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Computer simulation of wave collapses in the nonlinear Schrödinger equation

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The self-tuning numerical scheme for simulation of collapse phenomena in the framework of the nonlinear Schrodinger equation is developed The scheme allows to reach extremely high field amplitudes ($|\Psi|^2/|\Psi_0|^2 \sim 10^{15}$) without lack of validity Numerical results are compared with analytical predictions

1. Introduction

Wave collapses, i e. blowup-like processes of wave energy concentration in a decreasing volume, are realized in nonlinear media in many cases when the nonlinear effects of increasing gradients prevail over the linear effects of wave packets running off. From a mathematical point of view this means that the solutions of the nonlinear evolutionary wave equations have singularities arising for a finite time. Actually the singularity is as a rule limited on a defined level by mechanisms of dissipative type. In many cases wave collapse is an elementary structural unit of strong turbulence and a main and strongly nonlinear mechanism of wave energy dissipation in a medium with negligibly small linear dissipation. The dissipation of wave energy in the collapses defines their most important physical meaning for construction of a consistent theory of strong wave turbulence in nonlinear media and for explanation of the macroscopic manifestations of this turbulence.

The singularity structure near the collapse point defines essentially the effectiveness of the collapse as a nonlinear mechanism of dissipation. With a whole variety of physical examples of wave collapses (different types of self-focusing of quasimonochromatic waves, collapses of Langmuir, hybrid and electromagnetic waves in plasmas and many others [1]) only several qualitatively different regimes of dissipation are realized practically. It is said that collapse is strong, when a finite and strictly defined energy is captured into the singularity i e. a delta-like singularity is formed. After the dissipation of a fixed (not depending on the damping parameters) portion of this energy the collapse regime changes into a decay one. Postcollaptical decay also takes place in the case of a weak collapse, when the energy captured tends to zero when approaching the singularity. The energy absorbed in the weak collapse tends to zero with a decrease of the nonlinear damping coefficient and is actually defined by the amplitude level where the dissipation starts.

In the case of weak and strong collapses only the energy captured into the singularity at the moment of collapse dissipates. In many cases a principally different situation is realized when a dissipation zone of small size absorbing energy from the surrounding space is formed at the collapse point. In this case the collapse lifetime is longer than the characteristic time scale of the dissipation; the decay is absent and replaced by a quasistationary state. It is natural to call such type of collapse a "superstrong" collapse· the

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total energy absorption can exceed (over a fairly long time) the energy absorption in a single act of strong collapse.

All types of collapse enumerated above are realized in the framework of one of the most fundamental models of nonlinear modern physics, the nonlinear Schrödinger equation (NSE):

$$i\Psi_t + \Delta\Psi + f(|\Psi|^2)\Psi = 0. \tag{1}$$

This equation has a lot of physical applications. It describes, in particular, the evolution of a quasi-monochromatic wave packet in a conservative isotropic medium with a positive dispersion and inertialess local nonlinearity. Eq. (1) is derived by means of averaging over large frequencies, which corresponds to the wave packet center, of the initial equations describing a nonlinear medium. At $f(u) > Cu^{2/d}$, $C > 0$ (d is a space dimensionality), the nonlinear effects prevail over dispersional ones and eq. (1) describes the singularity formation for the finite time $t = t_0$ (the collapse of wave packets) in media which are stable to small perturbations. In such media the perturbation leading to the collapse should exceed some threshold.

The most intensive investigations of wave collapses were carried out in the model of NSE with maximal symmetry and power nonlinearity,

$$i\Psi_t + \Delta_r\Psi + |\Psi|^s\Psi = 0,$$

$$\Delta_r = \frac{d^2}{dr^2} + \frac{d-1}{r}\frac{d}{dr},$$

$$\Psi_r(0,t) = \Psi(\infty,t) = 0. \tag{2}$$

A lot of analytical investigations and computer experiments (see, for example, refs. [2–4] and references given therein) are devoted to the study of this equation. For the first time the collapsing solutions of eq. (2) were considered in connection with the investigation of stationary self-focusing of radiation in a medium with cubic nonlinearity ($s = d = 2$) [5–8]. On the basis of computer experiments the hypothesis of "moving foci" equivalent to the concept of collapse has been proposed in ref. [7]. The strict proof of the existence of the collapse in eq. (2) has been obtained (for $s = d = 2$) for the first time in ref. [9]. Except the stationary self-focusing this equation describes quite a number of wave phenomena (gravitational waves in deep water [10, 11], one-dimensional excitations in optical fibers [12] and molecular structures [13], etc.). Among these phenomena one should especially mention nonstationary self-focusing and subsonic collapse of Langmuir waves in plasmas [14, 15].

After early publications concerning self-focusing (see reviews [16, 17] and references given therein) a lot of works have been devoted to the investigation of the structure of the singularity arising in the collapse and adjacent questions (see, e.g., refs. [18–25]). The interest in the investigation of wave collapses in the model (2) has increased appreciably at present [2–4, 26–46]. This interest is due to, on the one hand, the universality of the NSE as a general model in the theory of wave collapses and, on the other hand, to remarkable contradictions in results obtained by different authors. Some debatable questions concerning the singularity structure have been kept vague up to the present.

Since in most cases the problems of wave collapse physics, including the ones described by eq. (2), are nonintegrable, computer simulation plays a rather important role in the solution of these problems. Under the highly limited possibilities of analytical methods by means of computer experiment the principal questions of the singularity structure and the effectiveness of collapse as a mechanism of energy dissipation are solved. At the same time the problems of computer simulation of wave collapses are very difficult because of two contradictory circumstances. The first one is to approach the singularity as much as possible without loss of accuracy and the second one is that the spatial and temporal gradients increase simultaneously when approaching the collapse point. To avoid these difficulties we have developed a special adaptive method with automatic reconstruction in the calculation of spatial–temporal scales. The method used allowed us to approach the singularity appreciably closer than in calculations of other authors and to get a description of the collapsing structures in the vicinity of the collapse point.

Our earlier works [4, 36, 37, 39, 42, 43] are devoted to these investigations and also include, in some cases, new analytical results. The present paper is a review of these works. The methods of calculations and obtained results are presented below. An analysis of the asymptotic behavior of solutions in different regions of parameters varying in the framework of problem (2) with power nonlinearity and classification of the main types of wave collapse have been carried out.

2. Statement of the problem and simulation method

First of all, it is necessary to point out the fundamental properties of the NSE essential for us.

Eq. (2) has integrals of motion: "the number of particles" (with an accuracy up to small terms coinciding with the energy of the wave packet),

$$N = \int_0^\infty |\Psi|^2 r^{d-1}\,\mathrm{d}r, \tag{3}$$

and the Hamiltonian,

$$H = \int_0^\infty \left(|\Psi_r|^2 - \frac{2}{s+2}|\Psi|^{s+2}\right) r^{d-1}\,\mathrm{d}r. \tag{4}$$

From (2) with account of (3), (4) one can get an important relationship (usually called the virial theorem),

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_0^\infty |\Psi|^2 r^{d+1}\,\mathrm{d}r = 8\left[H - \frac{2}{s+2}\left(\frac{sd}{4} - 1\right)\int_0^\infty |\Psi|^{s+2} r^{d-1}\,\mathrm{d}r\right], \tag{5}$$

from which it follows that the collapse in eq. (2) takes place at

$$sd \geq 4. \tag{6}$$

Taking into account the conservation of integral N one can also easily get this inequality comparing nonlinear and dispersive terms in eq. (2). While fulfilling (6) a sufficient condition of the collapse is that the Hamiltonian is negative. At $sd = 4$ (critical case) this condition is close to necessary, at $sd > 4$ (supercritical case) it can be exceedingly strong. Actually, the condition $H < 0$ restricts from below the value $\Psi_0^s l^2$ (Ψ_0 is the amplitude, l is the half-width of the initial wave packet). It means that collapse is possible for sufficiently intensive initial conditions.

