem with a shift.*

The nonlocal Riemann problem. Another special case of the problem (1) is a nonlocal Riemann problem. Let $\gamma=\lambda(\xi)$, $\xi\in\mathbb{R}$ be an oriented curve in a complex plane (may be not connected) and let the kernel of the problem (1) be concentrated on the product of couple of these curves in λ and in μ planes. In other words,

$$R(\lambda,\mu)=\delta_\gamma(\lambda)R_\gamma(\lambda,\mu)\delta_\gamma(\mu), \tag{7}$$

where $\delta_\gamma(\lambda)$ is a δ-function picking out points on γ. The solution ψ of the problem (1) with kernel (7) is rational outside γ and has boundary values ψ^+, ψ^- on γ. After regularizing δ_γ we obtain from the problem (1) with kernel (7) the nonlocal Riemann problem

$$\psi^+-\psi^-=\frac{1}{2}\int_\gamma(\psi^++\psi^-)R_\gamma(\lambda,\mu)d\mu, \tag{8}$$

the integration in (8) is performed along the curve γ.

The Riemann problem with a shift. A combination of these two special cases leads to the Riemann problem with a shift (or Carleman's problem). The shift function $\mu = \mu_i(\lambda)$ is defined now on the curve γ ($\lambda, \mu \in \gamma$). In this case

$$R_\gamma(\lambda,\mu) = \sum_i R^i_\gamma(\lambda)\delta_\gamma(\mu - \mu_i(\lambda))$$

and

$$\psi^+ - \psi^- = \frac{1}{2}\sum_i(\psi^+(\mu_i(\lambda) + \psi^-(\mu_i(\lambda))R^i_\gamma(\lambda), \tag{9}$$

where $\mu_i(\lambda)$ is a multi-valued shift function on the curve $\lambda(\xi)$. We shall write the problem (9) symbolically in the form

$$\Delta(\psi(\lambda(\xi))) = R_\gamma(\lambda, \mu(\lambda))\psi(\mu(\lambda(\xi))), \tag{10}$$

where $\gamma = \lambda(\xi)$ ($\xi \in \mathbb{R}$) is a curve in the complex plane, Δ is a jump of the function across the curve, the value of the function on the curve is the half-sum of the boundary values, $\mu(\lambda)$ is the shift function (may be multi-valued).

Corresponding integral equations. In all these three cases the problem is equivalent to a certain integral equation which can be obtained by a proper reduction of equations (3), (5). Let us do that for a Riemann problem with a shift. Introducing

$$\rho_\gamma(\lambda) = \psi^+ - \psi^-|_{\lambda\in\gamma},$$

we can restore the function ψ in the form

$$\psi = \eta + \frac{1}{2\pi i}\int_\gamma \frac{\rho_\gamma(\lambda')}{(\lambda - \lambda')}d\lambda'.$$

Hence

$$\frac{1}{2}(\psi^+ + \psi^-)|_{\lambda\in\gamma} = \eta(\lambda) + \frac{1}{2\pi i}v.p.\int \frac{\rho_\gamma(\lambda')}{(\lambda - \lambda')}d\lambda',$$

and from equation (9) we get

$$\rho_\gamma(\lambda) = \sum_i \left(\eta(\mu_i(\lambda)) + \frac{1}{2\pi i}v.p.\int \frac{\rho_\gamma(\lambda)}{(\mu_i(\lambda) - \lambda')}d\lambda'\right) R^i_\gamma(\lambda), \quad \lambda \in \gamma. \tag{11}$$

Let the curve γ consist of n connected branches $\gamma_i = \lambda_i(\xi)$, $\xi \in \mathbb{R}$, and $\rho_i(\xi)$ be the jump of the function ψ across the corresponding branch. Then the expression for the function ψ takes the form

$$\psi = \eta + \frac{1}{2\pi i}\sum_{i=1}^n \int \frac{\rho_i(\xi')}{(\lambda - \lambda_i(\xi'))}\frac{d\lambda_i}{d\xi'}d\xi', \tag{12}$$

and the integral equation (11) reads

$$\rho_k(\xi) = \sum_i \left(\eta(f_i(\xi)) + \frac{1}{2\pi i}\sum_{j=1}^n v.p.\int \frac{\rho_j(\xi')}{\mu_i(\lambda_k(\xi)) - \lambda_j(\xi'))}\frac{d\lambda_j}{d\xi'}d\xi'\right)R^{ik}(\xi). \tag{13}$$

Thus we have obtained the system of n singular integral equations.

The δ-functional kernels. Theré is one important special case of the nonlocal $\bar{\partial}$-problem which is exactly solvable, which correspons to soliton solutions and discrete spectrum (in some broad sense). This is the case of δ-functional kernels

$$R(\lambda, \mu) = 2\pi i \sum_{i=1}^{N} R_i \delta(\lambda - \lambda_i)\delta(\mu - \mu_i), \tag{14}$$

where λ_i, μ_i is a set of points in the complex plane, $\lambda_i \neq \mu_j$,

$$R_k = c_k \exp((K_i(\lambda) - K_i(\mu))x_i).$$

In this case the solution of the problem (1) is a rational function, and the problem (1) reduces to a system of linear equations. The formula for the solution normalized by $(\lambda - \mu)^{-1}$ is

$$\begin{aligned} \psi(\lambda, \mu) &= \frac{1}{\lambda - \mu} + ((A)^{-1})_{ij} \frac{R_j}{(\mu_j - \mu)(\lambda - \lambda_i)}, \\ A_{ij} &= \delta_{ij} - \frac{R_i}{\mu_i - \lambda_j}, \end{aligned} \tag{15}$$

where

$$R_k = c_k \exp((K_i(\mu_k) - K_i(\lambda_k))x_i),$$

or in a more symmetric form with respect to λ and μ,

$$\begin{aligned} \psi(\lambda, \mu) &= \frac{1}{\lambda - \mu} + ((A')^{-1})_{ij} \frac{1}{(\mu_j - \mu)(\lambda - \lambda_i)}, \\ A'_{ij} &= R_i^{-1}\delta_{ij} - \frac{1}{\mu_i - \lambda_j}, \end{aligned} \tag{16}$$

In the limit when a pair of poles λ_i, μ_j coincides, rational with respect to x_i factors appear in the formula for ψ. The limit $\lambda_i \to \mu_i$ for all $0 < i \leqslant N$ corresponds to a rational with respect to x_i solution.

2.2. Construction of equations integrable via the $\bar{\partial}$-dressing method. The nonlocal $\bar{\partial}$-problem and its special cases (the $\bar{\partial}$-problem with a shift, the nonlocal Riemann problem, the Riemann problem with a shift) are powerful tools for constructing integrable nonlinear wave equations and their solutions (see [3, 4, 5, 9]).

The algebraic scheme of constructing equations is based on the following property of the problem (1): if $\psi(\mathbf{x}, \lambda)$ is a solution of the problem (1), then the functions

$$u(\mathbf{x})\psi, \qquad D_i\psi = (\partial/\partial x_i + K_i)\psi \tag{17}$$

are also solutions. Combining this property with the unique solvability of the problem (1), one obtains differential relations between the coefficients of expansion of functions $\psi(\mathbf{x}, \lambda)$ in powers of $(\lambda - \lambda_p)$ at the poles of $K_i(\lambda)$. Let us outline the basic steps of this scheme for the equations which will be used in this work, i.e., for the KP equation and for the N-wave type system of equations [5].

