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Hidden conservation laws in hydrodynamics; energy and dissipation rate fluctuation spectra in strong turbulence

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The Hamiltonian formulation of hydrodynamics in Clebsch variables is used for construction of a statistical theory of turbulence. The statistics of the Clebsch field is assumed to be close to Gaussian. It is shown that the energy spectrum consists of two ranges with $E(k) \approx k^{-5/3}$ and $E(k) \approx k^{-1}$. The spectrum of the dissipation rate fluctuations has three scaling regimes: $E^{\epsilon}(k) \approx k^{-1}$; $k^{-1/3}$ and k^{0} at the large, intermediate and small scales, respectively. The origin of the exponential distribution of velocity differences is discussed. The new scaling regime corresponds to a hidden conservation law, discovered in the Clebsch formulation of hydrodynamics. It is shown that viscous effects are responsible for production of the conserved quantity. The theoretical predictions are compared with results of numerical simulations of decaying turbulence.

1. Introduction

Strong hydrodynamic turbulence is a complex system characterized by an intricate interplay between order and chaos, between local and non-local interactions leading to formation of strongly anisotropic small-scale coherent structures resembling "spaghetti", worms, two-dimensional sheets, etc. Our understanding of this fascinating complexity is far from complete, though some of the features of turbulent flows are quite accurately described using field-theoretical methods. For example, the second order structure function $S_2(x)$ is represented by the Kolmogorov law:

$$S_2 \equiv \overline{(\Delta u)^2} \equiv \overline{[u(X) - u(X+x)]^2} = C_K \bar{\epsilon}^{2/3} x^{2/3} . \tag{1.1}$$

However, the scaling behaviour of higher-order structure functions remains something of a mystery. Numerous experiments indicate that:

$$S_n = \overline{[u(X) - u(X+x)]^n} \propto x^{\xi_n} , \tag{1.2}$$

where the scaling exponents ξ_n deviate from the Kolmogorov values $\xi_n = n/3$. Further, it was found that the larger the order n, the stronger the deviation ξ_n from the predictions of the Kolmogorov theory.

Another unsolved problem is the shape of the probability distribution function (PDF) of the velocity differences $P(\Delta u)$. It is well established that the single-point PDF is Gaussian. In other words, $P(\Delta u) \propto \exp[-(\Delta u)^2/u_{\mathrm{rms}}^2]$ for $x \gg L$, where L is the integral scale of turbulence. For separations x corresponding to the inertial range $l_{\mathrm{d}} \ll x \ll L$, where l_{d} is the dissipation scale, the experimentally observed $P(\Delta u)$ is:

$$P(\Delta u) \propto \exp(-\alpha|\Delta u|/u_{\mathrm{rms}}) \tag{1.3}$$

with the dimensionless coefficient $\alpha = \mathcal{O}(1)$. The relation (1.3) represents the most dominant feature of the PDF of velocity differences. The finer details and possible corrections to (1.3) will be discussed in section 5.

No less interesting is the behaviour of the fluctuations of the local value of the kinetic energy dissipation rate $\epsilon = \nu(\partial u_i/\partial x_j)^2$. It has been suggested [1] that the ϵ-fluctuations may be responsible for the deviations from the Kolmogorov theory, which predicts:

$$S_2^\epsilon = \overline{[\epsilon(X) - \epsilon(X+x)]^2} \propto x^0 \,. \tag{1.4}$$

The relation (1.4) has never been observed either in numerical or in physical experiments. Instead, observations suggest

$$S_2^\epsilon \propto x^{-\mu} \,, \tag{1.5}$$

with the "intermittency exponent" μ ranging from 0.1 to $\simeq 1$ depending on the experimental conditions, Reynolds numbers etc. The relation (1.5) shows that the dissipation rate of kinetic energy is concentrated in the localized areas of the space having the "spotty" nature. This is often interpreted as spacial intermittency of strong turbulence. Theoretical understanding of this behaviour is a major challenge. Accurate experimental verification of (1.5) is very difficult and at the present time we cannot even be sure that a scaling relation of the type (1.5) exists at all, though it is clear that experimentally observed $S_2^\epsilon(x)$ decreases with x contrary to (1.4). Existing theories, usually based on one-loop renormalized perturbation expansions, when properly regularized, give satisfactory predictions of large-scale-dominated features of turbulence such as the Kolmogorov relation (1.1), including the numerical value of the Kolmogorov constant C_K, energy transfer, turbulent transport, etc. It is only when we are interested in the properties of the high-order moments S_n or statistics and scaling behaviour of the velocity differences and velocity derivatives that simple one-loop approximations fail, being unable to describe the small-scale strongly non-local coherent phenomena.

The velocity and velocity derivative fields are extremely complex. So, it is natural to ask the following question: does there exist an underlying field connected to the original, Navier–Stokes (Euler) variables by a non-linear, maybe even non-local, transformation, which has relatively simple dynamical structure ("quarks of turbulence" according to A.A. Migdal). If this field does exist, then the problem of constructing a turbulence theory stems from an unfortunate choice of variables, not from the intrinsic inability of one-loop approximations to describe the small-scale intermittency manifested in the relations (1.2) and (1.5). Thus, it is a major challenge to find a set of variables yielding the experimentally observed properties of turbulence (1.1)–(1.3) and (1.5).

This goal is clearly too ambitious, since finding a set of transformations of original velocity field which exactly yields the properties of turbulence discussed above, amounts to the solution of the turbulence problem. However, something more modest, but still producing experimentally observed features of turbulent flows (1.1)–(1.5) can be done: it will be shown in this work that the description of turbulence in terms of Clebsch variables leads, even in the simple one-loop approximation, to many non-trivial results which cannot be derived directly from the Navier–Stokes equations. The most remarkable property of these variables, discovered in the middle of the last Century, is that, being canonical variables, they enable one to represent the Euler equations of ideal hydrodynamics in Hamiltonian form. In Clebsch variables the equations of motion become even more non-linear, which makes the problem less tractable. However, it will be clear below that these equations have the conservation laws which are hidden in the conventional Navier–Stokes hydrodynamics. We will show that some of the most interesting features of turbulence correspond to these new conservation laws.

The main results of this paper can be summarized as follows: the one-loop renormalized perturbation expansion of the Euler equations, written in the Clebsch variables, gives many experimentally observed properties of turbulence such as: the Kolmogorov spectrum and another k^{-1}-energy range observed in several numerical simulations of decaying turbulence. This second inertial range corresponds to the inverse cascade of the hidden integral of motion discovered in the Clebsch description of hydrodynamics. We shall also demonstrate that the simplest assumption about the nature of statistics of the Clebsch fields leads to an understanding of the origin of the exponential distribution function of velocity differences. In addition, the properties of the dissipation rate correlation functions will be derived. All results are obtained using the ideas and methods of the Hamiltonian formulation of hydrodynamics, produced during last two decades by Zakharov and coworkers [2–5], combined with a mean field approximation and a new understanding of the central role of viscosity in the Clebsch formulation of the Navier–Stokes equations, developed in this work.