Eq. (2) has a family of soliton solutions:

$$\Psi(r,t) = \mathrm{e}^{i\lambda^2 t}\lambda^{2/s}R(\eta), \quad \eta = \lambda r, \tag{7}$$

where $R(\eta)$ is described by

$$\Delta_\eta R - R + R^{s+1} = 0, \quad R_\eta(0) = R(\infty) = 0. \tag{8}$$

The most interesting positive monotonously decreasing solution (8) exists for arbitrary s at $d \leq 2$ and for $s(d-2) < 4$ at $d > 2$. As for the solitary solution (7) it is stable only at $sd < 4$. At $sd > 4$ only linear instability of the soliton takes place, while in the critical case $sd = 4$ the soliton is neutrally stable in the linear approximation but unstable with respect to the finite disturbances

Eq. (2) permits also a self-similar substitution:

$$\Psi(r,t) = (t_0 - t)^{-1/s - i\kappa} g(\xi), \quad \xi = r(t_0 - t)^{-1/2}, \tag{9}$$

An analysis of the corresponding self-similar solution (see below) demonstrates that in the supercritical case $sd > 2$ the formation of an integrable singularity takes place at $t \to t_0$. This self-similar solution has physical sense only in the case of $sd > 4$. It corresponds to the weak collapse with zero energy flowing into the singularity.

It should be mentioned, finally, that eq. (2) is invariant under scale transformation

$$\Psi(r,t) \to f^{-2/s}\Psi(\xi,\tau),\quad \xi = r/f,\quad \tau = t/f^2,\quad f = \text{const.}; \tag{10}$$

in the critical case $sd = 4$ an additional invariance with linear function $f = C_1 t + C_2$,

$$\Psi(r,t) \to f^{-d/2}\Psi(\xi,\tau)\, e^{i f \dot{f} \xi^2/4},\quad \xi = r/f,\quad \frac{d\tau}{dt} = f^{-2}, \tag{11}$$

takes place. From this invariance it follows that there is an exact collapsing solution in the critical case:

$$\Psi(r,t) = (t_0 - t)^{-d/2} R\left(\frac{r}{t_0 - t}\right) e^{i(1 - r^2/4)/(t_0 - t)}, \tag{12}$$

which, however, is unstable with respect to finite disturbances and is not realized in calculations.

As was already mentioned, the question of the singularity structure near the collapse point is of principal interest from the point of view of the effectiveness of collapse as a mechanism of wave energy dissipation in strong turbulent processes Eq. (2) has been subjected to numerical integration for more than two decades, starting from ref. [7]. In this case the possibility to carry out the correct calculations in the direct vicinity of the singularity is extremely important. This problem is especially difficult in the critical case (see below). The exceeding $P = |\Psi/\Psi_0|^2$ of the field intensity over the initial one defines the degree of approach to the collapse point. Since, under the collapse, the wave field increases and the size of the characteristic region of its localization decreases in a blowup way, usual algorithms (without even the adaptation of the partial scheme to the varying field) are unsuitable to attain large exceedings. These algorithms cannot ensure the attainment of exceedings P_{max} larger than 10^3–10^4. The application in ref [20] of a nonuniform space grid with steps of integration depending on time allowed to reach the values $P_{max} \sim 10^7$. In ref. [19] in which the adaptive method of Lagrangian coordinates over one independent variable with time step decreasing was used in the calculations, and the value $P_{max} \sim 10^8$ was obtained. In the recently published preprint [42] the attainment of $P_{max} \sim 10^{15}$ by the method of ref. [19] is reported without mentioning some improvements of this method. Another adaptive method, used in refs [30, 33] (scale transformations (10) with function of time depending on the solution), allowed to reach $P_{max} \sim 10^9$.

Independently of the authors of ref. [30] (see footnote on p. 4 in ref. [4]) we have developed a method [4, 36] in many aspects analogous to the one used in ref. [30]: the calculations are carried out in a coordinate system compressing with the velocity of the field growth and a simultaneous nonlinear "straightening" of time transferring the moment of collapse into infinity. In the most difficult critical case we have managed to reach the record values $P_{max} \sim 10^{18}$ and in the simulation of weak collapse in a general case $sd > 4$ we have attained arbitrary large values P_{max} without loss of accuracy. Let us describe our method in detail.

The transition to the coordinate system of the collapsing region is accomplished by means of a nonlinear substitution:

$$\xi = rA(\tau),\quad \frac{d\tau}{dt} = A^2,\quad A(\tau) = |\Psi(0,\tau)|^{s/2}. \tag{13}$$

In these new variables problem (2) with initial condition $\Psi(r,0) = \Psi^0(r)$, $|\Psi^0|_r \le 0$, takes the form

$$\begin{aligned} & i\Psi_\tau + \Delta_\xi\Psi + i\xi a(\tau)\,\Psi_\xi + A^{-2}|\Psi|^s\Psi = 0, \\ & a(\tau) = \frac{d}{d\tau}\ln A = i\frac{sd}{4}\left(\frac{\Psi_{\xi\xi}}{\Psi}\bigg|_{\xi=0} - \text{c.c.}\right), \\ & \Psi_\xi|_{\xi=0} = \Psi|_{\xi=\infty} = 0, \\ & \Psi(\xi,0) = \Psi^0(\xi/\Psi_0^{s/2}),\quad \Psi_0 = |\Psi^0(0)|; \end{aligned} \tag{14}$$

the finite time interval $0 \le t < t_0$ maps itself into a semi-infinite $0 \le \tau < \infty$, and the moment of singularity formation is equal to

$$t_0 = \int_0^{\infty} |\Psi(0,\tau)|^{-s} \, d\tau \tag{15}$$

Problem (14) also conserves invariants (3), (4), which can be written in the form

$$N = A^{-d} \int_0^{\infty} |\Psi|^2 \xi^{d-1} \, d\xi, \tag{16}$$

$$H = A^{-d} \int_0^{\infty} \left(A^2 |\Psi_\xi|^2 - \frac{2}{s+2} |\Psi|^{s+2} \right) \xi^{d-1} \, d\xi. \tag{17}$$

The next rather important step introducing a qualitative difference between our method and other adaptive methods [19, 30] is to reformulate problem (14) for the finite region of integration $0 \le \xi \le L$. A constancy of the region of integration in ξ-space means a decrease in physical variables. We chose the length of the segment L in such a way that the ratio $|\Psi(0,\tau)/\Psi(L,\tau)|^2$ was sufficiently large. In the calculations it was more than three orders in the supercritical case (under the power space asymptotic behavior of the field) and more than six orders in the critical case when the field decreases exponentially (see below). Such a choice of the region size allows one to suppose that the field near the right boundary is "frozen":

$$\Psi_\tau + a(\tau)\, \xi \Psi_\xi = 0, \quad \xi \ge L \tag{18}$$

It is natural to consider this relationship at $\xi = L$ as a condition on the right boundary. It can also be represented in some other equivalent form. It is easy to get an analytical solution to (18) with initial solution $\Psi(\xi, \bar{\tau}) = \overline{\Psi}(\xi)$ which has at $\xi = L$ the form

$$\Psi(L,\tau) = \overline{\Psi}\left(L \frac{A(\bar{\tau})}{A(\tau)} \right), \quad \tau \ge \bar{\tau}. \tag{19}$$

Applying the explicit algorithms to the solution of the evolutionary problem when the field is defined immediately by its previous value, one can simply use a polynomial extrapolation to define the field in the additional grid point (necessary for Laplacian calculation at $\xi = L$) to the right of the boundary. This is equivalent to the supposition that the higher derivatives $\partial^m \Psi / \partial \xi^m$, $m > q$ (q is the degree of the approximating polynomial) tend to zero near the right boundary.