The KP equation. For the KP equation $D_1 = \partial/\partial x + i\lambda$, $D_2 = \partial/\partial y + \alpha^{-1}\lambda^2$, $(\alpha = 1; i)$, $D_3 = \partial/\partial t + i\lambda^3$. Let us introduce the solution of the problem (1) normalized by 1 $(\eta = 1)$,

$$\psi(\lambda, x, y, t)_{\lambda \to \infty} \to 1 + \psi_0(x, y, t)\lambda^{-1} + \ldots$$

The basis in the space of solutions of the problem (1) with the polynomial normalization is composed of the set of functions $D_1^n\psi$, $0 \leqslant n < \infty$. It follows from the unique solvability of the problem (1) that ψ satisfies the relations

$$(D_3 + D_1^3 + g(x,y,t)D_1 + h(x,y,t))\psi = 0, \tag{18}$$

$$(\alpha D_2 + D_1^2 + 2v(x,y,t))\psi = 0. \tag{19}$$

The successive use of the coefficients of expansion of these relations as $\lambda \to \infty$ allows us to define the functions v, g, h,

$$v = -i\frac{\partial}{\partial x}\psi_0, \qquad g = 3v, \qquad h_x = \frac{3}{2}\left(v_{xx} - \alpha v_y\right),$$

and to derive the KP equation for the first coefficient of expansion of the function ψ as $\lambda \to \infty$:

$$\frac{\partial}{\partial x}\left(v_t + \frac{1}{4}v_{xxx} + 3v_x v\right) = -\frac{3}{4}\alpha^2 v_{yy}. \tag{20}$$

The matrix KP equation via the scalar dressing. We would like to represent here a simple and interesting result important for the clasification of equations integrable by the $\bar{\partial}$-dressing method. The $\bar{\partial}$-dressing method establishes a correspondance between a triad of rational functions $K_i(\lambda)$ and an integrable equation (system of equations). There is a natural question: what is the equation corresponding to the triad $i\lambda^2$, $1/\alpha\lambda^4$, $i\lambda^6$? First, the dispersion law for this system coincides with the dispersion law of the KP equation [1]. But the accurate answer is that the matrix (2×2) KP equation correspons to this triad. Indeed, let us introduce the functions ψ_1 and ψ_2, normalized respectively by 1 and λ. Now the basis in the space of solution of the problem (1) with the polynomial normalization is composed of the set of functions $D_1^n\psi_1$ and $D_1^n\psi_2$, ψ_1 and ψ_2 are normalized respectively by 1 and λ. One can easily check that relations (19) and (18) hold in this case also, but now the potentials are (2×2) matrix functions

$$(\alpha D_2 + D_1^2 + 2v(x,y,t))\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = 0,$$

$$(D_3 + D_1^3 + g(x,y,t)D_1 + h(x,y,t))\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = 0$$

The matrix KP equation can be derived as a compatibility condition for these equations, it reads

$$\frac{\partial}{\partial x}\left(u_t + \frac{1}{4}u_{xxx} + \frac{3}{2}u_x^2\right) = -\frac{3}{4}\alpha^2 u_{yy} + 3[u_x, u_{xx} - \alpha u_y],$$

where $u_x = v$, $u = -i\psi_0$.

This observation allows us to construct explicitely a U–V pair corresponding to an arbitrary triad of rational functions. Indeed, let us take the triad

$$K_i(\lambda) = \lambda^{N_i} + \sum_{k=1}^{N_i-1} c_k\lambda^k, \quad i = 1,2,3. \tag{21}$$

Let N_1 be the smallest of N_i. In this case the basis in the space of solution of the problem (1) with the polynomial normalization is composed of the set of functions $D_1^n\psi_1, \ldots D_1^n\psi_{N_i}$, where $\psi, \ldots \psi_{N_i}$ are normalized respectively by $1, \ldots \lambda^{N_i}$. The U–V pair now is given by the expressions

$$\Big(D_2 - \sum_{k=1}^{[N_2/N_1]} u_k(x_1,x_2,x_3)D_1^k\Big)\begin{pmatrix}\psi_1\\ \vdots\\ \psi_{N_i}\end{pmatrix} = 0, \tag{22}$$

$$\Big(D_3 - \sum_{k=1}^{[N_3/N_1]} v_k(x_1,x_2,x_3)D_1^k\Big)\begin{pmatrix}\psi_1\\ \vdots\\ \psi_{N_i}\end{pmatrix} = 0, \tag{23}$$

where the potentials are $N_i \times N_i$ matrices and some of them may be equal to zero or have a special structure (some entries are equal to zero or a constant) depending on the powers N_1, N_2, N_3, $[M/N]$ denotes here $\min n : n \geqslant M/N$. So (22) and (23) give the U–V pair for the integrable system of equations corresponding to an arbitrary triad of polynomials (21). The solutions for this system can be found using the problem (1).

The N-wave type system of equations. To construct the N-wave type system of equations, we use the functions $K_i(\lambda)$ with an arbitrary number of simple and distinct poles:

$$K_i(\lambda) = \sum_{\alpha=1}^{n_i} \frac{a_i^\alpha}{\lambda - \lambda_i^\alpha}, \tag{24}$$

where $a_i^\alpha, \lambda_i^\alpha \in \mathbb{C}$, $1 \leqslant \alpha \leqslant n_i$, $\lambda_i^\alpha \neq \lambda_j^\beta$. Let us introduce the solutions $\psi_i^\alpha(\mathbf{x},\lambda)$ of the problem (1), normalized by $(\lambda - \lambda_i^\alpha)^{-1}$. These solutions satisfy the relations

$$D_i\psi_j^\beta - K_i(\lambda_j^\beta)\psi_j^\beta - \psi_{ji}^{\beta\alpha}a_i^\alpha\psi_i^\alpha = 0, \quad \psi_{ji}^{\beta\alpha}(\mathbf{x}) = \psi_j^\beta(\mathbf{x},\lambda_i^\alpha), \tag{25}$$

where $i \neq j$, summation over β is meant. The leading order of expansion of relation (25) as $\lambda \to \lambda_k^\gamma$, $i \neq j \neq k$ yields the equation

$$\frac{\partial}{\partial x_i}\psi_{jk}^{\beta\gamma} + (K_i(\lambda_k^\gamma) - K_i(\lambda_j^\beta))\psi_{jk}^{\beta\gamma} - \psi_{ji}^{\beta\alpha}a_i^\alpha\psi_{ik}^{\alpha\gamma} = 0, \tag{26}$$

summation over β is meant. If different permutations ijk and substitutions of the indices β, γ are taken into account, (26) is a closed set of equations for the functions $\psi_{ji}^{\beta\alpha}(\mathbf{x})$. The system of equations (26) is formally Lagrangian with the action density

$$\mathcal{L}(\mathbf{x}) = \varepsilon_{ijk}\Big(\frac{1}{2}\psi_{ij}^{\alpha\beta}a_j^\beta\partial_k\psi_{ji}^{\beta\alpha}a_i^\alpha + K_k(\lambda_i^\alpha)\psi_{ij}^{\alpha\beta}a_j^\beta\psi_{ji}^{\beta\alpha}a_i^\alpha + \frac{1}{3}a_i^\alpha\psi_{ik}^{\alpha\gamma}a_k^\gamma\psi_{kj}^{\gamma\beta}a_j^\beta\psi_{ji}^{\beta\alpha}\Big), \tag{27}$$

summation is over α, β, γ as well as over i, j, k.