2. The model

We consider a fluid flow driven at the very large scales $l \gg L \to \infty$. Somewhere at the smallest scales $l \to 0$ an energy sink is assumed, so that a statistically steady state can be achieved. It is customary to describe the dynamics of the intermediate scales by the Euler equation (the density $\rho = 1$):

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p \,,$$

$$\nabla \cdot \boldsymbol{v} = 0 \,. \tag{2.1}$$

The Clebsch variables are defined as:

$$\boldsymbol{v} = \lambda \nabla \mu + \nabla \phi \,. \tag{2.2}$$

Using the incompressibility condition, the potential ϕ can be expressed through λ and μ:

$$\phi = -\nabla^{-2} \nabla \cdot (\lambda \nabla \mu) \,,$$

and thus:

$$\boldsymbol{v} = -\nabla^{-2} \nabla \times (\nabla \lambda \times \nabla \mu) \,, \tag{2.3}$$

$$\boldsymbol{\omega} = \nabla \lambda \times \nabla \mu \,. \tag{2.4}$$

The Clebsch variables are transported by the flow and the Euler equation can be represented as:

$$\mathcal{D}\mu = \frac{\partial \mu}{\partial t} + \boldsymbol{v} \cdot \nabla \mu = 0 \,;$$

$$\mathcal{D}\lambda = \frac{\partial \lambda}{\partial t} + \boldsymbol{v} \cdot \nabla \lambda = 0 \,. \tag{2.5}$$

It follows from eq. (2.3) that the velocity field does not uniquely define the Clebsch field ($\lambda(x, t)$, $\mu(x, t)$). In fact, a set of pairs of the Clebsch variables ($\lambda_i(x, t)$, $\mu_i(x, t)$) can be used to express the velocity $\boldsymbol{v}(x, t)$:

$$\boldsymbol{v} = \sum_{i=1}^{M} \lambda_i \nabla \mu_i + \nabla \phi \,, \tag{2.6}$$

and

$$\boldsymbol{\omega} = \sum_{i=1}^{M} \nabla \lambda_i \times \nabla \mu_i \,, \tag{2.7}$$

where M is the number of Clebsch pairs necessary for the complete representation of velocity field. The equations of motion for each pair (λ_i, μ_i), given by (2.5) with the subscript i specifying the pair, can be written in a Hamiltonian form since λ and μ are canonical variables. The Hamiltonian and corresponding equations of motion are given below. The minimal number of canonical pairs needed to describe an arbitrary flow depends on the topology of the field $\boldsymbol{v}$. It is a plausible conjecture that $M=2$ is sufficient to represent a wide class of turbulent flows. Indeed, the velocity field in a three-dimensional incompressible flow has two independent components. This field, however, cannot be described by one pair of Clebsch variables due to the constraint $\boldsymbol{v}\cdot\boldsymbol{\omega}=0$ which tells us that, in fact, we have only one independent Clebsch variable. Introducing the second Clebsch pair we create two independent variables, sufficient for the description of the general velocity field with the non-zero values of the local helicity. In this paper we will discuss only the case of $M=1$ which corresponds to zero helicity $\int \boldsymbol{v}\cdot\boldsymbol{\omega}\,\mathrm{d}x \equiv 0$ which follows directly from the definition (2.4). This restricts applicability of the analysis to flows in which the vortex lines do not have any knots. However, the results of this work may be readily generalized to the case of $M=2$ corresponding to an arbitrary topology.

Introducing the complex variables $a(\boldsymbol{k})$ and $a^*(\boldsymbol{k})$

$$\mu(\boldsymbol{k}) = \frac{1}{\sqrt{2}}\,[a(\boldsymbol{k}) + a^*(-\boldsymbol{k})]\,,$$
$$\lambda(\boldsymbol{k}) = \frac{\mathrm{i}}{\sqrt{2}}\,[a(\boldsymbol{k}) - a^*(-\boldsymbol{k})] \tag{2.8}$$

the Euler equation can be written in a Hamiltonian form:

$$\mathrm{i}\,\frac{\partial a(\boldsymbol{k})}{\partial t} = \frac{\delta H}{\delta a^*(\boldsymbol{k})}\,, \tag{2.9}$$

where the Hamiltonian H is:

$$H = \frac{1}{4}\int T_{12,34}a^*(\boldsymbol{k}_1)\,a^*(\boldsymbol{k}_2)\,a(\boldsymbol{k}_3)\,a(\boldsymbol{k}_4) \times \delta(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3\,\mathrm{d}\boldsymbol{k}_4\,. \tag{2.10}$$

The interaction potential is

$$T_{12,34} \equiv T(\boldsymbol{k}_1\boldsymbol{k}_2,\,\boldsymbol{k}_3\boldsymbol{k}_4) = \varphi_{13}\varphi_{24} + \varphi_{14}\varphi_{23}\,, \tag{2.11}$$

where

$$\varphi(\boldsymbol{k}_1,\boldsymbol{k}_2) \equiv \varphi_{12} = \boldsymbol{k}_1 + \boldsymbol{k}_2 - (\boldsymbol{k}_1-\boldsymbol{k}_2)\,\frac{k_1^2-k_2^2}{|\boldsymbol{k}_1-\boldsymbol{k}_2|^2}\,. \tag{2.12}$$

In these variables:

$$\boldsymbol{v} = \int \varphi_{\boldsymbol{q},\boldsymbol{k}-\boldsymbol{q}}\,a^*_{\boldsymbol{q}}a_{\boldsymbol{k}-\boldsymbol{q}}\,\mathrm{d}\boldsymbol{q}\,. \tag{2.13}$$

The function $\varphi(\boldsymbol{k}_1,\,\boldsymbol{k}_2)$ is a discontinuous function at $\boldsymbol{k}_1=\boldsymbol{k}_2$ since the diagonal elements of $\varphi(\boldsymbol{k},\,\boldsymbol{k})$ determine an arbitrary mean velocity in the flow $\boldsymbol{v}(\boldsymbol{k}=0)$. So, in what follows we set $\varphi(\boldsymbol{k},\,\boldsymbol{k})=0$.

Substituting (2.10)–(2.12) into (2.9) the equation of motion for the "creation-annihilation" operators $a(\boldsymbol{k})$ is readily derived:

$$\mathrm{i}\,\frac{\partial a(\boldsymbol{k})}{\partial t} = \frac{1}{2}\int T(\boldsymbol{k}\boldsymbol{k}_2,\,\boldsymbol{k}_3\boldsymbol{k}_4)\,a^*(\boldsymbol{k}_2)\,a(\boldsymbol{k}_3)\,a(\boldsymbol{k}_4) \times \delta(\boldsymbol{k}+\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4)\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3\,\mathrm{d}\boldsymbol{k}_4\,. \tag{2.14}$$

Eq. (2.14) conserves total energy, since it is a Hamiltonian equation of motion. In addition, they conserve an infinite number of integrals of motion $\int F(\lambda,\,\mu)\,\mathrm{d}\boldsymbol{r} = \text{constant}$. These integrals do not have simple interpretation in terms of the velocity field. In the present paper we concentrate only on one of the integral of motion:

$$N = \frac{1}{2}\int (\lambda^2+\mu^2)\,\mathrm{d}x = \int a^*(\boldsymbol{k})\,a(\boldsymbol{k})\,\mathrm{d}\boldsymbol{k} = \text{constant}\,. \tag{2.15}$$

The parameter N has the dimensionality of action and can be called the "hydrodynamic action" or number of quasi-particles (elementary excitations) describing turbulent flow. The relation (2.15) has the most important impact on what follows, so the elucidation of the physical meaning of the "quasi-particles" or waves and of the topological consequences of this conservation law remains a very important task.