For the finite region of integration, the wave energy and Hamiltonian

$$N_L = A^{-d} \int_0^{L} |\Psi|^2 \xi^{d-1} \, d\xi,$$

$$H_L = A^{-d} \int_0^{L} \left(A^2 |\Psi_\xi|^2 - \frac{2}{s+2} |\Psi|^{s+2} \right) \xi^{d-1} \, d\xi \tag{20}$$

are no longer integrals of the problem. However, with an account of the exchange with the periphery it is easy to get modified integrals. So instead of (16) we have

$$N = N_L + L^d \int_0^{\tau} [A(\tau')]^{-d} |\Psi(L,\tau')|^2 \times \left[a(\tau') + 2L^{-1} \Phi_\xi(L,\tau') \right] d\tau',$$

$$\Phi = \arg \Psi \tag{21}$$

We do not present here the expression for the Hamiltonian because it is too bulky. The modified integrals of motion were used, in particular to control the correctness of the calculation.

The nonstationary problem (14) can be presented in the following form:

$$\Psi_\tau = f(\Psi_{\xi\xi}, \Psi_\xi, \Psi, \xi), \quad \Psi = \Psi(\xi,\tau) \tag{22}$$

Solving this problem numerically we used, as a rule, an explicit predictor–corrector method of the second order of the accuracy over $\Delta\tau$ (n is the time step number):

$$\Psi^{n+1/2} = \Psi^n + \tfrac{1}{2} \Delta\tau f^n,$$

$$\Psi^{n+1} = \Psi^n + \Delta\tau f^{n+1/2}$$

Spatial differential operators in the right-hand side of (22) have been approximated on a uniform grid $\xi_j = j\Delta$, $L = L/M$, $0 \le j \le M$ with second order of accuracy, here the polynomial extrapolation has been used to calculate the Laplacian on the right boundary $\xi = L$ ($j = M$). For example, for $q = 3$ we have

$$\Psi_{M+1} = 4\Psi_M - 6\Psi_{M-1} + 4\Psi_{M-2} - \Psi_{M-3}.$$

Besides the observation of the modified integrals of the problem, we realized the formal control of the correctness of calculations consisting in the comparison of physically identical variants which differ by the parameters M, $\Delta\tau$ and L. This control completely confirmed the adequateness of the simulation method used. Furthermore, in the investigations we have confirmed many theoretical and numerical results obtained by other authors.

We shall note the main differences of our simulation method from the one used in ref. [30]. only independent variables are transformed, the function $A(\tau)$ is chosen otherwise, the transformed equation is approximated in some other way. The main difference, however, is that we seek to resolve in detail the region of "spike" at the cost of the refusal to consider the wave packet as a whole, i.e. throwing back the "tail" of the distribution. It explains most likely the approach of the exceedings larger than in ref. [30] and also in ref. [19], where another method of simulation of the wave packet as a whole has been used

A transition from problem (2) to problem (14) makes it possible to study in detail the singularity near the center $r = 0$ with an automatic tuning of the steps of integration (which being constant in the ξ, τ variables, decrease continuously in the physical variables) to the rate of the field change. Integration of initial problem (2) in physical variables with a decrease of spatial and temporal steps (or with a decrease of one of these two steps) in the process of calculation is not equivalent to our statement of the problem and cannot lead to such exceedings due to the inevitable loss of accuracy starting from some moment, caused by finite word length in computers and leading to scheme violation.

To study the evolution of localized distributions the following functions possess a sufficient community and they are used in our calculations as the initial condition:

$$\Psi(r,0) = \Psi_0 \exp\left[-(r/l)^n\right], \quad \operatorname{Im}\Psi_0 = 0, \qquad (23)$$

allowing one to simulate both Gaussian ($n = 2$) and, in terms of refs. [21, 22], "plateau-like" ($n \gg 1$) distributions. The calculation of the invariants (3), (4) for function (23) gives

$$N = \Psi_0^2 l^d \frac{\Gamma(d/n)}{n 2^{d/n}},$$

$$H = N\left[D l^{-2} - \Psi_0^s \left(\frac{2}{s+2}\right)^{(d+n)/n}\right], \qquad (24)$$

where

$$D(n,d) = \tfrac{1}{4} n^2 2^{2/n} \Gamma\left(\frac{2n+d-2}{n}\right) \Big/ \Gamma\left(\frac{d}{n}\right)$$

(it coincides for the Gaussian packet with space dimensionality), and $\Gamma(x)$ is the gamma function. Taking into account (24) a sufficient condition of collapse takes the form

$$\Psi_0^s l^2 > D(n,d) \left(\frac{2}{s+2}\right)^{(d+n)/n}. \qquad (25)$$

The calculations were made for a large number of variants with a wide variation of the parameters of the initial condition (23):

$$1 \le \Psi_0 \le 10, \quad 1 \le l \le 8, \quad 2 \le n \le 10.$$

However, we observed that for the initial condition satisfying (25), the concrete values of these parameters did not practically influence the general picture of the process obtained while attaining sufficiently large exceedings $P = |\Psi/\Psi_0|^2$. Naturally, at moderate exceedings when the asymptotic regime is not attained yet, the influence of the initial parameters is possible. Thus, in calculations [28] the following parameter is qualitatively essential:

$$\epsilon = N/N^{\text{th}},$$

which represents an exceeding by the energy over the conditional (increased) "threshold" of the collapse N^{th} defined from the condition $H = 0$. In the critical case the sufficient condition $H < 0$ is the nearest to the necessary one and the introduction of the parameter ϵ equal for the initial condition (23) to

$$\epsilon = \Psi_0^2 l^d \left[D(n,d) \left(\frac{d+2}{d} \right)^{(d+n)/n} \right]^{-d/2} \tag{26}$$

(in ref. [28] $n = d = 2$, $\epsilon = \Psi_0^2 l^2/8$) is qualitatively justified. We shall emphasize that in our calculations the established dynamics of the collapsing structures did not depend on that parameter (except several details mentioned below in section 4).

3. A weak self-similar collapse

Acting similarly to refs. [4, 29], consider the self-similar substitution (9) for eq. (2). The function $g(\eta)$ satisfies the equation

$$g_{\eta\eta} + g_\eta \left(\frac{d-1}{\eta} + \frac{\mathrm{i}}{2}\eta \right) + \left(\frac{\mathrm{i}}{s} - \kappa \right) g + |g|^s g = 0,$$

$$g_\eta(0) = g(\infty) = 0. \tag{27}$$

The solution of this equation is defined with an accuracy up to a multiplication by a constant phase multiplier, which allows to consider $\operatorname{Im} g(0) = 0$. From all the solutions of (24), we are interested (for each pair of the parameters s, d) in a regular solution monotonously decreasing on the semiaxis $0 \le \eta < \infty$. Its asymptotic behavior is

$$g \xrightarrow[\eta \to \infty]{} C\eta^{-2(1/s + \mathrm{i}\kappa)}.$$

This means that at $t \to t_0$ the formation of the finite power solution

$$|\Psi|^2 \to C^2 r^{-4/s} \tag{28}$$

takes place in every point with coordinate r. In the real physical space the self-similar solution is carried out in the restricted region $r < r_0$ in the center of which the singularity (28) "arises" at $t \to t_0$. At $sd \le 4$ this singularity is nonintegrable (wave energy integral diverges in a lower limit) and solution (9), (27) has no physical sense. In the supercritical case $sd > 4$ (the most important partial case $s = 2$, $d = 3$ corresponds to the subsonic collapse of Langmuir waves in plasma [14]) is integrable and one can attach physical sense to the solution discussed A formal divergency of the integral N in the upper limit does not matter since in the arbitrary finite region $r < r_0$ the value of the integral remains constant. Actually, for the region $r < r_0$ we have

$$N = (t_0 - t)^{2(sd/4 - 1)/s} \int_0^{r_0 (t_0 - t)^{-1/2}} |g|^2 \eta^{d-1}\, \mathrm{d}\eta. \tag{29}$$

It is easy to see that the integral in (29) diverges in the upper limit at $t \to t_0$, but the number of quanta N remains finite,

$$N \sim \frac{s}{sd-4} r_0^{(sd-4)/s}$$

Since at sufficiently large r_0, N should be close to its value in the region $r < r_0$ at the moment of collapse, the relationship

$$\int_0^\infty \left(|g|^2 - C^2 \eta^{-4/s} \right) \eta^{d-1}\, \mathrm{d}\eta = 0$$

is correct. This relationship has been earlier obtained in ref [29] for $s = 2$, $d = 3$. The constructed solution corresponds to the weak collapse: as the singularity is reached the energy captured in the collapse decreases so that, formally speaking, zero energy is absorbed into the singularity $r = 0$. The actual energy absorbed in the weak collapse depends on the level of the field amplitude on which the stabilizing mechanisms begin to act and the conditions of applicability of initial equation (2) are not fulfilled.

