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Surface singularities of ideal fluid

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We show that the equations of motion of an ideal fluid with a free surface due to inertial forces only can be effectively solved in the approximation of small surface angles. For almost arbitrary initial conditions the system evolves to the formation of singularities in a finite time. Three kinds of singularities are shown to be possible: the root ones for which the process of the singularity formation represents some analog of the wave breaking; singularities in the form of wedges on the interface; the floating ones associated with motion in the complex plane of the singular points of the analytical continuation of the surface shape.

1. Introduction

The formation of singularities in a wave system in a finite time, or in other words, wave collapse, is one of the basic phenomena in nonlinear physics. The collapses play an essential role in various fields of physics. In many cases the collapse is the most effective mechanism of wave energy dissipation.

From the mathematical point of view, collapse means that the solution of the Cauchy problem for some evolution PDE exists only for finite time until some definite moment $t=t_0$ and cannot be continued for $t>t_0$. At the moment $t=t_0$ the solution loses its initial smoothness and a singularity appears. What kind of singularities will arise depends on the physical model. For example, for the self-focusing of light [1,2] or for the collapse of Langmuir waves [3], the amplitude of electromagnetic waves tends to infinity. In another case, that of wave breaking in gas dynamics described by the well-known Riemann solution (see, for example, ref. [4]), the first derivative of the velocity becomes infinite at the moment of breaking. For sea surface waves the analogous phenomenon leads to an infinite second derivative of the surface profile (so that angles or cones appear on the surface). Checking analyticity violation is the most sensitive tool for studying that set of collapses. Loss of analyticity of vortex sheets in the nonlinear stage of the Kelvin–Helmholz instability [5] is an example of that. Various aspects of singularity formation for vortex sheet motion have so far been studied in a number of papers, both numerically and analytically [5–8]. The recent paper [8] should be mentioned in particular. It provides considerable numerical evidence of the occurrence of an infinite surface curvature in a finite time. The root (in space) character of the arising singularity has been checked in ref. [8] too. As for analytical considerations, though they show the existence of singularities, in our opinion, a demonstration of explicit collapsing solutions is still lacking. The question also remains open whether root singularities are generic for the Cauchy problem in this system.

In this paper we will consider how the singularities appear as a result of the analyticity breaking on the interface between two ideal liquids in the absence of both gravity and surface tension. This question is very important, also, for understanding the evolution of the boundary between two fluids while studying sea surface waves and the nonlinear stage of the Rayleigh–Taylor instability resulting in the finger structure (see, for instance, ref. [9], and references therein). We present the analytical solution of the problem based both on the perturbation approach, assuming small angles of the interface variations, and using the Hamiltonian formalism for the description of the interface motion. For the case of a liquid with a free surface, the problem was formulated [10] by one of the authors of the present paper (V.Z.). It is supposed that the singularity formation on a free surface of an ideal fluid or in the more general case, for the boundary between two ideal fluids, is mainly connected with inertial forces, other factors giving a minor correction. This means that if one considers, for instance, the motion of an ideal liquid drop (without both gravity and surface tension) then on the surface of the drop there will appear a singularity of the wedge type. This idea was later confirmed by direct numerical integration of the Euler equation for the case of deep water [11].

Adopting only the small slope approximation, we give the solution of the Cauchy problem for the motion of the boundary between two liquids.

The main conjecture of this paper is as follows. The formation of singularities on the interface for the small angle approximation can be considered as wave breaking in the complex plane to which the solution can be extended. This results in the motion of both branch points of the analytical continuation of the velocity potential and singular points of the analytical extension of the surface elevation. When for the first time the most "rapid" singular point will reach the real axis, then the singularity will appear. Three kinds of singularities are possible. For the first kind, at the moment the tangent velocity touches the interface, it has an infinite first derivative and simultaneously the second space derivative of the interface coordinate $z=\eta(x, t)$, i.e. η_{xx}, also turns to infinity. These are weak singularities of root character ($\eta_{xx}\sim|x|^{-1/2}$) which can be assumed to serve as a source of more powerful singularities, observed in numerical experiments [11], or to represent a separate type of singularities. This kind of singularities turns out to be consistent with the assumption of small surface angles. It is shown that the interaction of two movable branch points to the tangent velocity can lead under some definite conditions to the formation of the second type of singularities, wedges on the surface shape. Close to the collapse time the self-similar solution for such singularities happens to be compatible with the complete system of equations describing arbitrary angle values. The third type is caused by the initial analytical properties of $\eta_0(x)$ resulting in the formation of a strong singular interface profile.