3. One-loop approximation; kinetic equation

Let us single out the diagonal contributions to the equation of motion (2.14):

$$\mathrm{i}\,\frac{\partial a(\boldsymbol{k})}{\partial t} - \omega(\boldsymbol{k})\,a(\boldsymbol{k}) = \int' T(\boldsymbol{k}, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4)\,a^*(\boldsymbol{k}_2)\,a(\boldsymbol{k}_3)\,a(\boldsymbol{k}_4) \times \delta(\boldsymbol{k} + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4)\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3\,\mathrm{d}\boldsymbol{k}_4\,, \tag{3.1}$$

where

$$\omega(\boldsymbol{k}) = \int T(\boldsymbol{k}\boldsymbol{k}_2, \boldsymbol{k}\boldsymbol{k}_2)\,a^*(\boldsymbol{k}_2)\,a(\boldsymbol{k}_2)\,\mathrm{d}\boldsymbol{k}_2\,, \tag{3.2}$$

and the symbol $'$ in the integral in (3.1) means that the diagonal contributions with $\boldsymbol{k} = \boldsymbol{k}_3$, $\boldsymbol{k}_2 = \boldsymbol{k}_4$ are not included. It will be shown in what follows that the integral

$$\bar{\omega}(\boldsymbol{k}) = \int T(\boldsymbol{k}, \boldsymbol{k}_2, \boldsymbol{k}, \boldsymbol{k}_2)\,n(\boldsymbol{k}_2)\,\mathrm{d}\boldsymbol{k}_2\,, \tag{3.3}$$

with

$$n(\boldsymbol{k}) = \langle a^*(\boldsymbol{k})\,a(\boldsymbol{k})\rangle \tag{3.4}$$

converges when calculated on the solutions $n(\boldsymbol{k})$ of the kinetic equation derived below. This means that the main contribution to (3.3) comes from the region $\boldsymbol{k} \simeq \boldsymbol{k}_2$. In this work we are interested in statistically steady solutions $n(\boldsymbol{k})$, so $\omega(\boldsymbol{k}) = \text{constant}$ is time-independent. Thus, we introduce the mean-field approximation [6]:

$$\mathrm{i}\,\frac{\partial a(\boldsymbol{k})}{\partial t} - \bar{\omega}(\boldsymbol{k})\,a(\boldsymbol{k}) = S \tag{3.5}$$

where the collision integral $S(\boldsymbol{k})$ is defined by the right side of eq. (3.1). In the zeroth order of the expansion in powers of the non-linear interaction S we have:

$$a^0(\boldsymbol{k}, t) = a(\boldsymbol{k})\,\mathrm{e}^{-\mathrm{i}\omega(\boldsymbol{k})t}\,. \tag{3.6}$$

The bar over $\bar{\omega}(\boldsymbol{k})$ defined by (3.3) is omitted in what follows. The statistical ensemble can be constructed by introducing an infinite set of realizations differing in the values of the initial phases $\varphi(\boldsymbol{k})$ in (3.5):

$$a^0(\boldsymbol{k}, t) = |a(\boldsymbol{k})|\,\mathrm{e}^{\mathrm{i}\omega(\boldsymbol{k})t + \mathrm{i}\varphi(\boldsymbol{k})}\,. \tag{3.7}$$

We assume further that all phases $\varphi(\boldsymbol{k})$ uncorrelated, i.e.:

$$\langle a(\boldsymbol{k})\rangle = \langle |a(\boldsymbol{k})|\,\mathrm{e}^{\mathrm{i}\varphi(\boldsymbol{k})}\rangle = 0\,,$$

$$\langle a(\boldsymbol{k})\,a(\boldsymbol{k}')\rangle = \langle |a(\boldsymbol{k})|\,|a(\boldsymbol{k}')|\,\exp[\mathrm{i}(\varphi(\boldsymbol{k}) + \varphi(\boldsymbol{k}'))]\rangle = 0\,,$$

$$\langle a(\boldsymbol{k})\,a^*(\boldsymbol{k}')\rangle = \langle |a(\boldsymbol{k})|\,|a(\boldsymbol{k}')|\,\exp[\mathrm{i}(\varphi(\boldsymbol{k}) - \varphi(\boldsymbol{k}'))]\rangle = n(\boldsymbol{k})\,\delta(\boldsymbol{k} - \boldsymbol{k}')\,,$$

All odd-order correlation functions of the fields $a(\boldsymbol{k})$ are equal to zero in this random phase approximation (RPA). As was mentioned above the averaging is performed over the ensemble of initial phases $\varphi(\boldsymbol{k})$.

The RPA, introduced above, assumes gaussian statistics for the zero-order solution (3.7). This does not mean that the solution of the full non-linear problem (3.5) is a Gaussian random field. In what follows we assume that the non-linear interaction is weak and consider the one-loop approximation only, which is the simplest possible case. Still, it will be shown that, despite the simplicity of the assumed dynamics of the Clebsch fields, the dynamical picture corresponding to the velocity field is extremely complex.

To derive equations of motion for the "occupation numbers" $n(\boldsymbol{k})$, let us multiply (3.4) and the corresponding equation for $a^*(\boldsymbol{k})$ by $a^*(\boldsymbol{k})$ and by $a(\boldsymbol{k})$, respectively. Then, the equation of motion for $n(\boldsymbol{k})$ reads:

$$\frac{\partial n(\boldsymbol{k}, t)}{\partial t} = \mathrm{Im} \int T_{\boldsymbol{k}\boldsymbol{k}_2,\boldsymbol{k}_3\boldsymbol{k}_4} J_{\boldsymbol{k}\boldsymbol{k}_2,\boldsymbol{k}_3\boldsymbol{k}_4} \times \delta(\boldsymbol{k} + \boldsymbol{k}_1, -\boldsymbol{k}_2 - \boldsymbol{k}_3)\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3\,\mathrm{d}\boldsymbol{k}_4\,, \tag{3.8}$$

where

$$J_4 = J_{kk_2,k_3k_4} = \langle a^*(k)\, a^*(k_2)\, a(k_3)\, a(k_4)\rangle \,. \quad (3.9)$$

Writing the equation of motion for J_4 as:

$$\frac{\partial J_4}{\partial t} = \left\langle \frac{\partial}{\partial t}\,(a^*(k)\, a^*(k_2)\, a(k_3)\, a(k_4)) \right\rangle , \quad (3.10)$$

and expressing the time-derivatives in (3.10) using (3.4) we obtain in the random-phase approximation:

$$\frac{\partial n(k,t)}{\partial t} = \frac{\pi}{2}\int |T_{kk_2,k_3k_4}|^2 J_4\, \delta(k + k_2 - k_3 - k_4) \times \delta(\omega(k) + \omega(k_2) - \omega(k_3) - \omega(k_4))\, dk_2\, dk_3\, dk_4 \,, \quad (3.11)$$

with:

$$J_4 = n_3 n_4 (n_2 + n_k) - n_2 n_k (n_3 + n_4) \quad (3.12)$$

Here $n(k_i) = n_i$.