A mode structure of the weak collapse is defined by eq. (27), for which the requirement of regularity and monotonous decrease of the solution at $0 \le \eta < \infty$ removes the ambiguity in the choice of the constants κ and $g_0 = g(0)$, which are the eigenvalues of the nonlinear boundary problem (27). In refs. [4, 29] this problem is solved by the shooting method It was found that $\kappa = 0.545$, $g_0 = 1.39$ at $s = 2$, $d = 3$ [29] and $\kappa = 0.359$, $g_0 = 0.976$ for $s = 6$, $d = 1$ [4].

In a set of works [3, 26, 29, 41] variants of collapse dynamics alternative to the solution constructed above have been obtained in the supercritical case Mainly, it is a quasiclassical regime of three-dimensional wave collapse in a medium with cubic nonlinearity proposed in ref. [26] (the consistent theory has been developed in ref. [29]) with generalization in the arbitrary supercritical case $sd > 4$ [3] (see also ref. [37]). Quasiclassical theory predicts strong collapse of the wave as a whole which, however, is unstable to small-scale perturbations On the other hand, the calculations in ref [41] confirmed (at $s = 2$, $d = 3$) the weak self-similar character of the field singularity and asymptotic behavior (28). But as a self-similar variable a value $\eta = r(t_0 - t)^{-1/3}$ different from (9) has been determined. Finally, the calculations carried out at $s = 2$, $d = 3$ up to the exceedings $P_{max} \sim 10^9$ not only confirmed the going out of the solution to the regime (9), (27) but also the values of the constants κ, g_0 obtained earlier by the shooting method for eq. (27) in ref. [29][#1].

[#1] In ref [30] the values $K = 0.917$, $Q_0 = 1.885$ are obtained from the solution of the evolutionary problem Taking into account the relationships $K = 1/2\kappa$, and $Q_0 = g_0\kappa^{-1/2}$, it completely corresponds to the values κ, g_0 obtained in ref [29] by the shooting method

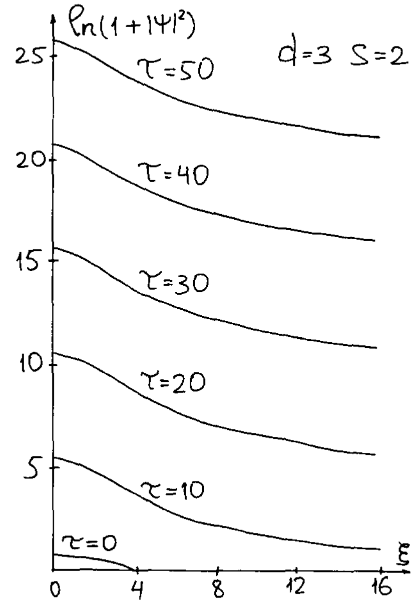

Fig 1 Space dependence of amplitude for different moments of time in the case $d = 3$, $s = 2$, $\Psi(r,0) = \exp(-r^2/16)$

The stated clearly shows the actuality of the questions of the practical realizability of solutions possible in the supercritical case and their interaction in the process of nonlinear evolution, especially of the weak self-similar collapse stability

Our calculations [4, 37, 42] carried out up to exceedings larger than in all enumerated works definitely confirm that the self-similar solution (9), (27) describes the general case of the supercritical collapse (before turning on the dissipative or some other stabilizing mechanisms). The calculations were carried out mainly for the cases $s = 2$, $d = 3$ and $s = 6$, $d = 1$.

The results of calculations showed that the initial distribution of the field is rapidly captured into a stable self-similar regime without changing the form in the process of the further evolution (figs 1, 2) To define the degree of correspondence between a stable process observed and a solution of a weak collapse (9), (27), we shall write this solution in variables of our simulation (13). For the amplitude $|\Psi|$ and phase Φ it is

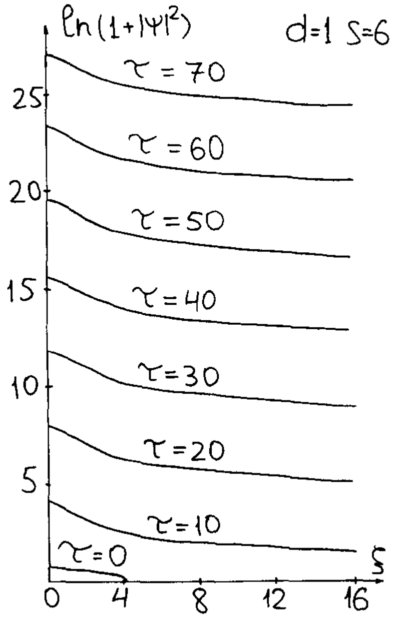

Fig 2 Space dependence of amplitude for different moments of time in the case $d = 1$, $s = 6$, $\Psi(r,0) = \exp(-r^2/16)$

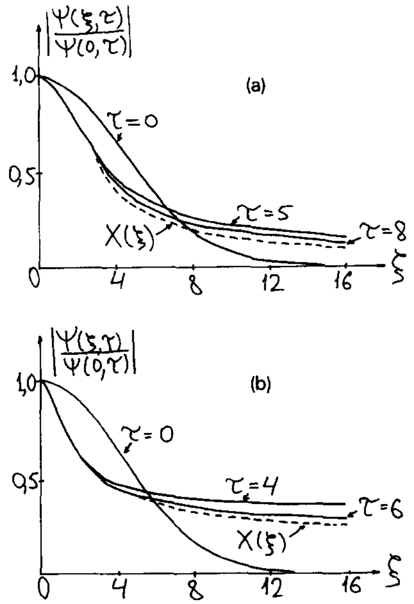

Fig 3 The going out of the solution amplitude to the mode of weak self-similar collapse $X(\xi)$, (a) $d = 3$, $s = 2$, $\Psi(r,0) = \exp(-r^2/36)$, (b) $d = 1$, $s = 6$, $\Psi(r,0) = \exp(-r^2/36)$

easy to get

$$|\Psi(\xi,\tau)| = C_1 g_0 \exp\left(i\frac{\alpha}{s}\tau\right) X(\xi),$$

$$\Phi(\xi,\tau) = \alpha\kappa\tau + \omega(\xi) + C_2, \qquad (30)$$

where $\alpha = g_0^{-s}$, and X and ω are amplitude and phase of the function

$$y(\xi) = g_0^{-1} g(\xi g_0^{-s/2}),$$

described according to (27) by the following equation:

$$\Delta_\xi y + \alpha\left[\frac{i}{2}\xi y_\xi + \left(\frac{1}{s} - \kappa\right) y\right] + |y|^s y = 0,$$

$$y(0) = 1, \quad y_\xi(0) = 0 \qquad (31)$$

From (30) it follows that

$$\left|\frac{\Psi(\xi,t)}{\Psi(0,\tau)}\right| = |X(\xi)|,$$

$$\Phi(\xi,\tau) - \Phi(0,\tau) = \arg\omega(\zeta).$$

From figs. 3, 4 one can see clearly the going out of the solution to a universal mode $y(\xi)$, calculated from (27) (dashed lines) The calculation of the parameters κ and α over the results of simulation (it can be done using (30) or comparing the asymptotic behavior of $y(\xi)$ at $\xi \to 0$ and $\xi \to \infty$ with the solution in the self-similar regime) gave the values coinciding with the result obtained by the shooting method. Thus, it follows from (30) that $dQ/d\tau = \alpha$ ($Q = \ln A^2$), $d\Phi(0,\tau)/d\tau = \kappa\alpha$. From fig. 5 one can see well the going out of both functions to the constants equal to the values $\alpha = 1.160$, $\kappa\alpha = 0.417$. It corresponds exactly to the values κ, g_0 obtained independently by the shooting method at $s = 6$, $d = 1$