2. Model

Let us consider two ideal fluids with mass densities ρ_1 and ρ_2, respectively. Let $z=\eta(x, y, t)$ be the coordinate of the interface between these two liquids so that the first liquid occupies the region $-\infty<z\leqslant\eta(x, y, t)$, the second $\eta(x, y, t)<z\leqslant\infty$. Implying the liquid velocities to be potential ones, $\boldsymbol{v}_i=\nabla\Phi_i$ ($i=1, 2$) in the absence of both gravity and surface tension, the potential Φ satisfies the equation

$$\rho_i\left(\frac{\partial\Phi_i}{\partial t}+\tfrac{1}{2}(\nabla\Phi_i)^2\right)+p_i=0\,, \tag{1}$$

which combines in a complete closed system, when amplified with the incompressibility equation $\triangle\Phi_i=0$, the kinematic relation on the free surface

$$\frac{\partial\eta}{\partial t}=\left(\frac{\partial\Phi_i}{\partial z}-\nabla\eta\cdot\nabla\Phi_i\right)\bigg|_{z=\eta}=v_{ni}\sqrt{1+(\nabla\eta)^2}\,, \tag{2}$$

and the boundary conditions $(p_1=p_2)|_{z=\eta}$, $\Phi|_{|z|\to\infty}\to0$. Here v_{ni} is the velocity component normal to the interface $z=\eta(x, y, t)$ and the surface tension is neglected.

Equations (1), (2) render standard the Hamiltonian form with elevation η as a general coordinate and

$$\Psi=(\rho_1\Phi_1-\rho_2\Phi_2)|_{z=\eta} \tag{3}$$

as a general momentum [12],

$$\frac{\partial\eta}{\partial t}=\frac{\delta H}{\delta\Psi}, \tag{4}$$

$$\frac{\partial\Psi}{\partial t}=-\frac{\delta H}{\delta\eta}, \tag{5}$$

where the Hamiltonian

$$H=\tfrac{1}{2}\rho_1\int \mathrm{d}\boldsymbol{r}_\perp\int_{-\infty}^{\eta}\mathrm{d}z\,(\nabla\Phi_1)^2+\tfrac{1}{2}\rho_2\int \mathrm{d}\boldsymbol{r}_\perp\int_{\eta}^{\infty}\mathrm{d}z\,(\nabla\Phi_2)^2 \tag{6}$$

coincides with the total (kinetic) energy of the liquids. Such canonical variables were first introduced in ref. [13] for the case of one liquid with a free surface.

The potential Φ (and consequently $\nabla\Phi$), being the solution of the Laplace equation with boundary conditions, represents some functional of Ψ and η and can be determined with the help of the corresponding Green function.

Assuming $|\nabla\eta| \ll 1$ let us find an expansion of the Hamiltonian in a power series of canonical variables. We will restrict ourselves only to quadratic and cubic terms in the Hamiltonian. In order to find them it is convenient to rewrite H as an integral over the free surface, then to find the solution of the Laplace equations by means of the Fourier transform with respect to $\boldsymbol{r}_\perp = (x, y)$ and after that to perform the needed iterations. As a result, the Hamiltonian H in this approximation has the following form,

$$H = \tfrac{1}{2}(\rho_1+\rho_2)\left(\int \Psi\hat{k}\Psi \, \mathrm{d}\boldsymbol{r}_\perp + A\int [(\nabla\Psi)^2 - (\hat{k}\Psi)^2]\eta \, \mathrm{d}\boldsymbol{r}_\perp\right). \tag{7}$$