The kinetic equation (3.10)–(3.11) has been analyzed and solved by Zakharov [4]. It has been shown that if

$$\omega(k) \propto k^\alpha \,, \quad (3.13)$$

there exist four scaling solutions:

$$n(k) = \text{constant}\,, \quad n(k) \propto \frac{1}{\omega(k)}\,, \quad (3.14)$$

and

$$n(k) \propto k^{-x}\,, \quad (3.15)$$

with

$$x_1 = \frac{4}{3} + d\,, \qquad x_2 = \frac{4-\alpha}{3} + d\,. \quad (3.16)$$

The solutions (3.14) correspond to a fluid in thermodynamic equilibrium, while the relations (3.15), (3.16) describe a non-equilibrium flow. From the definition of $\omega(k)$ given by (3.3) we find readily:

$$\alpha = -x + d + 2 = -x + 5 \quad (d = 3)\,, \quad (3.17)$$

and the expressions for $n(k)$ can be obtained in a closed form:

$$n \propto k^{-13/3}\,, \qquad \alpha = 2/3\,, \quad (3.18)$$

and

$$n \propto k^{-4}\,, \qquad \alpha = 1\,. \quad (3.19)$$

It can be checked easily that the total energy can be evaluated from the following relation:

$$E = \frac{1}{2}\int \omega(k)\, n(k)\, dk\,, \quad (3.20)$$

which defines the energy spectra in terms of the Clebsch variables:

$$E(k) = 2\pi k^2 \omega(k)\, n(k)\,.$$

The relations (3.18) and (3.19) generate two solutions:

$$E(k) \propto \epsilon^{2/3} k^{-5/3}\,, \quad (3.21)$$

and

$$E_n(k) \propto P k^{-1}\,, \quad (3.22)$$

where P denotes the "particle" flux in the wave-number space. It has also been shown [2] that, while the energy flux is positive, i.e., the energy is cascading from the largest to the smallest scales, the flux of particles is in the opposite direction: from small to large scales (inverse cascade). The importance of this fact will be discussed below. Thus, as follows from (3.21) and (3.22), the small- and large-scale dynamics in turbulent flows are characterized by two different energy spectra. This is a completely new development which is discovered due to our use of the Clebsch variables. It is clear that the energy spectrum (3.22) can readily be obtained from dimensional considerations which do not require one-loop approximation and kinetic equation (3.10)–(3.11). However, in this case we have to postulate existence of the scaling regimes corresponding to constant fluxes in wave-number space of each conserved quantity. Then, the particle and energy conservation laws lead immediately to (3.21) and (3.22). To conclude this section we would like to reiterate that the assumption about close-to-Gaussian statistics of the Clebsch variables cannot be easily justified. It is interesting, however, that such an assumption leads to so many experimentally observed facts. It has to be stressed that the gaussianity of the Clebsch field does not imply the gaussianity of the velocity field which is known to be grossly incorrect. Moreover, it will be shown below that this assumption leads to the experimentally observed exponential distribution of the velocity differences.

4. Energy spectra in turbulent flow

The energy spectrum (3.22) corresponds to the flux in wave-number space of the integral of motion (2.13) resembling the total number of quasi-particles or elementary excitations in condensed matter physics. The physical meaning of these particles or waves is not clear since we do not have a representation of (2.13) in terms of observables. Moreover, we even dò not have conclusive ideas about the source and dissipation mechanisms contributing to the dynamics of these excitations. However, knowing the direction of the flux in wave-number space, we can assume that the particles are created at the smallest scales $l \to 0$. By the construction of the model, the range of scales $l \to 0$ corresponds to the energy sink, necessary for maintenance of a statistically steady state. Thus, the particle production in the Navier–Stokes hydrodynamics is dominated by molecular viscosity ν. The only additional parameter of the problem is the power of the external source which in the statistically steady state is equal to the dissipation rate ϵ. On the basis of these considerations we come to an assumption that the energy sink is the source where the particles are born. This statement will be verified below. In the Navier–Stokes description the small-scale viscous effects leading to the energy dissipation, simultaneously generate and "emit" these particles (waves). Then, the nonlinear interaction leads to the formation of the large-scale excitations $a(k)$ by an inverse cascade mechanism.

From dimensional considerations we have:

$$E_n(k) = c_n(\epsilon\nu)^{1/2}k^{-1} , \tag{4.1}$$

where c_n is the second Kolmogorov constant.

To verify the qualitative reasoning leading to (4.1) let us represent the Navier–Stokes dynamics in the Clebsch variables. We consider the equations of motion

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\nabla p + \nu\,\nabla^2\boldsymbol{v} + \boldsymbol{F} ,$$

where the large-scale stirring force $|\boldsymbol{F}(k, t)|^2 = 0$ for all $k > 1/L \to 0$. The vorticity equation is:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla\times(\boldsymbol{v}\times\boldsymbol{\omega}) = \nu\,\Delta\boldsymbol{\omega} , \tag{4.2}$$

where $(\nabla\times\boldsymbol{F})^2 \to 0$ has been neglected as small. Recalling the definition of vorticity in terms of the Clebsch variables (2.4) we have:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla\times(\boldsymbol{v}\times\boldsymbol{\omega}) = \nabla\lambda\times\nabla\mathscr{D}\mu + \nabla\mathscr{D}\lambda\times\nabla\mu , \tag{4.3}$$

where $\mathscr{D}\lambda$ and $\mathscr{D}\mu$ are defined in (2.5). Comparing (4.3) with (4.2) leads to:

$$\nabla\lambda \times \nabla\mathscr{D}\mu + \nabla\mathscr{D}\lambda \times \nabla\mu = \nu\,\Delta\boldsymbol{\omega}\,. \tag{4.4}$$

Calculating the scalar product of (4.4) with $\nabla\mu$ gives:

$$(\nabla\mu \times \nabla\lambda)\cdot\nabla\mathscr{D}\mu = -\boldsymbol{\omega}\cdot\nabla\mathscr{D}\mu = \nu\,\nabla\mu\cdot\Delta\boldsymbol{\omega}\,. \tag{4.5}$$

This relation can be simplified since:

$$\boldsymbol{\omega}\cdot\nabla\mathscr{D}\mu = \omega\,\frac{\partial}{\partial l}\,\mathscr{D}\mu\,, \tag{4.6}$$

where the derivative $\partial/\partial l$ is taken along the vortex line. This leads to:

$$\mathscr{D}\mu = \frac{\partial\mu}{\partial t} + \boldsymbol{v}\cdot\nabla\mu = -\nu\int_{-\infty}^{r}\frac{\nabla\lambda\cdot\Delta\boldsymbol{\omega}}{\omega}\,\mathrm{d}l\,, \tag{4.7}$$

and

$$\mathscr{D}\lambda = \frac{\partial\lambda}{\partial t} + \boldsymbol{v}\cdot\nabla\lambda = -\nu\int_{-\infty}^{r}\frac{\nabla\mu\cdot\Delta\boldsymbol{\omega}}{\omega}\,\mathrm{d}l\,. \tag{4.8}$$