The results obtained demonstrate that in the case of general position $sd > 4$ a stable regime of weak self-similar collapse defined by function (27) is realized. However, this conclusion is completely correct only for the evolution in the inertial interval preceding to the absorption. It is shown in section 5 that with account of postcollaptical dynamics the physical picture besides

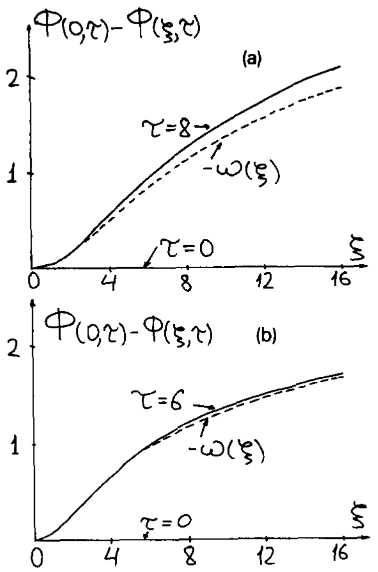

Fig 4 The going out of the solution amplitude to the mode of weak self-similar collapse $\omega(\xi)$, (a) $d=3$, $s=2$, $\Psi(r,0)=\exp(-r^2/36)$, (b) $d=1$, $s=6$, $\Psi(r,0)=\exp(-r^2/36)$

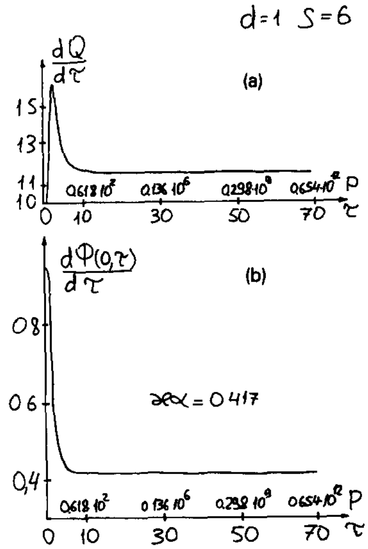

Fig 5 The dependence on time of the functions (a) $dQ/d\tau$ ($Q=\ln A^2$) and (b) $d\Phi/d\tau(0,\tau)$ for $d=1$, $s=6$, $\Psi(r,0)=\exp(-r^2/36)$

critical collapse is enriched by a new class of solutions different from (9), (27).

4. Critical wave collapse

The case $sd=4$ including, in particular, the most important problem of stationary two-dimensional self-focusing ($s=d=2$) is called critical. It is well known that this case is especially difficult both for analytical and for numerical investigation (see, for example, refs. [2–4] and the references given).

In the critical case the self-similar solution (9), (27) leads at $t \to t_0$ to the power profile $|\Psi|^2 \to C^2 r^{-d}$ on which the wave energy integral N diverges logarithmically. Hence, the strictly self-similar solution is useless to describe critical collapse. Let us consider the solitary solution (7) with the shape described in the critical case by equation

$$\left(\Delta_\eta - 1 + R^{4/d}\right)R = 0, \quad R_\eta(0) = R(\infty) = 0. \tag{32}$$

In the one-dimensional case $R(\eta) = 3^{1/4} \times (\mathrm{ch}\, 2\eta)^{-1/2}$; at $d=2$ the function $R(\eta)$ has been calculated in ref. [49].

For the solitary solution $|\Psi|^2 = \lambda^d R^2(\lambda r)$ and it is easy to see that the energy integral does not depend on the parameter λ which defines the soliton size:

$$\begin{aligned} N &= \int_0^\infty R^2(\eta)\eta^{d-1}\,d\eta \\ &\equiv N_{cr} = \pi\sqrt{3}/4 \quad d=1, \\ &= 1.86 \quad d=2. \end{aligned} \tag{33}$$

Neutral stability of the soliton is an argument in favor of the following hypothesis: The singularity in the critical case is a compressing soliton It has been proposed in ref. [18] that

$$\begin{aligned} \Psi &\underset{t\to t_0}{\longrightarrow} f^{-d/2} R\left(\frac{r}{f}\right) \\ &\times \exp\left\{ i\left[\int \left(\frac{1}{f^2} + \frac{r^2}{4}\frac{d}{dt}\ln f \right) dt \right] \right\}, \\ f &= f(t_0 - t), \quad f(0) = 0. \end{aligned} \tag{34}$$

The asymptotic behavior in (34) means, in particular, that critical collapse is strong. the energy N_{cr}

depending only on the space dimension (see (33)) is captured into the singularity. This value defines the necessary condition of the collapse $N > N_{cr}$, i.e. the threshold energy in the critical case.

The spatial distribution near the singularity in the form of the soliton (32) and the capture of energy in the collapse close to the critical one are suggested in numerous computer experiments Refs. [21, 22] are exceptions in which it is stated that under a special choice of initial conditions ("plateau-like" distributions with $N \gg N_{cr}$) an energy substantially exceeding the critical one is captured into the collapse The main (and rather essential) disagreements between authors corresponds to the form of the function $f(t_0 - t)$ characterizing the collapse velocity.

At $f(t) \sim t$ solution (34) is exact but, as was mentioned above, unstable. The result $f(t) \sim t^{2/3}$ obtained in ref [13] on the basis of variational estimations has been confirmed in some works (see, for example, ref. [24]). However, more precise experiments under the approximation by power function $f(t) \sim t^{\mu}$ give the value $\mu = 1/2$ obtained for the first time in ref. [19] The proximity of μ to 1/2 has been pointed out in many papers and there are no doubts at present that it is the most exact result in the framework of power approximation. This numerical result is rather natural and at the same time perfectly unsatisfactory from a theoretical point of view. Actually, as we have seen, low $f(t) \sim t^{1/2}$, which we would have for a strictly self-similar solution, leads to "increasing" of the nonintegrable singularity and cannot be realized in the process of nonlinear evolution. On the other hand as the energy integral diverges weakly (logarithmically) on the self-similar solution, the asymptotic behavior (34) should be "quasi-self-similar" This means, in particular, that time dependence requires more precise approximations than a power approximation: $f^2(t) \sim t/b(t)$, where $b(t)$ varies essentially slower than linearly. The presence of weak dependence $b(t)$ confirmed numerically in many papers (including works with adaptive models) does not give rise to doubt Nevertheless, recently, in ref [41] the conclusion $f^2(t) \sim t$ has been drawn again

The law $b(t) \sim |\log t|$ proposed and numerically verified in refs. [20, 25] (see also ref. [3]) has not been confirmed in adaptive calculations [30, 33], which have demonstrated a slower behavior of $b(t)$. The authors of ref. [30] have the opinion that the same conclusion is correct for the approximation

$$b(t) \sim |\log t|^{\gamma}. \tag{35}$$

Finally, in refs. [27, 34, 35, 44] in a distinct asymptotical region the following result:

$$b(t) \sim \log|\log t|, \tag{36}$$

was obtained analytically and with some reserves supported (see below) by results of calculations presented in ref. [34].