Here $\hat{k}$ is the integral operator with the difference kernel, whose Fourier transform is the modulus of the wave vector $\boldsymbol{k}$, and $A = (\rho_1-\rho_2)/(\rho_1+\rho_2)$. It is convenient now to renormalize Hamiltonian (7) and variables Ψ, η as follows,

$$\Psi = \frac{\rho_1+\rho_2}{A}\tilde{\Psi}, \quad \eta = \frac{1}{A}\tilde{\eta}, \quad H = \frac{\rho_1-\rho_2}{A^2}\tilde{H},$$

so that our problem transforms into that for one liquid with a free surface [14] (here and below we omit tildes),

$$H = \tfrac{1}{2}\int \Psi\hat{k}\Psi \, \mathrm{d}\boldsymbol{r}_\perp + \tfrac{1}{2}\int [(\nabla\Psi)^2 - (\hat{k}\Psi)^2]\eta \, \mathrm{d}\boldsymbol{r}_\perp . \tag{8}$$

The equations of motion (4), (5), corresponding to Hamiltonian (8), are

$$\frac{\partial\eta}{\partial t} = \hat{k}\Psi - [\hat{k}(\eta\hat{k}\Psi) + \nabla\cdot(\eta\nabla\Psi)], \tag{9}$$

$$\frac{\partial\Psi}{\partial t} = \tfrac{1}{2}[(\hat{k}\Psi)^2 - (\nabla\Psi)^2]. \tag{10}$$

The remarkable property of these equations is the separation of eq. (10), which involves only the variable Ψ, from that of (9), which governs the behavior of the elevation η. Such a separation is a peculiarity of the used perturbation order and is lost in the next orders, when η appears in eq. (10) as well. Since we assume $|\nabla\eta| \ll 1$, it is possible to omit the second term on the r.h.s. of eq. (9),

$$\frac{\partial\eta}{\partial t} = \hat{k}\Psi. \tag{11}$$

To study the dynamics of this system and for the sake of simplicity we will consider the one-dimensional case when the functions Ψ and η depend only on x (and t) and the operator $\hat{k}$ may be presented in the form

$$\hat{k} = -\frac{\partial}{\partial x}\hat{H},$$

where

$$(\hat{H}f)(x) = \frac{1}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty} \frac{f(x')}{x'-x}\,\mathrm{d}x'$$

is the Hilbert transform. By introducing a new function $v = \partial\Psi/\partial x$, which has the meaning of the tangent velocity on the interface, eqs. (10), (11) can be rewritten as

$$\frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x}[(\hat{H}v)^2 - v^2], \tag{12}$$

$$\frac{\partial\eta}{\partial t} = -\hat{H}v. \tag{13}$$

We exploit further the property of the Hilbert transform $\hat{H}$ that two operators $\hat{P}^\pm = \tfrac{1}{2}(1 \mp \mathrm{i}\hat{H})$ are projection operators. Namely, they decompose a function into the sum of two, $v = v^{(+)} + v^{(-)}$, with $v^{(\pm)} = \hat{P}^\pm v$ a function analytically extendable into the upper (lower) complex half-plane. Then, the Hilbert transform acts as follows,

$$\hat{H}v = \mathrm{i}(v^{(+)} - v^{(-)}). \tag{14}$$

Relation (14) should be substituted into both eq. (13) for η and eq. (12) for v. As a result, the latter decomposes into separate equations for the upper ($v^{(+)}$) and lower ($v^{(-)}$) analytical parts of v,

$$\frac{\partial v^{(\pm)}}{\partial t} + 2v^{(\pm)}\frac{\partial v^{(\pm)}}{\partial x} = 0. \tag{15}$$

Equations (15) look like those for the motion of a free particle and can be solved by the standard method of characteristics,

$$v^{(\pm)}=F^{(\pm)}(x_0)\,, \tag{16}$$

$$x=x_0+2F^{(\pm)}(x_0)t\,, \tag{17}$$

where the functions $F^{(\pm)}$ are defined from initial conditions. On the real axis the functions $v^{(\pm)}$ are complex conjugate, so it is enough to find a solution only for $v^{(+)}$, for example.