§3. Solutions with special properties

3.1. The decreasing solutions. A solution given by the problem (1) in the general case is defined only locally in the vicinity of the point $\mathbf{x} = 0$, where the $\bar\partial$-problem is uniquely solvable. Solvability may be lost on some manifold in a space (x_1, x_2, x_3), where the solution has a singularity. To get "good enough" solutions having no singularities and bounded (decreasing) as $|\mathbf{x}| \to \infty$ one should impose some restrictions on the kernel $R(\lambda,\mu)$. These restrictions were discussed in our article [1]. The main result of this article can be formulated as follows. Let us choose a unit vector n_i ($\sum n_i^2 = 1$) defining a direction in the x-space. The solution given by the problem (1) is regular in a neighborhood of straight line $x_i = n_i\xi$ and decreasing along this line as $\xi \to \pm\infty$ if the condition

$$\operatorname{Re}\sum_{i=1}^{3} n_i(K_i(\lambda) - K_i(\mu)) = 0 \tag{28}$$

is satisfied (this condition is in fact the condition for the kernel $R(\lambda,\mu)$, it means that we should use the kernel localized on the manifold (28)), and the kernel $R(\lambda,\mu)$ is "small enough".

To get a solution which is "good enough" in a neighborhood of some plane, defined by two vectors n_i, m_i, the following two conditions must be satisfied:

$$\operatorname{Re}\sum_{i=1}^{3} n_i(K_i(\lambda) - K_i(\mu)) = 0,$$

$$\operatorname{Re}\sum_{i=1}^{3} m_i(K_i(\lambda) - K_i(\mu)) = 0.$$

In the generic case the pair of conditions (28) define a manifold of real dimension 2 in the space $\mathbb{C}^2$ of complex variables λ, μ.

Let us illustrate this result by a simple example of the KP equation. To obtain a small nonsingular solution decreasing in the plane (x, y) it is sufficient to use the problem (1) with kernel localized on the manifold defined by the system of conditions (28)

$$\operatorname{Im}(\lambda - \mu) = 0, \tag{29}$$

$$\operatorname{Re}\alpha^{-1}(\lambda^2 - \mu^2) = 0. \tag{30}$$

If $\alpha = i$, the system (30), (29) has a solution $\lambda, \mu \in \mathbb{R}$, which defines a nonlocal Riemann problem on the real axis. So the small decreasing solutions of the KP1 equation are given by the nonlocal Riemann problem

$$\psi^+ - \psi^- = \int_\gamma (\psi^+ + \psi^-) R_\gamma(\lambda,\mu)\exp(\varphi_i x_i)\,d\mu, \tag{31}$$

that was originally used by Manakov [10] to integrate the KP1 equation.

If $\alpha = 1$, the solution of the system (30) is $\mu = -\bar{\lambda}$. Thus small decreasing solutions of the KP2 equation are given by the $\bar{\partial}$-problem with conjugation

$$\bar{\partial}\psi(x,y,t,\lambda) = R(\lambda,-\bar{\lambda})\exp(\varphi_i x_i)\psi(x,y,t,-\bar{\lambda}), \tag{32}$$

and we reproduce the result of Ablowitz, Bar Yaacov, and Fokas [11].

The technique of the $\bar{\partial}$-problem is flexible enough to construct solutions with a different given type of asymptotic behavior. For instance, to get a solution, periodic in x and decreasing in y, one can use the kernel localized on the countable system of manifolds

$$\lambda - \mu = \frac{2\pi n}{X}, \quad -\infty < n < \infty, \tag{33}$$

$$\operatorname{Re}\alpha^{-1}(\lambda^2 - \mu^2) = 0. \tag{34}$$

If $\alpha = i$, the system (34) has a solution $\lambda, \mu \in \mathbb{R}$, $\lambda - \mu = \pm\frac{2\pi n}{X}$, and the solutions of the KP1 equation, periodic in x and decreasing in y, are given by the Riemann problem with a shift on the real axis with the shift function (33).

If $\alpha = 1$, the system (34) has a solution $Re\lambda, \mu = \pm\frac{\pi n}{X}$, $\lambda - \mu = \pm\frac{2\pi n}{X}$. Thus the solutions of the KP2 equation, periodic in x and decreasing in y, are given by the Riemann problem with a shift on the system of lines $Re\lambda, \mu = \pm\frac{\pi n}{X}$ with the shift function $\mu = -\bar{\lambda}$. This interesting problem is quite complicated, but we shall not treat it here in detail.

3.2. The $(1+1)$-dimensional case. The solutions independent of the variable x_j can be obtained from the problem (1) with kernel localized on the manifold

$$K_j(\lambda) - K_j(\mu) = 0. \tag{35}$$

This observation allows us to use the $(2+1)$-dimensional dressing method for $(1+1)$-dimensional equations and leads us naturally to the $\bar{\partial}$-problem with a shift and, for decreasing solutions, to the Riemann problem with a shift. Let us consider this observation in more detail.

If we have a (2+1)-dimensional integrable equation defined by the functions $K_i(\lambda)$, we can descend to the (1+1)-dimensional case, using the condition (35) for some coordinate x_i in the original or rotated coordinate system. For example, the y-independent KP equation gives the KdV equation

$$\left(v_t + \frac{1}{4}v_{xxx} + 3v_x v\right) = 0.$$

In this case condition (35) reads

$$\lambda^2 - \mu^2 = 0,$$

and the solutions of the KdV equation are given by the $\bar{\partial}$-problem with a shift [4]

$$\bar{\partial}\psi(\lambda) = R(\lambda, -\lambda)\exp(\varphi_i x_i)\psi(-\lambda), \tag{36}$$

the shift function for this case is quite simple ($\mu = -\lambda$), and it is easy to transform the problem (36) to the local matrix (2×2) Riemann problem.

We may also consider the case of the t-independent KP equation, which corresponds to the simplified Boussinesq equation

$$\frac{3}{4}\alpha^2 v_{yy} = -\left(\frac{1}{4}v_{xx} + \frac{3}{2}v^2\right)_{xx}. \tag{37}$$

In this case condition (35) reads

$$\lambda^3 - \mu^3 = 0,$$

and the solutions of the simplified Boussinesq equation (37) are given by the $\bar{\partial}$-problem

$$\bar{\partial}\psi(\lambda) = \sum_{i=1}^{3} R_i\psi(e_i\lambda),$$

where $e_i^3 = 1$. A simplified version of the Boussinesq equation was considered in [12]. Let us show that for decreasing solutions our approach leads us to the Riemann problem with a shift for the functions analytic in sectors (such a geometry for the local matrix Riemann problem arose in [12] from the analytic properties of the direct scattering problem). Combining condition (35) with condition (28)

$$\mathrm{Im}(\lambda - \mu) = 0,$$

we obtain

$$\begin{cases} \lambda - e_i\mu = 0, \\ \lambda - \mu = \xi, \quad \xi \in \mathbb{R}. \end{cases}$$

The solution of this system is

$$\lambda = \xi(1 - e_i)^{-1},$$
$$\mu = -\xi(1 - e_i^{-1})^{-1},$$

it defines the Riemann problem with a shift on the pair of straight lines with the vectors $\exp(i\pi/6)$, $\exp(-i\pi/6)$, the shift function is $\mu = -\bar\lambda$. So we arrive at the problem for the function analytic in corresponding sectors.