In the a-variables we thus have

$$\frac{\partial a(\boldsymbol{k})}{\partial t} + \mathrm{i}\,\frac{\partial H}{\partial a^*(\boldsymbol{k})} = -\nu f(\boldsymbol{k})\,, \tag{4.9}$$

where $f(\boldsymbol{k})$, the Fourier transform of the function

$$f(r) = \int_{-\infty}^{r}\frac{\nabla a\cdot\Delta\boldsymbol{\omega}}{\omega}\,\mathrm{d}l \tag{4.10}$$

describes the source of the particles (waves). Simple power counting gives for $f(\boldsymbol{k})$

$$f(\boldsymbol{k}) \propto k^2\xi a(\boldsymbol{k})\,,$$

where the random function $\xi = \mathscr{O}(1)$ comes from the ratio $\Delta\boldsymbol{\omega}/\omega$. This relation tells us that the viscous term in the Navier–Stokes equation corresponds to a random source driving the Clebsch field. Then:

$$F(\boldsymbol{k}) = \nu^2\,\frac{\langle f(\boldsymbol{k})\,f^*(\boldsymbol{k}')\rangle}{\delta(\boldsymbol{k}-\boldsymbol{k}')} \approx \nu^2k^4n_k\,, \tag{4.11}$$

where $F(\boldsymbol{k})$ is the source correlation function which generates the corresponding production term in the kinetic equation:

$$\frac{\partial n}{\partial t} \approx \frac{\nu^2F_k}{\omega_k} \approx \frac{\nu^2}{k}\,. \tag{4.12}$$

The relation

$$\omega_k \approx k^5n_k\,,$$

following from (3.3) was used in derivation of (4.12).

The relation (4.12) is defined on the interval $k < k_n$ where k_n is the dissipation cut-off of the k^{-1}-spectrum. The production rate of the quasi-particles is:

$$P \approx \frac{\partial}{\partial t}\int n\,\mathrm{d}k \approx \nu^2k_n^2\,. \tag{4.13}$$

The energy spectrum at the scales larger than the dissipation scale is:

$$E_k = c_{\mathrm{K}}\epsilon^{2/3}k^{-5/3} + c_nPk^{-1}\,. \tag{4.14}$$

Here c_{K} and c_n are the Kolmogorov constants. The value of k_n can be found from the energy balance equation

$$\int \nu k^2E_k\,\mathrm{d}k = \epsilon\,. \tag{4.15}$$

If $c_{\mathrm{K}} \approx c_n \approx 1$ both terms in (4.14) have the same order at $k \sim k_n \sim k_{\mathrm{d}}$. Here $k_{\mathrm{d}} \approx \epsilon^{1/4}/\nu^{3/4}$ is the Kolmogorov scale. In this case the Kolmogorov spectrum $\epsilon^{2/3}k^{-5/3}$ prevails over the entire inertial range $k < k_{\mathrm{d}}$. But if for some reason $c_n \gg c_{\mathrm{K}}$,

$k_n \approx c_n^{-1/4} k_d$ and $P \approx c_n^{-1/2}(\epsilon\nu)^{1/2}$. In this case in the region $c_n^{-3/4} < k/k_d < c_n^{-1/4}$ the inverse cascade k^{-1}-spectrum dominates.

Now we can make the following prediction. Consider stationary Kolmogorov turbulence driven by a large-scale force. This flow is dominated by the 5/3-spectrum (3.20). At time $t = 0$ let us turn off the energy source and follow the process of turbulence decay. In this case the energy is transferred to the small scales and is dissipated there into heat. At the same time the source of waves (viscosity) keeps supporting the k^{-1}-spectrum. As a result the spectrum (4.1) must dominate the later stages of turbulence decay.

Analyzing the existing literature on high-resolution direct numerical simulations (DNS) of isotropic homogeneous turbulence we were able to find ample evidence of the existence of the k^{-1}-energy spectrum at the later stages of turbulence decay. Fig. 1 shows the results of simulations performed by Panda and coworkers [7]. The microscale Reynolds number in these simulations was $R_\lambda \approx 60$ and 256^3 Fourier modes were accurately resolved. The wide range of $E(k) \approx k^{-1}$ is clearly seen at the later stages $(1.6 < t < 4)$ of the decay process. The Kolmogorov range is hardly noticeable in fig. 1. The k^{-1}-spectrum was also observed by Jackson and She [8] who studied turbulence decay. The theoretical possibility of a k^{-1}-spectrum was discussed by Orszag [9], who analyzed the one-loop Dyson equations for the propagator G and the velocity correlation function U. The spectrum is obtained if it is assumed that the response function G is dominated by viscous effects (negligibly small non-linearity) while the equation for the correlation function is governed exclusively by non-linear effects. However, examining the Dyson equations leads to the conclusion that these assumptions yielding the k^{-1}-energy spectrum appear to be dynamically inconsistent and this is the reason why this range was never found in numerical solutions of the regularized one-loop Dyson equations.

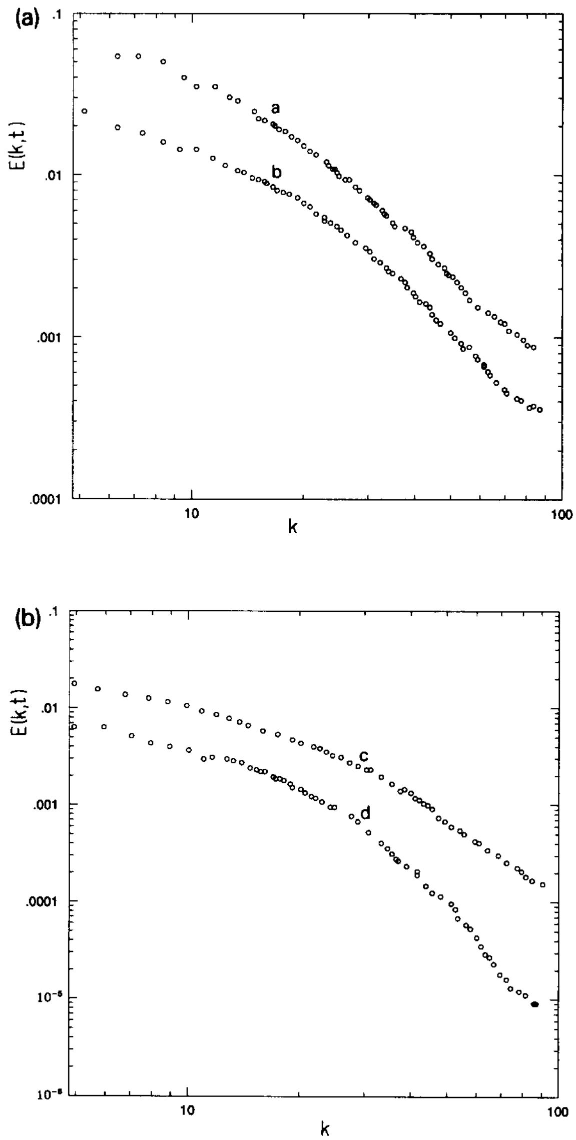

Fig. 1. The energy spectra in decaying turbulences [7]. a, b, c, d correspond to dimensionless times $t = 0.08$; 1.6; 3; 4 respectively.