3. General solution

Let in (15) $F^{(+)}(x_0)$ be some analytical function in the upper half-plane of complex x_0 with its singularities in the lower half-plane. To find the solution of eqs. (15) one needs to resolve first eq. (17) with respect to x_0. The mapping $x\rightarrow x_0$ becomes ambiguous in the points where

$$\frac{\partial x}{\partial x_0}=1+2F^{(+)\prime}(x_0)t=0\,. \tag{18}$$

The solution of (18) gives some trajectory on the complex plane x_0: $x_0=x_0(t)$. The roots of (18) together with (16) define the corresponding movable branch points of the function $v^{(+)}(x, t)$,

$$z_{\rm br}(t)=x_0(t)+2F^{(+)}(x_0(t))t\,. \tag{19}$$

These points should be connected with a set of cuts, providing for the uniqueness of the function $v^{(+)}(x, t)$. The choice of these cuts has to be made in such a way that at the moment $t=0$ $v^{(+)}(x, t)$ would have the initial singularities. These movable branch points originate from the singularities of the function $F^{(+)}(z_0)$. At the moment the most "rapid" branch touches the real axis, the analyticity of $V^{(+)}(x, t)$ breaks down, and, respectively, a singularity appears in the solution of system (15).

First, define the touching time t_0 from the requirement that $z_{\rm br}$ is real, $z_{\rm br}=x_{\rm br}$. Assuming $\tau=t_0-t\ll t_0$, and considering a small vicinity of $z=x_{\rm br}$, expansion of (17) up to the leading order gives

$$F''t_0(\delta x_0)^2-2F'\tau\delta x_0-2F_0\tau-x'=0\,, \tag{20}$$

where $F''=F''(x_0(t_0))$, $\delta x_0=x_0-x_0(t_0)$, $x'=x-x_{\rm br}$, $F_0=F^{(+)}(z_0(t_0))$.

From this equation we find

$$x_0=x_0(t_0)+\frac{F'\tau}{F''t_0}+\sqrt{\left(\frac{F'\tau}{F''t_0}\right)^2+\frac{2F_0\tau+x'}{F''t_0}}\,. \tag{21}$$

If $F_0\neq 0$ the leading term in the square root is the linear one with respect to τ. Therefore with the needed accuracy

$$x_0=x_0(t_0)+C(x'+2F_0\tau)^{1/2}\,, \tag{22}$$

where $C=[F^{(+)\prime\prime}(x_0(t_0))]^{-1/2}$.

In the vicinity of $x=x_{\rm br}$ and $t=t_0$, such a general form of x_0 provides self-similar singular dependences for $\partial v/\partial x$ and η_{xx} which follow after substitution of (22) into (16) and forth integrating eq. (13). The first step gives for the tangent velocity with the same accuracy as for (22),

$$v=2\,{\rm Re}[F_0-(1/t_0)C(x'+2F_0\tau)^{1/2}]\,. \tag{23}$$

Hence we get for the first derivative of v,

$$\frac{\partial v}{\partial x}=-\frac{1}{t_0}{\rm Re}\left(\frac{C}{\sqrt{x'+2F_0\tau}}\right). \tag{24}$$

So, close to the touching time t_0, v_x behaves in a self-similar way, $x'\sim\tau$, increasing as $\tau^{-1/2}$. In the limit $\xi=x'/\tau\rightarrow\infty$ this function does not depend on τ,

$$\frac{\partial v}{\partial x}\sim|x'|^{-1/2}\,. \tag{25}$$

This means this profile is formed first at the periphery and then propagates to the center ($x'=0$), resulting in a singularity at $\tau=0$.

The curvature η_{xx} demonstrates the same self-similar behavior. In fact, the elevation η^+, governed by eq. (13), can be presented in the following form,

$$\eta^{(+)}=-{\rm i}\left(tF(x_0)-\int_x^{x_0(x,t)}\frac{(x-x_0)}{2F^{(+)}(x_0)}F'^{(+)}(x_0)\,{\rm d}x_0\right),$$

where the dependence $x_0(x, t)$ is defined by means of (17). Thereafter, differentiating $\eta^{(+)}$ with respect to x yields an explicit expression for η_x,