For an arbitrary rational function $K_i(\lambda)$ condition (35) defines a multi-valued shift function $\mu_i(\lambda)$, and the corresponding $\bar\partial$-problem reads

$$\bar\partial\psi(\lambda) = \sum_{i=1}^{n} R_i\psi(\lambda_i(\mu)). \tag{38}$$

§4. The Boussinesq equation

Let us introduce a modification of the KP equation

$$\frac{\partial}{\partial x}\Big((v_t - \beta v_x) + \frac{1}{4}v_{xxx} + 3v_x v\Big) = -\frac{3}{4}\alpha^2 v_{yy}, \quad \beta^2 = 1. \tag{39}$$

The solutions of this equation are given by the problem (1) with the dependence of the kernel on the variables x, y, t defined by the expressions (compare (2), (17))

$$\begin{aligned} D_1 &= \partial/\partial x + i\lambda, \\ D_2 &= \partial/\partial y + \alpha^{-1}\lambda^2 \quad (\alpha = 1; i), \\ D_3 &= \partial/\partial t + i\lambda^3 + i\beta\lambda, \end{aligned} \tag{40}$$

the derivation of this statement for the KP equation is given in §2, in this case it is completely analogous (or otherwise one may treat equation (39) as a KP equation with transformed variables x, y, t).

The time-independent solutions of equation (39) satisfy the Boussinesq equation

$$\Big(\frac{3}{4}\alpha^2 v_{yy} - \beta v_{xx}\Big) = -\Big(\frac{1}{4}v_{xx} + \frac{3}{2}v^2\Big)_{xx}. \tag{41}$$

Such solutions are given by the problem (1) ($v = -i\frac{\partial}{\partial x}\psi_0$) if the support of the kernel $R(\lambda, \mu)$ belongs to the manifold defined by condition (35)

$$(\lambda^3 + \beta\lambda - \mu^3 - \beta\mu) = 0, \quad \lambda \neq \mu \tag{42}$$

or

$$\lambda^2 + \lambda\mu + \mu^2 + \beta = 0.$$

This relation defines a $\bar\partial$-problem with a shift

$$\begin{aligned} \bar\partial\psi(\lambda, x, y) &= R(\lambda, \mu(\lambda))\exp(\varphi_i x_i)\psi(\mu(\lambda), x, y), \\ \mu(\lambda) &= \frac{1}{2}(-\lambda \pm (4\beta - 3\lambda^2)^{\frac{1}{2}}), \end{aligned} \tag{43}$$

The solutions of the Boussinesq equation, given by the problem (43) ($v = -i\frac{\partial}{\partial x}\psi_0$), are defined locally in a neighborhood of the point $x = 0$, $y = 0$. We consider the Boussinesq equation as a dynamical equation with respect to the variable y. To obtain decreasing solutions as $|x| \to \infty$, we should investigate the intersection of the manifold (35) with the manifold defined by condition (28):

$$\operatorname{Im}(\lambda - \mu) = 0. \tag{44}$$

Conditions (42), (44) define the Riemann problem with a shift (the Carleman's problem) which is a proper tool to solve Boussinesq's equation. Introducing $\xi = \frac{1}{2}(\lambda - \mu)$, $\nu = -i\frac{1}{2}(\lambda + \mu)$, $\xi \in \mathbb{R}$, one can get

$$\beta + \xi^2 - 3\nu^2 = 0. \tag{45}$$

On the reduction. Let us make a remark on the reduction. For $\alpha = 1$, $v(x, y)$ is real if the kernel of the problem (1) satisfies the condition

$$R(\lambda, \mu) = \bar{R}(-\bar{\lambda}, -\bar{\mu}), \tag{46}$$

and the condition

$$R(\lambda, \mu) = \bar{R}(\bar{\mu}, \bar{\lambda}) \tag{47}$$

if $\alpha = i$.

Soliton solutions. In the case of Boussinesq equation, formula (15) gives the determinant expression for the solution (for a similar expression for the KP, see [2])

$$\begin{aligned} v &= \frac{\partial^2}{\partial x^2} \ln \det(A), \\ A_{ij} &= \delta_{ij} - \frac{R_i}{\mu_i - \lambda_j}, \end{aligned} \tag{48}$$

where

$$R_k = c_k \exp\left(i(\mu_k - \lambda_k)\left(x - \frac{i}{\alpha}(\mu_k + \lambda_k)y\right)\right),$$

the pairs (λ_k, μ_k) should satisfy equation (42) and $\lambda_k \neq \mu_j$. The reductions (46) or (47) also are to be taken into account.

We shall not investigate formula (48) and its degenerate cases in more detail here, because this subject seems to be well discussed in literature (see, for example, [6]).

4.1. The "plus" Boussinesq equation. One can see that the properties of the Boussinesq equation depend significantly on the sign of β. Let $\beta = 1$. The corresponding equation (the plus Boussinesq equation) is of the form

$$\frac{3}{4}\alpha^2 v_{yy} - v_{xx} + \frac{1}{4}v_{xxxx} + \left(\frac{3}{2}v^2\right)_{xx} = 0. \tag{49}$$

In the case $\alpha^2 = 1$, it is a nonlinear wave equation, having in a linear approximation the monochromatic solution

$$v \simeq e^{i(\omega y + kx)}, \quad \omega^2 = \frac{4}{3}\left(k^2 + \frac{1}{4}k^4\right).$$

In the case $\alpha^2 = -1$ it is a nonlinear elliptic equation. In both cases equation (49) can be solved by the following Riemann problem with a shift:

$$-3\nu^2 + \xi^2 + 1 = 0, \quad \lambda = -\bar{\mu}, \ \lambda = \xi + i\nu, \ \mu = -\xi + i\nu \tag{50}$$