Before giving and analyzing the results of our simulation [4, 36, 42] we must emphasize that in the critical case the virial theorem (5) allows one to define an upper bound of the collapse t_0^* which at $\operatorname{Im}\Psi^0(\boldsymbol{r}) = 0$ is equal to

$$t_0^* = \left(-\frac{1}{4H} \int r^2 |\Psi^0|^2 \, d\boldsymbol{r} \right)^{1/2}$$

and for the initial conditions of the form (23) is expressed through the exceeding (26)

$$t_0^* = l^2 2^{-1-1/n} \times \left(D(n,d) \left(\epsilon^{2/d} - 1 \right) \frac{\Gamma(d/n)}{\Gamma((d+2)/n)} \right)^{-1/2} \tag{37}$$

Certainly, since the condition $H < 0$ is just sufficient, the real time of the collapse is always smaller than t_0^*. In addition, t_0 is sensitive to the structure of the initial profile of the packet [18]. Qualitatively, however, relationship (37) gives in most cases a correct dependence on the parameters and can be useful for estimations

We have performed a computer simulation for a large number of one- and two-dimensional vari-

Table 1

No	d	Ψ_0	l	n	N/N_{cr}	ϵ	t_0^*	t_0
1	2	4	1	2	2 15	2	0 250	0 146
2	2	10	1	8	20 5	8	0 053	0 036
3	2	1 34	8	10	24 71	8	3 136	1 711
4	1	1	4	2	1 84	1 76	2 774	1 378
5	2	8	1	2	8 60	8	0 094	0 043

ants. The illustrative material given below corresponds to the variants presented in table 1, in the last column of which the values of the moments of singularity formation, obtained by the results of simulation from relationship (15), are also presented.

In all variants the going out to the asymptotic behavior (34) has been surely observed. At a sufficient approach to the collapse point a spike compressed self-similarly has been formed in the center; its form coincides with the form of the stationary soliton (fig. 6). From (13), (34) with account of $sd = 4$ it follows that

$$\left|\frac{\Psi(\xi,\tau)}{\Psi(0,\tau)}\right| \to \frac{1}{C_0} R\left(\xi / C_0^{-2/d}\right),$$

$$C_0 = R(0) = 3^{1/4}, \qquad d = 1,$$

$$= 2.2062, \quad d = 2. \tag{38}$$

From fig. 7 one can clearly see in the example of variants 1, 4 the going out of the solution to mode (32). Independent of the initial conditions, the energy N_L captured into the collapse goes rapidly to critical value (33) (fig. 8). Thus, hypothesis (34) is effectively confirmed. The variance with the results of papers [21, 22], in which a relatively small proximity to the collapse has been attained, can be explained by the fact that in these papers there has been observed an intermediate asymptotic behavior depending on the form of the initial conditions. This asymptotic behavior corresponds to the following physical picture.

At large exceedings over critical threshold the plateau-like field distribution is under the influence of a modulational instability, as a result of the development of which the field distribution is

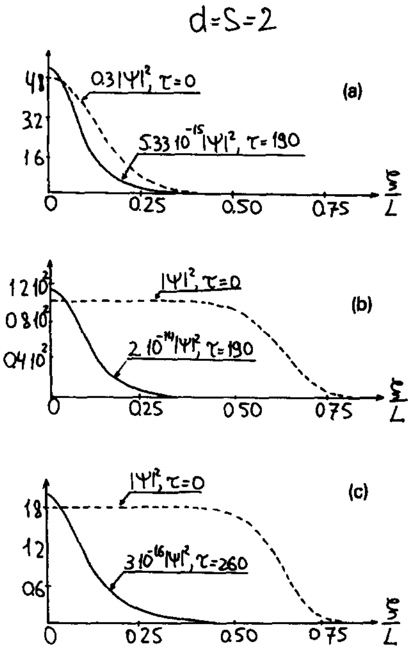

Fig. 6 The established space dependence of the solution and the initial conditions (dashed line) (a) variant 1, (b) variant 2, (c) variant 3

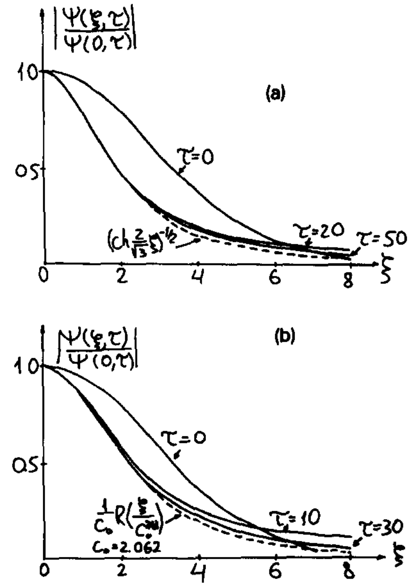

Fig. 7 The going out of the solution amplitude to the form of the stationary soliton for (a) variant 4, (b) variant 1

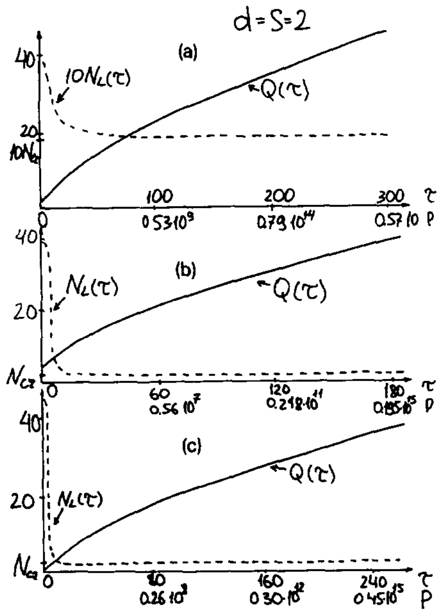

Fig 8 The dependence on time of the captured energy $N_L(\tau)$ and the function $Q(\tau)$, (a) variant 1, (b) variant 2, (c) variant 3

transformed into a system of compressed "rings" Each ring energy exceeds the critical one A ring is reduced in a quasi-self-similar way while conserving its width. When the most rapid comes to the center at a distance of the order of its width, the values of the exceedings of the intensity over the initial one ($P \sim 10^2$–10^3) are attained in the center. At this stage, in papers [21, 22] a nonlinear damping absorbing energy has been included. However, in the conservative system the collapse takes place according to the usual scenario with the formation of a compressed soliton. We must emphasize that the picture described is unstable relative to angle perturbations; thus, in a real two-dimensional situation, when the axial symmetry is not artificially pressed, it will not be realized. This instability has been surely demonstrated, for example, in calculations [48].

Thus, the spatial structure of the singularity near the center at the critical collapse is completely defined by the form of the stationary soliton (32)

The question of time dependence is very difficult. An asymptotical regime of the behavior of $f(t)$ is attained at t very close to t_0, characteristic times (in τ representation) are very large in the critical case. To obtain reliable results it is necessary to attain rather large exceedings P_{max}. As was pointed out above we have attained $P_{max} \sim 10^{18}$. This exceeds the best results of other authors Another important question is to process correctly the time dependence of the results. From our point of view, it should be carried out only in the τ variable. An alternative variant even under the condition of very exact t_0 calculation can distort essentially the picture since the computation of $t_0 - t$ used in general formulas of processing is a source of mistakes[#2] starting from some moment of time t. We processed time characteristics constructing for each investigated model $f(t)$ such a functional at $\xi = 0$, $F[A(\tau)]$, which in the frameworks of the given model should be a linear function of τ The following thorough processing on the basis of a computer experiment of $F(\tau)$ (see details in ref [4]) allowed us to draw clear conclusions about the degree of adequateness of the considered model $f(t)$.

The calculations have demonstrated that $f^2(t)$ is close to a linear function. Under $f^2(t) \sim t$ $F(A) = \log A^2$ (or $\log \mathrm{d}A^2/\mathrm{d}\tau$) In all variants we have obtained that function $Q(\tau) = \log A^2$ is nearly linear over τ (fig. 8), but as the processing has demonstrated, the function $Q(\tau)$ increases slightly slower than linearly. At the same time even an insignificant increase of the power $\mu > 1/2$ assuming $f(t) = t^{\mu}$ ($F(A) = A^{2-1/\mu}$) leads to an increase of $F(\tau)$, which is essentially faster than the linear one.