$$\eta_x={\rm Im}\log\frac{F^{(+)}(x)}{F^{(+)}(x_0)}\,. \tag{26}$$

This formula together with (22) leads to the same solution as that which we have obtained for v_x, η_{xx}

becomes infinite while approaching the singularity,

$$\eta_{xx}=\frac{1}{\tau^{1/2}}h(x'/\tau)\,, \tag{27}$$

where

$$h(\xi)=-\sqrt{2}\left(\frac{1+\sqrt{1+2\xi^2}}{1+2\xi^2}\right)^{1/2}.$$

At the critical moment $\tau=0$, η_{xx} looks like

$$\eta_{xx}\sim-|x|^{-1/2}\,, \tag{28}$$

which gives after integration the following behavior, $\eta\sim\frac{4}{3}|x|^{3/2}+$regular terms. In so doing both functions η and η_x remain finite at the singular point. The singularities, thus obtained, are the general ones for system (12) and (13).

Now, let us show how the general formulas work for a simple example when $F^{(+)}(x_0)$ is a rational function with one simple pole in the lower half-plane,

$$F^{(+)}(x_0)=\frac{A}{x_0+\mathrm{i}a}\,,$$

where $\mathrm{Re}\,a>0$. Then the dependence $x_0=x_0(x,t)$ can be readily found by means of (17),

$$x_0+\mathrm{i}a=\tfrac{1}{2}(x+\mathrm{i}a)+\sqrt{\tfrac{1}{4}(x+\mathrm{i}a)^2-2At}\,. \tag{29}$$

Thus, instead of the initial pole at the point $x=-\mathrm{i}a$ there appears a cut, connecting two moving branch points $x_{1,2}=-\mathrm{i}a\pm2\sqrt{2At}$.

The points $x_{1,2}(t)$ move (except for positive A) under some angle to the real axis. If, for instance, $A=-\frac{1}{8}$ and $a=1$, the cut spreads in the vertical direction axis and reaches the real axis at the moment of breaking $t=t_0=1$ at the point $x_{\mathrm{br}}=0$. In the vicinity of $\tau=0$ and $x=0$ expressions for $\partial v/\partial x$ and η_{xx} can be represented in the form

$$2\eta_{xx}=\frac{\partial v}{\partial x}$$
$$\approx-\frac{1}{\sqrt{4x^2+\tau^2}}\left[\tfrac{1}{2}\left(\tau+\sqrt{4x^2+\tau^2}\right)\right]^{1/2}. \tag{30}$$

Thus, at the critical moment $\tau=0$ the velocity derivative looks like

$$\frac{\partial v}{\partial x}\approx-\tfrac{1}{2}|x|^{-1/2}\,. \tag{31}$$

Evidently, formulas (30), (31) fully correspond to the general ones, (24), (27).

4. Wedges

Let us show that system (12), (13) has a special solution which describes another type of singularity. This solution arises if $F_0=0$. For this particular case formula (21) transforms into

$$x_0=z_0(t_0)+\frac{F'\tau}{F''t_0}+\sqrt{\left(\frac{F'\tau}{F''t_0}\right)^2+\frac{x'}{F''t_0}}$$

and, as a sequence, v can be approximately written in the form

$$v\approx[x_0-z_0(t_0)]F'\,. \tag{32}$$

Such dependence gives a new kind of self-similar behavior, $x\sim\tau^2$, which provides a surface singularity of wedge type. Indeed, when substituting (26) into (32) and considering the asymptotics of η_x for $x'/\tau^2\to\infty$, one gets

$$\eta_x\to-\tfrac{1}{4}\pi\,\mathrm{sgn}(x')\,,$$

which corresponds to a wedge surface profile with an angle $\alpha=2\arctan(4/\pi)\approx103.7°$. This angle is far from π and our assumption about small surface angles breaks down close to the singularity. However, the solution obtained above appears to be meaningful, because, first, the angle α is close to that calculated by Stokes for a critical stationary gravity surface wave on deep water and, second, self-similarity of the type $x\sim\tau^2$ is retained even by the complete system of equations (4), (5). It is worth noting that $F_0=0$ can be got from the initial conditions with two poles,

$$F^{(+)}(z)=\mathrm{i}\mu\left(\frac{a}{z+\mathrm{i}a}-\frac{a^*}{z+\mathrm{i}a^*}\right),$$

where $\mathrm{Re}\,a<0$, $\mathrm{Im}\,\mu=0$.