5. Probability distribution of the velocity differences

Let us consider the probability density of the velocity difference

$$U(x, r) = u(x) - u(x + r),$$

where u is the x-component of the velocity field and r is the displacement from the point x. If r denotes displacement in the x-direction, then it is well known the odd moments U are not equal to zero and the PDF $P(U) \neq P(-U)$. However this asymmetry is not very strong and it is absent when r denotes displacement in the direction perpendicular to the axis x. In this section we shall discuss the origins of the exponentially observed exponential PDF of the velocity differences, leaving more accurate theory to future publications. It is clear that in the random phase approximation:

$$\overline{U^2} = \int \varphi^2_{k_1,k_2} n(k_1)\, n(k_2) \times \{2 - 2\cos[(k_1 - k_2)r]\}\, \mathrm{d}k_1\, \mathrm{d}k_2 \,. \tag{5.1}$$

If $k_1 \gg k_2$ then

$$\varphi^2_{k_1,k_2} \sim k_2^2 \,, \tag{5.2}$$

which makes the ultra-violet convergence of the integral (5.1) obvious. To consider the infra-red properties of the integral let us set $\beta k_1 = k_2$ with $\beta \ll 1$. In this case $\varphi^2_{12} \sim k_2^2$ and the contribution to the integral from the interval $0 < k_2 < \beta k_1$ is estimated readily:

$$\int_0^\infty n(k_1)(2 - 2\cos k_1 r)\, \mathrm{d}k_1 \int_0^{\beta k_1} n(k_2)\, k_2^2\, \mathrm{d}k_2 \simeq \beta^{2/3} \int_0^\infty k_1^{-11/3}(2 - 2\cos k_1 x)\, \mathrm{d}^3 k_1 \,.$$

This integral converges in both limits. Thus, the main contribution to (5.1) comes from the interval where $k_1 \simeq k_2$. The probability distribution function $P(U)$ is defined as:

$$P(U) \propto \int \delta[U - u(x) + u(x + r)]\, \mathrm{d}x \,, \quad \text{or}$$

$$P(U) \propto \int_{-\infty}^{\infty} \mathrm{d}\alpha\, \mathrm{e}^{\mathrm{i}\alpha U} \int \mathrm{e}^{-\mathrm{i}\alpha U(x,r)}\, \mathrm{d}x \,.$$

Introducing the ensemble of fluctuating Clebsch fields and assuming that the space and ensemble averaging procedures are equivalent we have:

$$P(U) = \int_{-\infty}^{\infty} \mathrm{d}\alpha\, \mathrm{e}^{\mathrm{i}\alpha U} \langle \mathrm{e}^{-\mathrm{i}\alpha U(0,r)} \rangle \,. \tag{5.3}$$

Assuming further the Gaussian statistics of the Clebsch field:

$$I = \langle \mathrm{e}^{-\mathrm{i}\alpha U(0,r)} \rangle = \int \mathrm{d}a_{k_1}\, \mathrm{d}a^*_{k_2} \frac{1}{\pi\, n(k_i)} \times \exp\left[\int \mathrm{d}k_1\, \mathrm{d}k_2 \left(-\mathrm{i}\alpha\varphi_{k1,k2}(\mathrm{e}^{\mathrm{i}(k_1 - k_2)r} - 1) - \frac{\delta(\boldsymbol{k}_1 - \boldsymbol{k}_2)}{n(k_1)} \right) a_{k_1} a^*_{k_2} \right] .$$

The integral can be formally evaluated with the result:

$$I = \frac{1}{\operatorname{Det} A_{k_1,k_2}} \,,$$

where the matrix

$$A_{k_1,k_2} = \left(-\mathrm{i}\alpha\varphi_{1,2}(\mathrm{e}^{\mathrm{i}(k_1 - k_2)r} - 1) - \frac{\delta(\boldsymbol{k}_1 - \boldsymbol{k}_2)}{n(k_1)} \right) \prod n(k_{1i}) \,.$$

The determinant is equal to:

$$\operatorname{Det} A_{k_1,k_2} = \operatorname{tr}(\ln \mathrm{e}^{A_{k_1,k_2}}) \,.$$

The diagonal elements $A_{k,k} = n(k)^{-1}$. Expanding the expression for the Det A_{k_1,k_2} in powers of $k_1 - k_2$ we derive in the first non-vanishing order:

$$\langle \mathrm{e}^{-\mathrm{i}\alpha U(r)} \rangle \simeq \frac{1}{1 + \alpha^2 \overline{U^2}} \,, \tag{5.4}$$

with $\overline{U^2}$ given by (5.1). Substituting (5.4) into (5.3) and evaluating simple integral we have:

$$P(U) \propto \exp(-U/U_{\mathrm{rms}}) \tag{5.5}$$

where root-mean-square $U_{\mathrm{rms}} = (\overline{U^2})^{1/2}$. This result does not take into account the higher order in powers of $k_1 - k_2$ terms. The same answer is obtained if the contribution to the matrix element $a_{k_1,k_2} = \varphi_{1,2}(2 - 2\cos[k_1 - k_2]r)$ only weakly depends on the wave-vectors k_1, k_2. In this case the result (5.4) is exact to all orders of the perturbation expansion. The approximate expressions (5.4) and (5.5) demonstrate the origin of the exponential probability distributions in turbulence. Rigorous proof and evaluation of the corrections to (5.4) is a very interesting and important problem.

The result (5.4) is obtained under the assumption of Gaussian statistics of the Clebsch field, which naturally leads to the exponential distribution function $P(U)$, since $U \simeq \mathcal{O}(aa^*)$. Two kinds of corrections to (5.4) are expected. First of all, more accurate treatment of the integrals involved in the derivation of (5.4) might lead to modifications of $P(U)$ in the region $U \to 0$. It is not impossible to assume that the exact expression for the PDF does not scale with U/U_{rms}, but in addition has some r-dependent contributions. In this case, all high-order moments will also depend on the separation r. The second reason for corrections to (5.4) has a much deeper physical basis. The weak interaction approximation used in this work is an assumption yielding quite interesting results. However, the deviations from the Gaussian statistics of the Clebsch variables might lead to substantial changes in the shape of PDF's of different structure functions, especially of those dominated by small-scale processes. At the present time we do not have any idea about the nature of the deviations of statistics of the Clebsch variables from Gaussian. Still, some speculations about their consequences are possible. For example, exponential tails in the distribution function of the Clebsch variables corresponds to

$$P(\Delta) \sim \mathrm{e}^{-(\Delta/\Delta_{\mathrm{rms}})^{1/2}}$$

where $\Delta = \Delta(x) - \Delta(x + r)$ the difference of some, as yet unspecified physical property of turbulent flow (vorticity, vorticity derivatives, etc. . . .).