Equation (50) defines a hyperbola with the branches belonging respectively to the upper and to the lower half-planes (Figure 1, the dashed line). The shift is the change of the sign of the real part of λ. Let us introduce

$$\rho_\pm(\xi) = \Delta\psi \mid_\pm,$$

the jumps of the function $\psi(\lambda)$ across the upper and lower branches of the hyperbola. The function ψ can be represented in the form

$$\psi = 1 + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\rho_+(\xi')}{(\lambda - \lambda_+(\xi'))}\frac{d\lambda_+}{d\xi'}d\xi' + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\rho_-(\xi')}{(\lambda - \lambda_-(\xi'))}\frac{d\lambda_-}{d\xi'}d\xi',$$

where

$$\lambda_{\pm}(\xi) = \xi \pm i\sqrt{\frac{1+\xi^2}{3}}.$$

The Riemann problem with a shift (50) is equivalent to the system of two integral equations (13)

$$\rho_+(\xi) = \left(1 + \frac{1}{2\pi i} v.p. \int_{-\infty}^{\infty} \frac{\rho_+(\xi')}{(\lambda_+(-\xi) - \lambda_+(\xi'))} \frac{d\lambda_+}{d\xi'} d\xi' + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\rho_-(\xi')}{(\lambda_+(-\xi) - \lambda_-(\xi'))} \frac{d\lambda_-}{d\xi'} d\xi'\right) \times R^+(\xi) \exp\left(\frac{4i}{\sqrt{3}\alpha}\xi\sqrt{1+\xi^2}y + 2i\xi x\right),$$

$$\rho_-(\xi) = \left(1 + \frac{1}{2\pi i} v.p. \int_{-\infty}^{\infty} \frac{\rho_-(\xi')}{(\lambda_-(-\xi) - \lambda_-(\xi'))} \frac{d\lambda_-}{d\xi'} d\xi' + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\rho_+(\xi')}{(\lambda_-(-\xi) - \lambda_+(\xi'))} \frac{d\lambda_+}{d\xi'} d\xi\right) \times R^-(\xi) \exp\left(\frac{-4i}{\sqrt{3}\alpha}\xi\sqrt{1+\xi^2}y - 2i\xi x\right).$$

The solution of the Boussinesq equation is given by the formula

$$u = -\frac{\partial}{\partial x}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\rho_+(\xi)\frac{d\lambda_+}{d\xi} + \rho_-(\xi)\frac{d\lambda_-}{d\xi}\right)d\xi.$$

4.2. The "minus" Boussinesq equation. This equation

$$\frac{3}{4}\alpha^2 v_{yy} + v_{xx} + \frac{1}{4}v_{xxxx} + \left(\frac{3}{2}v^2\right)_{xx} = 0$$

arises upon putting $\beta = -1$. The reduced $\bar{\partial}$ problem for this equation is described by the conditions

$$\lambda^2 + \lambda\mu + \mu^2 = 1 \tag{51}$$

(time independence) and

$$\mathrm{Im}(\lambda - \mu) = 0 \tag{52}$$

(decreasing in the x-direction). There are two possibilities to satisfy these conditions.

1. λ and μ are real ($\lambda^2 < \frac{4}{3}$, $\mu^2 < \frac{4}{3}$) and

$$\mu = -\frac{1}{2}\lambda \pm \sqrt{1 - \frac{3}{4}\lambda^2}. \tag{53}$$

We have a Riemann problem on the cut $-\sqrt{\frac{4}{3}} < \mathrm{Re}\,\lambda < \sqrt{\frac{4}{3}}$ with the twofold shift (53).

2. λ and μ are complex, $\lambda = \nu + i\xi$, $\mu = -\nu + i\xi$, $\xi, \nu \in \mathbb{R}$,

$$\nu^2 - 3\xi^2 = 1 \tag{54}$$

Both λ and μ are placed on the hyperbola (see Figure 1, the continuous line). The shift with respect to the "plus" Boussinesq equation is the reflection with respect to the imaginary axis.

Let us parameterize the curves on which the solution ψ of the Riemann problem with a shift has a discontinuity in the following way:

$$\gamma_+ = \lambda_+(\xi) = i\xi + \sqrt{1+3\xi^2}, \quad -\infty < \xi < \infty,$$
$$\gamma_- = \lambda_-(\xi) = i\xi - \sqrt{1+3\xi^2}, \quad -\infty < \xi < \infty,$$
$$\gamma_0 = \lambda_0(\xi) = \xi, \quad \xi^2 < \frac{4}{3},$$

and introduce the jumps $\rho_+(\xi), \rho_-(\xi), \rho_0(\xi)$ of the function ψ across the respective curves. Then the function ψ can be represented in the form

$$\psi = 1 + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\rho_+(\xi')}{(\lambda - \lambda_+(\xi'))}\frac{d\lambda_+}{d\xi'}d\xi'$$
$$+ \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\rho_-(\xi')}{(\lambda - \lambda_-(\xi'))}\frac{d\lambda_-}{d\xi'}d\xi' + \frac{1}{2\pi i}\int_{-\sqrt{\frac{4}{3}}}^{\sqrt{\frac{4}{3}}} \frac{\rho_0(\xi')}{(\lambda - \xi')}d\xi'.$$

The Riemann problem in this case is equivalent to the system of three integral equations

$$\rho_0(\xi) = 1 + \Big(\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\rho_+(\xi')}{(\mu_+(\xi) - \lambda_+(\xi'))}\frac{d\lambda_+}{d\xi'}d\xi'$$
$$+ \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\rho_-(\xi')}{(\mu_+(\xi) - \lambda_-(\xi'))}\frac{d\lambda_-}{d\xi'}d\xi' + \frac{1}{2\pi i}\int_{-\sqrt{\frac{4}{3}}}^{\sqrt{\frac{4}{3}}} \frac{\rho_0(\xi')}{(\mu_+(\xi) - \xi')}d\xi'\Big)$$
$$\times R_0^+(\xi)e^{i(\frac{1}{2}\xi - \sqrt{1-\frac{3}{4}\xi^2})x + \frac{1}{\alpha}(\frac{3}{2}\xi^2 - \xi\sqrt{1-\frac{3}{4}\xi^2} - 1)y}$$
$$+ \Big(\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\rho_+(\xi')}{(\mu_-(\xi) - \lambda_+(\xi'))}\frac{d\lambda_+}{d\xi'}d\xi'$$
$$+ \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\rho_-(\xi')}{(\mu_-(\xi) - \lambda_-(\xi'))}\frac{d\lambda_-}{d\xi'}d\xi' + \frac{1}{2\pi i}\int_{-\sqrt{\frac{4}{3}}}^{\sqrt{\frac{4}{3}}} \frac{\rho_0(\xi')}{(\mu_-(\xi) - \xi')}d\xi'\Big)$$
$$\times R_0^-(\xi)e^{i(\frac{1}{2}\xi + \sqrt{1-\frac{3}{4}\xi^2})x + \frac{1}{\alpha}(\frac{3}{2}\xi^2 + \xi\sqrt{1-\frac{3}{4}\xi^2} - 1)y)},$$

where

$$\mu_\pm = \frac{1}{2}\xi \pm \sqrt{1 - \frac{3}{4}\xi^2};$$

$$\rho_+(\xi) = 1 + \Big(\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\rho_+(\xi')}{(\lambda_-(\xi)-\lambda_+(\xi'))}\frac{d\lambda_+}{d\xi'}d\xi' + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\rho_-(\xi')}{(\lambda_-(\xi)-\lambda_-(\xi'))}\frac{d\lambda_-}{d\xi'}d\xi' + \frac{1}{2\pi i}\int_{-\sqrt{\frac{4}{3}}}^{\sqrt{\frac{4}{3}}} \frac{\rho_0(\xi')}{(\lambda_-(\xi)-\xi')}d\xi'\Big) \times R^+(\xi)e^{2i\sqrt{1-3\xi^2}x+\frac{4i}{\alpha}\xi\sqrt{1-3\xi^2}y},$$

$$\rho_-(\xi) = 1 + \Big(\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\rho_+(\xi')}{(\lambda_+(\xi)-\lambda_+(\xi'))}\frac{d\lambda_+}{d\xi'}d\xi' + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\rho_-(\xi')}{(\lambda_+(\xi)-\lambda_-(\xi'))}\frac{d\lambda_+}{d\xi'}d\xi' + \frac{1}{2\pi i}\int_{-\sqrt{\frac{4}{3}}}^{\sqrt{\frac{4}{3}}} \frac{\rho_0(\xi')}{(\lambda_+(\xi)-\xi')}d\xi'\Big) \times R^-(\xi)e^{-2i\sqrt{1-3\xi^2}x-\frac{4i}{\alpha}\xi\sqrt{1-3\xi^2}y}.$$