The dynamics of the branch points generated by these two poles is also interesting: at the initial moment of time the poles produce two pairs of branch points, two of which move towards the imaginary axis and collide; after collision the points move along the imaginary axis in opposite directions; the touching of the real axis by one of them produces the appearance of the singularity.

5. Floating singularities

A new type of singularities is associated with the possibility of exact integration of eq. (9) taking into account the second term on its r.h.s. With this aim let us separate from (9) the equation for $\eta^{(+)}(x,t)$,

$$\frac{\partial\eta^{(+)}}{\partial t}+2\hat{P}^{(+)}(v^{(-)}\eta^{(+)})_x=-\mathrm{i}\,\Psi_x^{(+)}\,. \qquad (33)$$

Introducing instead of $\eta^{(+)}$ a new function $\xi^{(+)}$ by means of $\eta^{(+)}=\partial\xi^{(+)}/\partial x$ and integrating (33) once one can get

$$\hat{P}^{(+)}\left(\frac{\partial\xi}{\partial t}+2v^{(-)}\frac{\partial\xi}{\partial x}\right)=-\mathrm{i}\,\Psi^{(+)}\,. \qquad (34)$$

Here ξ is a function for which $\hat{P}^{(+)}\xi=\xi^{(+)}$. Omitting then on both sides of (34) the operator $\hat{P}^{(+)}$, we arrive at the equation for ξ,

$$\frac{\partial\xi}{\partial t}+2v^{(-)}\frac{\partial\xi}{\partial x}=-\mathrm{i}\,\Psi^{(+)}+\Phi^{(-)}\,, \qquad (35)$$

where $\Phi^{(-)}$ is some lower analytical function (for which $\hat{P}^{(+)}\Phi^{(-)}=0$). This equation can be integrated along the characteristics defined by (17). The general solution to (35) consists of two parts, $\xi=\xi_1+\xi_2$, where

$$\xi_1=-\mathrm{i}\int_0^t \psi^{(+)}(x(x_0,t'),t')\,\mathrm{d}t' + \int_0^t \Phi^{(-)}(x(x_0,t'),t')\,\mathrm{d}t' \qquad (36)$$

is the solution of the inhomogeneous equation with zero initial condition, and $\xi_2=f(x_0)$ is that of the homogeneous one, presenting simply the initial shape of ξ. The effect of ξ_1 is defined by the analytical properties of the tangent velocity only, while that of ξ_2 results from the interference of the tangent velocity effect and the intrinsic peculiarities of the initial elevation $\eta_0(x)$.

Analyzing the first term, ξ_1, we first stress that the integration of the function $\Phi^{(-)}$ along characteristics (17) with forthcoming application of the operator $\hat{P}^{(+)}$ gives zero. It is enough, therefore, to integrate only $\Phi^{(+)}$ in (35). In fact, the situation is even simpler, because we are interested in the behavior of the solution only close to the moment t_0. Omitting details, we note only that taking into account the convective term in (33), as compared with the simplified equation (11), though giving rise to some additional motion, does not change, in fact, the character of singularity in the elevation ($\eta_{xx}\sim|x'|^{-1/2}$).

It is very important that the singularities obtained belong to the weak ones (see (28)), which do not destroy our basic assumption about small values of angles, $|\nabla\eta|\ll 1$. Note also that self-similar asymptotics of form (27) is admitted by the complete set of equations (4) and (5).