6. Spectra of the dissipation rate fluctuations

In this section we will evaluate the dissipation rate structure functions S_2^ϵ defined by the relation (1.4). To begin the discussion, let us first recall the derivation of the Kolmogorov result $S_2^\epsilon \approx k^{-1}$. The dissipation fluctuation spectrum is:

$$E^\epsilon(k) \approx 4\pi k^2 \frac{\overline{\epsilon(k)\epsilon(k')}}{\delta(\boldsymbol{k} + \boldsymbol{k}')} . \tag{6.1}$$

Substituting the relation

$$\epsilon(k) = 2\nu \int_0^\infty q_i(k - q)_i \, v_j(q) \, v_j(k - q) \, \mathrm{d}^3 q \tag{6.2}$$

into (6.1) we obtain:

$$E^\epsilon(k) \approx \nu^2 k^2 \int_0^\infty [q_i(k - q)_i]^2 U(q) \, U(k - q) \, \mathrm{d}^3 q , \tag{6.3}$$

where $U(k) = E(k)/4\pi k^2$. It is easy to see that the integral converges in both limits, yielding $E^\epsilon(k) \approx k^{5/3}$, which is grossly incorrect. The mistake can be traced to the assumption of Gaussian statistics of the velocity field involved in the derivation of this relation. However, this assumption is plausible in the renormalization group sense, i.e., the deviations from Gaussian statistics can be treated as small, provided the effective (renormalized) transport coefficients are used in the evaluation of the corresponding flow features. This statement is rather clear since in the inertial range of fully developed turbulent flow the characteristic time-scale of the velocity fluctuations at the scale $l \approx 1/k$ is dominated by the non-linear interactions and can be represented using effective (turbulent) viscosity ν_K.

The ϵ-spectrum is given by:

$$E^{\epsilon}(k) \approx \nu_{\mathrm{K}}^{2}(k) k^{2} \int_{0}^{\infty} [q_i(k-q)_i]^2 U(q) \times U(k-q)\,\mathrm{d}^3 q\,, \tag{6.4}$$

with $\nu_{\mathrm{K}} \approx \epsilon^{1/3} k^{-4/3}$ in accord with the Kolmogorov theory. The integral (6.4) converges and (6.4) and (6.1) give the dissipation fluctuation spectrum in Kolmogorov turbulence:

$$E_{\mathrm{K}}^{\epsilon}(k) \approx \epsilon^{2} k^{-1}\,. \tag{6.5}$$

Using the same procedure we can calculate $E_n^{\epsilon}(k)$ corresponding to the k^{-1}-spectrum. The only fundamental difference between this case and the one considered above is that the integral (6.3) calculated for the k^{-1}-spectrum is ultraviolet divergent and, thus, is equal to a constant which strongly depends on the value of k_{d}. Then, it is easy to find the dissipation rate spectrum in the gaussian random field: $E^{\epsilon} \approx k^2$. In a turbulent flow dominated by a k^{-1}-spectrum, a different procedure should be used. It follows from the relation (3.19) and the energy spectrum (4.1) that the effective viscosity ν_n in this case is:

$$\nu_n(k) \approx (\epsilon\nu)^{1/4} k^{-1}\,, \tag{6.6}$$

and we derive readily:

$$E_n^{\epsilon}(k) \approx k^{0}\,. \tag{6.7}$$

In the general case of high Reynolds number statistically stationary turbulence the largest fraction of energy is contained in the 5/3-Kolmogorov range dominating the large-scale velocity fluctuations. Indeed, in this case $E_{\mathrm{K}}/E_n \approx \mathrm{Re}/\ln \mathrm{Re} \to \infty$ when the Reynolds number $\mathrm{Re} = u_{\mathrm{rms}} L/\nu \to \infty$. The subscripts n and K hereafter denote parameters of the flow corresponding to the k^{-1} and Kolmogorov spectra, respectively. It is easy to see that $\epsilon_{\mathrm{K}}/\epsilon_n = \mathcal{O}(1)$ and thus, the energy dissipation is more or less equally distributed between two spectra. To estimate the ϵ-correlation function let us assume that the total dissipation rate $\epsilon = \epsilon_{\mathrm{K}} + \epsilon_n$. This assumption is plausible within the framework of our weakly non-linear theory. Then we have:

$$E_{\mathrm{total}}^{\epsilon} = E_{\mathrm{K}}^{\epsilon} + E_n^{\epsilon} + E_{\mathrm{K}n}^{\epsilon}\,, \tag{6.8}$$

where in the three-dimensional case:

$$E_{\mathrm{K}n}^{\epsilon} = k^2 \nu_{\mathrm{K}}(k)\, \nu_n(k) \times \int_{0}^{\infty} [q_i(k-q)_i]^2 U_{\mathrm{K}}(k)\, U_n(k)\,\mathrm{d}^3 q \approx k^{-1/3}\,. \tag{6.9}$$

Statistical independence of the velocity fields described by the two spectra was assumed in the derivation of this relation. Thus, the Kolmogorov scaling (1.4) is "contaminated" by contributions coming from the second solution of the kinetic equation. This is a manifestation of small-scale intermittency. The predicted spectrum of the dissipation rate fluctuations consists of three ranges, characterized by the k^{-1}-spectrum at the largest scales and k^0-spectrum in the vicinity of the dissipation cut-off. The intermediate $k^{-1/3}$-range might be too narrow to be found in the available experimental data. It should be mentioned that the simple linear superposition of the dissipation rates corresponding to different spectral ranges introduced above is an approximation which can be rather crude. In a non-linear system a weighted superposition can modify the relative importance of different spectral contributions to the total dissipation fluctuation spectrum.

To test our predictions E. Jackson and Z.S. She [8] have generated a Gaussian random field having a k^{-1}-spectrum and used it as an initial condition to simulate the process of turbulence decay. The results of this remarkable experiment are presented in fig. 2. One can see that, while

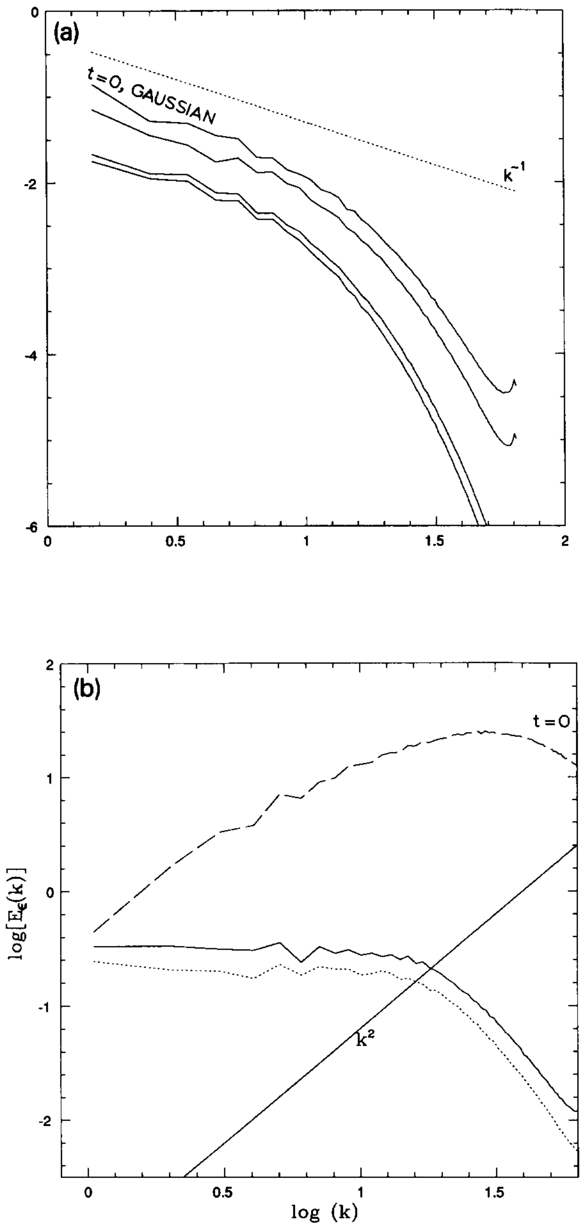

Fig. 2. Evolution of the k^{-1}-spectrum from the Gaussian initial conditions [8]. For the explanation see text.

the energy spectrum remained unchanged during the entire experiment, the dissipation fluctuation spectrum demonstrated a dramatic transition from $E^\epsilon(k) \approx k^2$ to $E^\epsilon(k) \sim k^0$ in accord with the theory developed in this paper.