The results obtained surely confirm the correctness of the law $f^2(t) \sim t/b(t)$, where $b(t)$ is a slowly varying function.

[#2] We suppose that this circumstance is one of the reasons causing a sharp discrepancy between the results of ref [41] and the theoretical and numerical results of other authors both in the critical and in the supercritical case

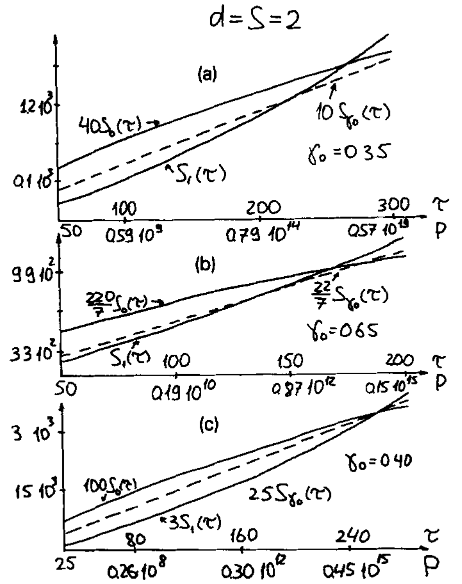

Fig 9 The dependence on time of the functions $S_\gamma(\tau)$, (a) variant 1, (b) variant 2, (c) variant 3

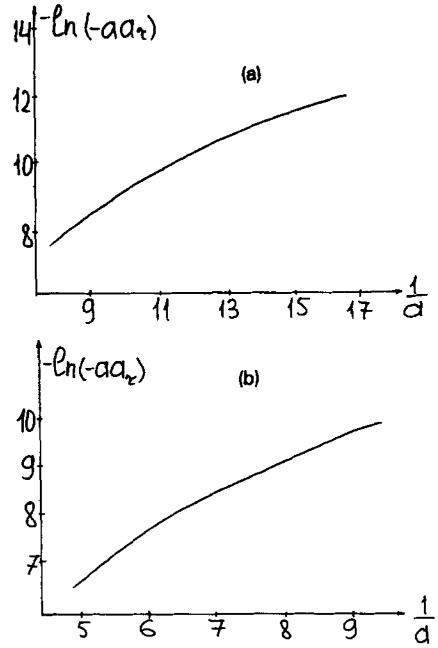

Fig 10 The dependence of $-\ln(-aa_\tau)$ on $1/a$ for (a) variant 1, (b) variant 2

Assuming (35) one gets the relationship $F(\tau) \equiv S_\gamma(\tau)[B - \gamma/B + \frac{1}{2}(\gamma/B)^2 + \ \ldots]^{1+\gamma}$, $B(\tau) = \log(\mathrm{d}A^2/\mathrm{d}\tau) \gg 1$. The processing of functions $S_\gamma(\tau)$ for $0 \le \gamma \le 1$ showed that at $\gamma = 1$ [20, 25] model (35) describes an exceedingly rapid growth of the field amplitude. The value $\gamma = 0$ considered above corresponds to the contrary case of slower field growth than true. Considering intermediate values γ we have that there is a value γ_0 for which the difference between $S_{\gamma_0}(\tau)$ and a linear function is practically absent even under special processing. These conclusions are obviously illustrated by fig. 9, where the curves $S_\gamma(\tau)$ are represented at $\gamma = 0$, $\gamma = 1$ and $\gamma = \gamma_0$. It turned out that although the values γ_0 obtained from the computer experiments depend on the initial conditions they are within the limits $0.35 \le \gamma_0 \le 0.65$. It means that attained law (35) in the frames of exceedings with the pointed out moderate values γ satisfactorily approximates the function $b(t)$.

Let us consider, finally, the "double logarithm" law (36) describing weaker growth of the field amplitude than (35). This law follows, in particular, from the asymptotic equation

$$\frac{\mathrm{d}a}{\mathrm{d}\tau} = -\frac{C_1}{a}\exp\left(-\frac{C_2}{a}\right), \qquad (39)$$

obtained in ref. [34] for a slowly decreasing function $a(\tau)$. Here $a(\tau)$ coincides, up to a constant factor, with the function $a(\tau)$ introduced in eq. (14). From (39) it follows that at $\tau \to \infty$, $a \sim C_2/\log\tau$ and since $a \sim -\mathrm{d}\log f/\mathrm{d}\tau$, law (36) is correct. The processing of results of our simulation showed, however, that the function $-\log(-aa_\tau)$ increases with the growth of $1/a$ during large intervals of this value varying slower than linear (fig. 10). Thus, our calculation does not support formula (36): we observed a slightly more intensive field growth. To explain this divergence it is necessary to develop a theory which explains formula (39). A summary of this theory has been published in ref. [43]. (Variants of this theory have been developed with different modifications in refs. [27, 34, 35, 44].)

Let us consider in eq (2) with $s = 4/d$ a transition to new variables:

$$\xi = \frac{r}{f}, \quad \frac{d\tau}{dt} = f^{-2},$$

$$f(t) = \left(\frac{d}{dt}\arg \Psi(0,t)\right)^{-1/2},$$

and introduce the functions

$$\varphi(\xi,\tau) = f^{-d/2} l^{-i\tau}\Psi, \quad Z = \varphi \exp\left(\tfrac{1}{4} i a \xi^2\right),$$

$$a = -\frac{d}{d\tau}\log f \underset{\tau\to\infty}{\longrightarrow} 0$$

These functions satisfy the equation

$$i\varphi_\tau + \Delta_\xi \varphi + i a\left(\tfrac{1}{2} d\varphi + \xi\varphi_\xi\right) + |\varphi|^{4/d}\varphi - \varphi = 0,$$

$$i Z_\tau + \Delta_\xi Z + |Z|^{4/d} Z - Z(1 - \epsilon\xi^2) = 0,$$

$$\epsilon = \tfrac{1}{4}\left(a^2 + a_\tau\right). \qquad (40)$$

This equation is a nonstationary Schrödinger equation with an effective potential

$$U = -|Z|^{4/d} + (1 - \epsilon\xi^2).$$

In the exact self-similar case $\epsilon = 0$ and one can manage with the stationary solution of the Schrödinger equation. In our case the asymptotic behavior is quasi-self-similar, $\epsilon > 0$. The time derivatives of the functions U and Z are small as before and one can neglect them in the first step. However, the Schrödinger equation (40) has no localized eigenvalues now since $U \to -\infty$ at $\xi \to \infty$. To find a way out let us suppose that

$$Z = \chi \exp\left(\int_\tau^\infty \nu(a)\, d\tau\right), \quad \frac{\nu(a)}{a} \underset{a\to 0}{\longrightarrow} 0,$$

$$\chi = \chi_0 + \chi_1 + \dots, \quad |\chi_1| \ll |\chi_0|.$$

Here χ_0 is the solution of a non-self-conjugate nonlinear eigenvalue problem with an explicit eigenvalue $i\nu(a)$ For an appropriate function φ_0 ($\varphi = \varphi_0 + \varphi_1 + \dots$) this problem takes the form

$$\Delta_\xi\varphi_0 + i a\left(\tfrac{1}{2} d\varphi_0 + \xi\varphi_{0_\xi}\right) + |\varphi_0|^{4/d}\varphi_0$$

$$-\varphi_0 - i\nu(a)\varphi_0 = 0,$$

$$\varphi_{0_\xi}|_{\xi=0} = 0, \quad \varphi_0(\xi) \underset{\xi\to\infty}{\sim} \xi^{-d/2+(\nu-i)/a} \qquad (41)$$

Boundary problem (41) defines the imaginary part of $\nu(a)$ uniquely. Such type boundary problems are well known in the theory of α-decay (see ref [49]).