The solution of the Boussinesq equation is given by the formula

$$u = -\frac{\partial}{\partial x}\frac{1}{2\pi}\Big[\int_{-\infty}^{\infty}\Big(\rho_+(\xi)\frac{d\lambda_+}{d\xi} + \rho_-(\xi)\frac{d\lambda_-}{d\xi}\Big)d\xi + \int_{-\sqrt{\frac{4}{3}}}^{\sqrt{\frac{4}{3}}} \rho_0(\xi)\frac{d\lambda_0}{d\xi}d\xi.\Big]$$

In this case the spectral data R_γ split into two parts; the short-wave part of the continuous spectrum is localized on the hyperbola (54), and the long-wave part of the spectrum on the segment of the real axis (in fact on the covering of this segment); see Figure 1, the continuous line. For $\alpha = 1$ the hyperbola corresponds to the stable part of the spectrum (the exponent (2) for y is imaginary) and the segment to the unstable part (the exponent is real), for $\alpha = i$ the situation is reversed, i.e., for $\alpha = 1$ the long-wave instability holds, and for $\alpha = 1$ we have the short-wave instability.

§5. The relativistically-invariant systems

The systems considered in this part were first integrated by Zakharov and Mikhailov [7], using the *matrix local* Riemann problem. In our work we use the *scalar nonlocal* problem and obtain some new spectral information for these equations.

Let us consider the solutions of the N-wave type system of equations (26) independent of the variable x_1. Such solutions are given by the $\bar{\partial}$-problem with a shift defined by condition (35) and obey the relation

$$\psi_{jk}^{\alpha\gamma} = (K_1(\lambda_k^\gamma) - K_1(\lambda_j^\alpha))^{-1}\psi_{j1}^{\alpha\beta}a_1^\beta\psi_{1k}^{\beta\gamma},$$

where $j \neq k$, $j,k = 2,3$. Substituting this relation in equations (26) with the derivatives ∂_2 and ∂_3 we obtain the closed set of equations for the functions $\psi_{13}^{\alpha\gamma}$, $\psi_{12}^{\alpha\beta}$, $\psi_{31}^{\gamma\alpha}$, $\psi_{21}^{\gamma\beta}$:

$$\begin{aligned}&\Big(\frac{\partial}{\partial x_j} + (K_j(\lambda_k^\gamma) - K_j(\lambda_1^\alpha))\Big)\psi_{1k}^{\alpha\gamma} = (K_1(\lambda_k^\gamma) - K_1(\lambda_j^\beta))^{-1}\psi_{1j}^{\alpha\beta}a_j^\beta\psi_{j1}^{\beta\alpha'}a_1^{\alpha'}\psi_{1k}^{\alpha'\gamma},\\ &\Big(\frac{\partial}{\partial x_j} - (K_j(\lambda_k^\gamma) - K_j(\lambda_1^\alpha))\Big)\psi_{k1}^{\gamma\alpha} = -(K_1(\lambda_k^\gamma) - K_1(\lambda_j^\beta))^{-1}\psi_{j1}^{\beta\alpha}a_1^\alpha\psi_{k1}^{\gamma\alpha'}a_1^{\alpha'}\psi_{1j}^{\alpha'\beta},\end{aligned} \tag{55}$$

$k \neq j$, $k, j = 2, 3$, summation is over β, α'. We consider the case $i\alpha_i$, $\lambda_i \in \mathbb{R}$. In this case system (55) admits a reduction

$$\psi_{ik}^{\alpha\gamma} = -\bar{\psi}_{ki}^{\gamma\alpha}, \tag{56}$$

which corresponds to the following condition for the kernel of the problem (1):

$$R(\lambda, \mu) = \bar{R}(\bar{\mu}, \bar{\lambda}).$$

The linear part of equation (55) can be canceled preserving the reduction by the substitution

$$\psi_{1k}^{\alpha\gamma} = \exp(K_j(\lambda_1^\alpha)x_j + K_k(\lambda_1^\alpha)x_k - K_j(\lambda_k^\gamma)x_j))\tilde{\psi}_{1k}^{\alpha\gamma}, \tag{57}$$

and the coefficients a_i^α can be made equal to 1 by the change

$$\hat{\psi}_{1k}^{\alpha\gamma} = \tilde{\psi}_{1k}^{\alpha\gamma}(-a_k^\gamma a_1^\alpha)^{\frac{1}{2}} \tag{58}$$

if $\operatorname{Im} a_i^\alpha > 0$. Further we use the notation $\psi^{\alpha\gamma}$ for the function $\hat{\psi}_{13}^{\alpha\gamma}$, $\varphi^{\alpha\beta}$ for the function $\hat{\psi}_{12}^{\alpha\beta}$ The set of equations (55) in this case reads

$$\begin{aligned} \frac{\partial}{\partial x_2}\psi^{\alpha\gamma} &= (K_1(\lambda_3^\gamma) - K_1(\lambda_2^\beta))^{-1}\varphi^{\alpha\beta}\bar{\varphi}^{\alpha'\beta}\psi^{\alpha'\gamma}, \\ \frac{\partial}{\partial x_3}\varphi^{\alpha\beta} &= -(K_1(\lambda_3^\gamma) - K_1(\lambda_2^\beta))^{-1}\psi^{\alpha\gamma}\bar{\psi}^{\alpha'\gamma}\varphi^{\alpha'\beta}, \end{aligned} \tag{59}$$

summation is over β, α' in the first equation and over γ, α' in the second equation. The Lagrangian density for this system is

$$\begin{aligned} \mathcal{L}(x_2, x_3) = i\big(&\bar{\psi}^{\alpha\gamma}\partial_2\psi^{\alpha\gamma} - \psi^{\alpha\gamma}\partial_2\bar{\psi}^{\alpha\gamma} - \bar{\varphi}^{\alpha\beta}\partial_3\varphi^{\alpha\beta} + \varphi^{\alpha\beta}\partial_3\bar{\varphi}^{\alpha\beta} \\ &+2(K_1(\lambda_2^\beta) - K_1(\lambda_3^\gamma))^{-1}\bar{\varphi}^{\alpha\beta}\bar{\psi}^{\alpha'\gamma}\psi^{\alpha\gamma}\varphi^{\alpha'\beta}\big), \end{aligned} \tag{60}$$

where summation is over α, β, γ, α'. In the special case of K_2 and K_3 having one pole, system (59) reduces to the relativistically-invariant Nambu system [7]:

$$\begin{cases} \partial_\eta\varphi^\alpha = i\psi^\alpha \sum_\beta \bar{\psi}^\beta\varphi^\beta, \\ \partial_\xi\psi^\alpha = i\varphi^\alpha \sum_\beta \psi^\beta\bar{\varphi}^\beta, \end{cases} \tag{61}$$

where $\eta = x_2$, $\xi = -x_3$ are the light cone variables, $\varphi^\alpha = A\psi_{13}^{\alpha 1}$, $\psi^\alpha = A\psi_{12}^{\alpha 1}$, $A = (\operatorname{Im}(K_1(\lambda_3) - K_1(\lambda_2))^{\frac{1}{2}}$ (without loss of generality, we suppose that $\operatorname{Im}(K_1(\lambda_3) - K_1(\lambda_2)) > 0$). In the general case, the complete relativistic invariance of system (59) can be obtained if K_2 and K_3 have equal number of poles.