Of greater interest now is the homogeneous part of the solution $\xi_2=f(x_0)$ (not considered in the previous sections at all). The corresponding upper analytical part of the elevation $\eta^{(+)}$ is defined as

$$\eta_2^{(+)}(x,t)=(\partial\xi_2/\partial x)^{(+)}=\hat{P}^{+}\left(\frac{\partial x_0}{\partial x}\frac{\mathrm{d}f}{\mathrm{d}x_0}\right).$$

Since at the initial moment $t=0$, $x=x_0$, $\partial x_0/\partial x=1$, the function $\mathrm{d}f/\mathrm{d}x_0$ coincides with $\eta_0^{(+)}(x_0)$, where $\eta_0(x)$ is the initial form of the interface. The exact form of $\eta_2^{(+)}$ may be written as

$$\eta_2^{(+)}(x,t)=\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\mathrm{d}x'}{x'-x-\mathrm{i}0}\frac{\partial x_0(x',t)}{\partial x'}\eta_0^{(+)}(x_0)\,.$$

Passing to x_0 as a new variable of integration, this integral reduces to the form

$$\eta_2^{(+)}(x,t)=\frac{1}{2\pi\mathrm{i}}\int_C\frac{\mathrm{d}x_0}{x'(x_0,t)-x-\mathrm{i}0}\eta_0^{(+)}(x_0)\,, \qquad (37)$$

with x' and the contour C both defined from (17). The contour C initially coincides with the real axis, then, in time, it is deformed in such a way that it is partially in the lower half-plane. The motion of the contour C towards singular points of $\eta^{(+)}(x_0)$ will define obviously the behavior and the singularity formation of the function $\eta(x,t)$ for real x. To clarify this situation let us assume that $\eta^{(+)}(x_0)$ has one pole in the lower half-plane,

$$\eta^{(+)}(x_0) = \frac{iB}{x_0 - b},$$

where B is real, and $\mathrm{Im}\, b < 0$. Then integral (37) is found explicitly,

$$\eta_2^{(+)}(x, t) = \frac{iB}{x - x^+(b, t)} = \frac{iB}{x - b - 2F^{(-)}(b)t}.$$

It is clear from this expression that the pole of $\eta^{(+)}$ is movable with "velocity" $2F^{(-)}(b)$, which is some regular function. Therefore if $F_2 = \mathrm{Im}\, F^{(-)}(b) > 0$ then there exists a moment in time t_c at which $\eta_2(x, t)$ becomes infinite. Evidently $t_c = -b_2/2F_2(b)$, where $b_2 = \mathrm{Im}\, b$. Close to this time $\eta(x, t)$ has a Lorentz form,

$$\eta(x, t) = -\frac{B(b_2 + 2F_2 t)}{[x - b_1 - 2F_1(b)t]^2 + (b_2 + 2F_2 t)^2},$$

which transforms at $t = t_c$ into the δ-function,

$$\eta(x, t) = B\pi\delta(x - b_1 + b_2 F_1/F_2).$$

Thus, the proper singularities of the analytical function $\eta^{(+)}$, not generated by the velocity field and existing initially, remain during the time evolution and are movable. This statement can be readily checked for an arbitrary case, not only for poles. It gives a new type of singularities of the free surface, generally speaking, of an arbitrary kind, appearing due to the proper analytical properties of the initial profile of the elevation. What kinds of singularities will appear first depends on the initial conditions. If, for instance, the initial elevation is equal to zero then we get the first kind of singularities with a root character. One should pay attention to the fact that for the second kind of singularities, our assumption about small surface angles breaks down. Close to the time $t = t_c$ one should use the complete system (4), (5) rather than the reduced equations (9), (10).

6. Conclusion

In this paper we did not touch the question of the stability problem of the collapsing regimes. According to the analysis performed in section 4, the first regime of root character is obviously stable in the framework of the truncated system (12), (13). For the complete system, however, this is an open question, as well as for two other regimes. It should be emphasized again, that from the very beginning we assumed the angle of the surface ($|\nabla\eta|$) to be small, and therefore, we cannot pretend to the full description of all types of possible singularities, as described by the complete system of equations (4), (5). However, the solutions corresponding to the weak singularity regime turn out to be consistent with the applicability condition of the truncated equations (10), (11).

In our opinion, there exist two possibilities for the role the root singularities may play in the general dynamics; either the singularities serve as the origin of more powerful ones observed in numerical experiments or they represent a new type of singularities. One should note also that the self-similar asymptotics for the wedge type of singularities are allowed by the exact system of equations. We believe therefore that just this type of singularity was observed in numerical experiments [11] (see also ref. [10]).

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