7. Summary and discussion

The first serious attempt to attack the turbulence problem using a field-theoretical approach was made by Kraichnan [10], who introduced the direct interaction approximation (DIA), a one loop renormalized perturbation expansion of the Navier–Stokes equations. It became clear soon after Kraichnan's first publication on the subject [11] that the DIA, combined with a proper regularization prescription, gives quite accurate predictions of a wide variety of the features of turbulence, such as energy spectra, turbulent transport, etc. However, despite the substantial successes of statistical theories in the description of the large-scale-dominated properties of the turbulent flows, their failure to explain the small-scale phenomena or the deviations from the Kolmogorov theory has led in recent years to a pessimistic assessment of the feasibility of field-theoretical methods in the development of the future theory of turbulence.

The reasons for the inability of one-loop perturbation expansions of the Navier–Stokes equations to describe the small-scale properties of turbulent flows can be easily understood in terms of the results of this work. The Clebsch variables, used for formulation of the theory, correspond to an extremely complicated non-linear transformation of the original velocity field. Thus, the one loop approximation in the Clebsch description is equivalent to an infinite resummation of the renormalized perturbation expansion of the Navier–Stokes theory. The fact that the hidden conservation law discovered in the Clebsch formulation of hydrodynamics leads to a new inertial range is not surprising if we recall the reasoning behind the well-known algebraic ranges in two and three-dimensional turbulent flows. Indeed, it is widely accepted that energy conservation is responsible for the 5/3-Kolmogorov range, while enstrophy conservation in 2D-hydrodynamics results in the k^{-3}-energy spectrum, observed in different situations. However, not all conservation laws are dynamically relevant. It is a plausible conjecture, adopted in this work, that in order to be dynamically significant, the conserved quantity must have an external source or production mechanism. In other words the integral of motion has to be affected by some external mechanisms. The relevance of the particle conservation law has been demonstrated in this paper where we have identified viscous mechanisms (the energy sink) with quasi-particle production. Thus, we believe, the k^{-1}-spectrum is dynamically justified. The existence of this new range is responsible for the strong deviations of the small-scale dynamics from the Kolmogorov description.

Comparison of our predictions with experimental data has to be regarded as preliminary. The results of the low Reynolds number numerical simulations supporting the k^{-1}-energy spectrum are most gratifying, although more numerical and experimental work is needed to understand the role of the new range in high-Reynolds number turbulent flows. Analysis of experimental data on the spectra of the dissipation rate fluctuations in atmospheric and laboratory boundary layers give rather strong indications in favor of the predictions of this paper, although more careful data processing is necessary to make more definitive statements.

To try to understand the reasons of apparent success of the Clebsch variables in describing both large and small-scale properties of turbulent flows, we would like to discuss the role of various dynamical constraints, following Euler equations of ideal fluid. First of all, the solutions must be Galileo invariant. The inability of the one-loop approximations to satisfy this important constraint resulted in the well-known infra-red divergencies which can be removed by the Lagrange transformations, the ϵ-expansion, infrared cut-offs etc. However, this is not sufficient for the development of the successfull theory of small-scale behaviour of the velocity fluctuations. The reason of this failure will be discussed below.

The velocity field must satisfy the Kelvin theorem or the law of conservation of circulation, which can be regarded as an infinite number of geometrical constraints on possible solutions of the problem. This basic property of the inviscid hydrodynamics must be preserved in the statistical theory. The low order renormalized perturbation expansions of the Navier–Stokes equations, though corresponding to an infinite subset of the diagrams, violate the Kelvin theorem. It is easy to speculate that the geometric constraints may be responsible for strong deviations from the gaussian statistics, which cannot be taken into account by one-loop approximations based on the velocity field. To illustrate this point, let us consider a spherical pendulum in the gaussian field. This system has an obvious geometric constraint: the fixed pendulum length. The trajectories of the pendullum can cover only the limited part of the space, sphere of a given radius, which results in the strong deviations from the Gaussian statistics. We believe that similar situation occurs in hydrodynamics: The evolution of any initial condition, obeying the Euler equation, does not cover all possible trajectories, but only those conserving the volume of the fluid element in accord with the Kelvin theorem. Thus, it is clear that this restricted random walk is not a Gaussian process. We speculate that this is the basic physical reason for the strong deviations of velocity difference distribution functions from Gaussian, observed in various experiments.

Another possibility, suggested by A. Newell, is that the quasiparticle source, derived in this paper is dominated by the viscous effects which tend to weaken the build up of coherence due to the non-linear interactions, thus leading to close-to-Gaussian statistics of the Clebsch field.

The advantage of the Clebsch variables is that they, by construction, automatically satisfy the Kelvin theorem of conservation of circulation and, as a consequence, must not obey the geometric constraints. Thus, their trajectories can cover much larger fraction of the phase space, allowing the close-to-Gaussian statistical behaviour.

The particle conservation responsible for the k^{-1}-range, leads to the inverse cascade in the wave-number space which resembles the similar process in 2D-turbulence due to the energy conservation. There exist still unanswered question about the fate of the energy, accumulated at the large scales as a result of this process. The existing experiments do not show any large peaks in the energy spectra at $k \approx 0$, so we would like to understand the details of the energy balance. We suspect that the inverse cascade leads to the formation of the large scale structures which are unstable. This instability acts as the large-scale energy source. The generated energy is then cascaded to the small scales and dissipated into heat. Similar mechanism has been discussed in ref. [12].

Now we can reassess the notion of "corrections to the Kolmogorov theory" which has been an elusive goal of turbulence theory. It is clear from the results of this work that renormalized perturbation expansions based on the Navier–Stokes equations are unable to describe the small-scale behaviour of the flow. The deviations from the Kolmogorov theory are not corrections at all: they correspond to a completely different physical reality governed by a conservation law hidden from the conventional hydrodynamics. Both $k^{-5/3}$ and k^{-1}-spectra have "equal rights" being manifestations of two different, but equally important, symmetries of the problem. It is an accident, though understandable, that the Navier–Stokes equations were the ones used for the description of all, including strongly turbulent, fluid flows. If the Clebsch variables were the only ones known to us, then the turbulence problem might have been much simpler than the problem of laminar flows, for which these variables are very incovenient. In this case, both spectra derived in this paper would have been discovered simultaneously since they appear as two different solutions of the same nonlinear equation.

The results obtained in this paper are derived in the mean field approximation, introduced in section 3. Investigation of the corrections to this approximation is a most interesting problem which is beyond the scope of this work. At the present time we can only hope that these corrections can be analyzed using field-theoretical methods. This is possible only if the theory presented here is indeed the mean field theory of turbulence. If there exist some yet unknown conservation laws or hidden symmetries, then we will again face an intrinsic inability of the familiar variables, this time the Clebsch variables, to describe some experimental data.

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