With an extra symmetry

$$\begin{aligned} K_2(\lambda) &= -K_2(-\lambda), \\ K_3(\lambda) &= -K_3(\lambda), \\ K_1(\lambda) &= \pm K_1(-\lambda) \end{aligned} \tag{62}$$

system (59) admits a reduction

$$\psi_{ik}^{\alpha\gamma} = -\bar{\psi}_{ik}^{\alpha'\gamma'}; \quad \lambda_i^\alpha = -\lambda_i^{\alpha'}, \ \lambda_i^\gamma = -\lambda_i^{\gamma'}, \tag{63}$$

which corresponds to the condition

$$R(\lambda, \mu) = \bar{R}(-\bar{\lambda}, -\bar{\mu}). \tag{64}$$

For the Nambu system, the symmetry (62) implies a special choice $K_2 = a_2\lambda$, $K_3 = a_3/\lambda$, $K_1(\lambda) = K_1(-\lambda)$ and leads to the reduction

$$\psi^\alpha = \bar{\psi}^{\alpha'}, \qquad \varphi^\alpha = \bar{\varphi}^{\alpha'}, \qquad \lambda_1^\alpha = -\lambda_1^{\alpha'}. \tag{65}$$

Thus we obtain the Gross-Neveu equations

$$\begin{cases} \partial_\eta \varphi^\alpha = i\psi^\alpha \sum_\beta (\bar{\psi}^\beta \varphi^\beta + \psi^\beta \bar{\varphi}^\beta), \\ \partial_\xi \psi^\alpha = i\varphi^\alpha \sum_\beta (\psi^\beta \bar{\varphi}^\beta + \bar{\psi}^\beta \varphi^\beta). \end{cases} \tag{66}$$

The Lagrangian density for systems (61) and (66) is given by the expression (60).

Now let us investigate the problem allowing us to construct the solutions with the asymptotic behavior resulting from the transform (57), (58); respectively, for systems (55), (59), (61), (66) as $\sqrt{\xi^2+\eta^2} \to \infty$:

$$\psi_{12}^{\alpha\beta} \to (\lambda_2^\beta - \lambda_1^\alpha)^{-1}, \tag{67}$$

$$\psi^{\alpha\beta} \to A_{12}^{\alpha\beta} \exp((K_3(\lambda_2^\beta) - K_3(\lambda_1^\alpha))x_3 - K_2(\lambda_1^\alpha)x_2), \tag{68}$$

$$\psi^\alpha \to A'^\alpha \exp\left(\frac{a_3\xi}{\lambda_1^\alpha - \lambda_3} - \frac{a_2\eta}{\lambda_1^\alpha - \lambda_2} - \frac{a_3\xi}{\lambda_2 - \lambda_3}\right), \tag{69}$$

$$\psi^\alpha \to A''^\alpha \exp\left(\frac{a_3\xi}{\lambda_1^\alpha} - a_2\eta\lambda_1^\alpha\right), \tag{70}$$

where

$$A_{12}^{\alpha\beta} = \frac{\sqrt{-a_2^\alpha a_2^\beta}}{\lambda_2^\beta - \lambda_1^\alpha}, \qquad A'^\alpha = \frac{\sqrt{-a_2^\alpha a_2}}{\lambda_2 - \lambda_1^\alpha}, \qquad A''^\alpha = \sqrt{\frac{-a_2^\alpha a_2}{\mathrm{Im}(K_1(0))}}.$$

Taking into account that $ia_i^\alpha, \lambda_i^\alpha \in \mathbb{R}$, the system of conditions (28)

$$\mathrm{Re}(K_2(\lambda) - K_2(\mu)) = 0, \ \mathrm{Re}(K_3(\lambda) - K_3(\mu)) = 0, \quad \lambda \neq \mu \tag{71}$$

has a solution $\lambda, \mu \in \mathbb{R}$ (which is unique in the generic case). Thus, the Riemann problem with a shift is set on the real axis. The shift function is defined by condition (35)

$$K_1(\lambda) - K_1(\mu) = 0, \quad \lambda \neq \mu, \tag{72}$$

$$K_1(\lambda) = \sum_{\alpha=1}^{n_1} \frac{a_1^\alpha}{\lambda - \lambda_1^\alpha}, \tag{73}$$

and it can be rather complicated. This problem gives solutions of the Nambu equations with asymptotic behavior (69).

In the presence of the extra symmetry (62), equations (71) have also a solution

$$\lambda = -\bar{\mu}. \tag{74}$$

The substitution of this solution in relation (73) gives an equation of the algebraic curve in a complex plane

$$K_1(\lambda) \pm K_1(\bar{\lambda}) = 0, \qquad \bar{K}_1(\bar{\lambda}) = -K_1(\lambda). \tag{75}$$

So in this case the Riemann problem with a shift is set on the curve consisting of the real axis and the algebraic curve (75), the shift functions are given respectively by (73) and (74). This problem gives solutions of the Gross-Neveu equations with asymptotic behavior (70).

A remark on soliton solutions. Soliton solutions for the equations considered in this section are given by formula (16), the pairs (λ_i, μ_i) should satisfy condition (35) for x_1, the reductions are to be taken into account.

§6. Inverse problems for a differential operator of an arbitrary order on a line

The developed method allows us to find a productive approach to the classical problem of analysis—the inverse problem for a differential operator of an arbitrary order (see [8, 6]). We consider the spectral problem

$$L\psi = \zeta\psi,$$

$$L = \partial^n + a_{n-1}\partial^{n-1} + \sum_{i=1}^{n-2} u_i(x)\partial^i,$$

$\infty < x < \infty$, satisfying the condition

$$u_i(x) \to a_i, \quad x \to \pm\infty,$$

where a_i are given complex constants. In other words, $L \to L_0$, $x \to \pm\infty$,

$$L^0 = \partial^n + \sum_{i=0}^{n-1} a_i\partial^i.$$

The problem is to restore the potentials u_i by some properly defined "scattering data".

One can construct the potentials with the corresponding wave functions, using nonlocal problems in the complex plane. Though we work in the scope of the dressing method and do not treat the direct scattering problem, we obtain an information (may be not complete) about the structure of the continuous spectrum. In fact we define *the inverse scattering transform* from the kernel of the Riemann problem with a shift to the small decreasing potentials of the corresponding operators, the wave functions are also given by this procedure.

Let us consider the nonlocal $\bar\partial$-problem (1) with $K_1 = \lambda$, $K_2 = \lambda^n + \sum_{i=1}^{n-1} a_i\lambda^i$, $a_i \in \mathbb{C}$, normalized by 1. This choice leads us to the relation

$$\Big(D_2 - \sum_{i=1}^{n} a_i D_1^i\Big)\psi(\lambda, x, y) = \sum_{i=1}^{n-2} u_i(x, y) D_1^i \psi(\lambda, x, y). \tag{76}$$

As usually, we can pass from the "prolonged" derivatives D_i to partial derivatives by the transform

$$\psi \to \psi \exp(K_i x_i).$$

The potentials u_i in the operator (76) can easily be expressed through the coefficients of expansion of the function $\psi(\lambda, x, y)$ as $\lambda \to \infty$. For the case $a_i \in \mathbb{R}$, the potentials are real if the kernel of the problem (1) satisfies the condition

$$R(\lambda, \mu) = \bar R(\bar\lambda, \bar\mu).$$

Now let us proceed to the one-dimensional case. To cancel the dependence on y, we should use the $\bar\partial$-problem (1) with the kernel $R(\lambda, \mu)$ localized on the manifold (35)

$$K_2(\lambda) - K_2(\mu) = 0$$

or

$$\lambda^n + \sum_{i=1}^{n-1} a_i\lambda^i - \mu^n - \sum_{i=1}^{n-1} a_i\mu^i = 0. \tag{77}$$

In this case we can solve the inverse scattering problem for the operator on the line

$$\Big(\partial^n + \sum_{i=1}^{n-1} a_i\partial^i + \sum_{i=1}^{n-2} u_i(x)\partial^i\Big)\psi(\lambda, x) = \zeta\psi(\lambda, x), \tag{78}$$

where

$$\zeta = \left(\lambda^n + \sum_{i=1}^{n-1} a_i \lambda^i\right).$$

Condition (77) defines a $\bar{\partial}$-problem with a shift

$$\bar{\partial}\psi(\lambda) = \sum_{i=1}^{n} R_i(\lambda)\psi(\mu_i(\lambda)). \tag{79}$$

The problem (79) gives the potentials with the wave functions locally near the point $x = 0$. To construct the decreasing potentials defined on all the line we should observe condition (28)

$$\mathrm{Re}(\lambda - \mu) = 0. \tag{80}$$

This condition, together with (77), defines a Riemann problem with a shift (10). The equation of the curve $\lambda(\xi)$, $\xi \in \mathbb{R}$, for this problem is given by the substitution of expression $\mu = \lambda - i\xi$ (compare (80)) in equation (77), the shift function is given by equation (77) (it is also useful to note that λ and μ have identical real part).

Thus, small decreasing potentials for the operator (78), together with the corresponding wave functions, are given by the Riemann problem with a shift

$$\Delta(\psi(\lambda(\xi))) = R_\gamma(\lambda, \mu(\lambda))\psi(\mu(\lambda(\xi))),$$

which reduces to the integral equations (13); this problem defines the transform from the kernel R to the potentials $u_i(x)$ (the inverse scattering transform for the continuous spectrum).

Let us consider a simple example $K_2 = \lambda^n$. In this case the operator (78) is of the form

$$\partial_1^n \psi(\lambda, x) = \sum_{i=1}^{n-2} u_i(x)\partial_1^i \psi(\lambda, x). \tag{81}$$

This class of operators was investigated in detail in [8]. We shall show now how our technique works in this case. The shift function (77) for this case is

$$\lambda^n - \mu^n = 0, \tag{82}$$

and the problem (79) reads

$$\bar{\partial}\psi(\lambda) = \sum_{i=1}^{n} R_i(\lambda)\psi(e_i\lambda), \quad e_i^n = 1. \tag{83}$$

If we take into account condition (80), we obtain

$$\begin{cases} \lambda - e_i\mu = 0, \\ \lambda - \mu = i\xi, \quad \xi \in \mathbb{R}. \end{cases}$$

The solution of this system is

$$\lambda = i\xi(1 - e_i)^{-1} := i\xi\alpha_i,$$

$$\mu = -i\xi(1 - e_i^{-1})^{-1} := -i\xi\bar{\alpha}_i, \quad e_i \neq 1.$$

In this case ψ has a discontinuity on $(n-1)$ lines with the angle π/n between them. Thus, we arrive at the Riemann problem with a shift for the function analytic in sectors, the shift function is $\mu = \bar{\lambda}$. The integral equations (13) for this case is of the form

$$\rho_k(\xi) = \left(1 + \frac{1}{2\pi i}\sum_{j=1}^{n-1} v.p. \int_{-\infty}^{\infty} \frac{\rho_j(\xi')}{\bar{\alpha}_k\xi - \alpha_j\xi'}\alpha_j d\xi'\right) R^k(\xi)\exp((\alpha_k - \bar{\alpha}_k)\xi),$$

where ρ_k is the jump of the function ψ across the corresponding line. In the general case, the Riemann problem with a shift may be defined on quite a general analytical curve in the complex plane. The symmetries of the function $K_2(\lambda)$ can simplify the investigation.

Acknowledgments

The authors are grateful to Dr. S. V. Manakov and Dr. A. V. Mikhailov for useful discussions.

References

[1] Bogdanov L. V., Zakharov V. E., St. Petersburg Math. J. **3** (1992), no. 3, 533.
[2] Zakharov V. E., Manakov S. V., Novikov S. P., Pitaevsky L. P., *Theory of Solitons: the Inverse Scattering Method*, Plenum Publ. Corp, New York, 1984.
[3] Zakharov V. E., Manakov S. V., Zap. Nauch. Sem. LOMI **133** (1984), 77.
[4] Zakharov V. E., Manakov S. V., Funk. Anal. Ego Prilozhen. **19** (1985), 11.
[5] Bogdanov L. V., Manakov S. V., J. Phys. A: Math. Gen. **21** (1988), L537.
[6] Caudrey P. J., Physica **6D** (1982), 51.
[7] Mikhailov A. V., Zakharov V. E., Comm. Math. Phys. **74** (1980), 21.
[8] Beals K., Deift P., Tomei C., *Direct and Inverse Scattering on the Line*, Math. Surveys and Monographs, vol. 28, AMS, Providence, RI, 1988.
[9] Zakharov V. E., *On the dressing method*, Inverse Methods in Action (P. C. Sabatier, ed.), Springer, Berlin, 1990.
[10] Manakov S. V., Physica **3D** (1981), 420.
[11] Ablowitz M. J., Bar Yaacov D., Fokas A. S., Stud. Appl. Math. **69** (1983), 135.
[12] Deift P., Tomei C., Trubowitz E., Comm. Pure Appl. Math. **35** (1982), no. 5, 567.

International Institute for Nonlinear Sciences
GSP-1 117940, 2 Kosygina, Moscow v-334, Russia

Поступило 13 апреля 1994 г.