From the solvability condition of the equation for χ_1 one can get [43]

$$\frac{da}{d\tau} = -C\frac{\nu(a)}{a}, \quad C = 4N_{cr}\Big/\int_0^\infty R^2\xi^{d+1}\, d\xi \qquad (42)$$

At $\tau \to \infty$, $a \to 0$ the imaginary eigenvalue $\nu(a)$ can be calculated using the quasiclassical formula for the coefficient of passing through the potential barrier,

$$\nu = \exp\left(-2\int_0^{\xi_0} U^{1/2}(\xi)\, d\xi\right), \quad U(\xi_0) = 0.$$

In a remote ($\xi \to \infty$) asymptotical limit $a \to 0$ neglecting small terms one can assume that $U(\xi) \approx 1 - \frac{1}{4}a^2\xi^2$. Then

$$\nu(a) \sim \exp(-\pi/a). \qquad (43)$$

So, from the stated follows not only eq. (39) but also the value of the constant $C_2 = \pi$. It was pointed out above that results represented in ref. [34] of adaptive calculations up to the exceedings $P_{max} \sim 10^9$ confirm the correctness of eq. (39). However, the value of the constant C_2 that was obtained is appreciably smaller than the theoretical one. In connection with this the authors of ref. [34] point out that the asymptotic regime has not been attained in their calculations.

Thus, we think that in the critical case even the attaining of record exceedings $P_{max} \sim 10^{18}$ does not allow to confirm surely $b(t) \sim \log|\log t|$, which, probably, is actually realized in a very small region near t_0. In the framework of exceedings obtained the approximation $b(t) \sim |\log t|^{\gamma}$, $0.35 \leq \gamma \leq 0.65$ is satisfactory. But a more exact answer in the broad vicinity of t_0 can be obtained solving numerically the eigenvalue problem (41).

5. Collapse and dissipation. Superstrong wave collapse

The central question of the physical theory of collapses is to estimate the effectiveness of collapse as a nonlinear mechanism of wave energy dissipation. For that it is necessary, first of all, to include in the equation describing the collapsing medium nonlinear dissipative terms. It is convenient to modify eq. (2) introducing nonlinear damping concentrated near the center $r \approx 0$:

$$\mathrm{i}(\Psi_t + \beta|\Psi|^m\Psi) + \Delta_r\Psi + |\Psi|^s\Psi = 0. \qquad (44)$$

At sufficiently large m eq. (44) has regular solutions for arbitrary small β. The energy absorbed in the collapse act is

$$I = \beta\int \mathrm{d}t \int_0^\infty \mathrm{d}r|\Psi|^{m+2}r^{d-1}. \qquad (45)$$

The behavior of integral (45) at $\beta \to 0$ defines energetic effectiveness of the collapse. In section 1 we pointed out that insensibility of I to varying the damping parameters is characteristic for the strong collapse. Numerical integration of eq. (44) at $sd = 4$ has confirmed [48] that critical wave collapse is strong. The energy absorbed in the collapse is over $(0.15–0.25)N_{cr}$. This value did not nearly change when decreasing β and weakly decreased when increasing m.

In the supercritical case $sd > 4$ the integrable singularity of wave energy density (28) is formed in the collapse point. Characteristic time Δt of the formation of the dissipation scale r_{min} on which the collapse is stopped is, according to (9), $\Delta t \sim r_{min}^2$. Substituting the self-similar solution in (45) we obtain an estimation

$$I \sim \beta r_{min}^{(sd+2s-4-2m)/s}. \qquad (46)$$

It is clear from (46) that the nonlinear damping is effective if $m > (sd + 2s - 4)/2$. Assuming that the whole energy concentrated in the collapse zone is absorbed,

$$\Delta N \sim \frac{s}{sd-4} r_{min}^{(sd-4)/s},$$

we have

$$r_{min} \sim \left(\frac{sd-4}{s}\beta\right)^{s/2(m-s)}$$

and

$$I \sim \left(\frac{sd-4}{s}\right)^{z-1}\beta^z, \quad z = \frac{sd-4}{2(m-s)}. \qquad (47)$$

From (47) it follows that $I \to 0$ at $\beta \to 0$, i.e. collapse is weak; at $sd \to 4$ the weak collapse transforms into the strong one.

The considerations presented are correct only for the absorbing in a single collapse act. The situation pointed out in section 1 is also widely spread when as a result of collapse in the vicinity of the singularity the stationary suck in point – the dissipation zone, which absorbs energy from the surrounding space with constant energy flux to the singularity. This effect of "hot spot" considered more or less in detail in various models in refs. [32, 38–40, 43, 45, 46, 50–52] has been termed a "funnel effect" [50], a "nucleation" [51], a "distributed collapse" [32, 40], "bi-self-similar collapse" [38], and "singular collapse" [46]. In refs. [39, 42] we have proposed the name "superstrong collapse" for this effect, which is extremely important from a variety of points of view. The reason was that the total energy absorption for the life-time of such a regime larger than Δt may be significantly greater than the energy absorption in a single collapse act

The existence of superstrong collapse in the frameworks of NSE means, in particular, that the limit (at $\beta \to 0$) of eq. (44) has a stationary singular solution with constant value of energy flux into the singularity

$$P = -\lim_{r\to 0} |\Psi|^2 r^{d-1} \frac{\mathrm{d}}{\mathrm{d}r} \arg \Psi. \tag{48}$$

We have shown and confirmed numerically in ref. [39] that the solutions of such type can actually be obtained. Let us demonstrate following ref. [39] that under the condition $sd \geq 2s+2$ (taking place at $d>2$) supercritical collapse is superstrong.

Consider $d>2$, $sd>2s+2$. Then the limit stationary equation

$$\Delta_r \Psi + |\Psi|^s \Psi = 0 \tag{49}$$

has the exact singular solution

$$\Psi = A r^{-2/s}, \quad A = \left[2(sd-2s-2)/s^2\right]^{1/s}. \tag{50}$$

One can use this equation as a first step for the construction of the singular solutions with a flux. For $2s+2 < sd < 2s+4$, solution (50) is supplemented by a family of stable singular solutions which have the following asymptotic behavior as $r \to 0$ with energy flux (48)

$$|\Psi| = A r^{-2/s}(1 + A_1 r^{\mu} + \ldots),$$

$$\mu = \frac{2}{s}(2s+4-sd) > 0, \quad A_1 = qP^2. \tag{51}$$

(We shall not reproduce the bulky expression for the constant $q = q(s,d) > 0$.) At $d=3$ the case pointed out takes place at nonlinearity indices $2<s<4$. In the case $sd = 2s+4$ ($s=4$ if $d=3$) eq. (49) has a single-parameter family of exact singular solutions for which we have

$$|\Psi| = B r^{-2/s}, \quad B^4\left[B^s - \left(\frac{2}{s}\right)^2\right] = P^2. \tag{52}$$

If $sd > 2s+4$, solution (50) is isolated, but eq. (49) has a single-parameter family of "quasiclassical" (in the sense of ref. [29]) solutions with asymptotic behavior

$$|\Psi| = C r^{-\gamma}(1 + C_1 r^{\nu} + \ldots), \tag{53}$$

where $C = P^{\alpha}$, $\gamma = \alpha(d-1)$, $\nu = \alpha(sd-2s-4) > 0$, $\alpha = 2/(s+4)$, $C_1(s,d) > 0$.

Finally, at the lower limit of the parameter region under consideration, $sd = 2s+2$ (the case $d=3$, $s=2$ corresponding to the subsonic Langmuir collapse, which is a particular version of this limit, but physically the most important one, was analyzed in refs. [32, 38, 40]), the following special stationary solution can take place:

$$|\Psi| = \left(2/s^2\right)^{1/s} r^{-2/s} |\log r|^{-1/s} \tag{54}$$

Near this solution there is also a family of singular solutions with a flux. The existence of stationary singular solutions of eq. (2) with a nonzero flux is analogous to the case of "falling on a center" in quantum mechanics (see ref. [53], for example).

To test the fact that stationary regimes of superstrong collapse are established, we carried out a numerical integration of eq. (44) in variables (13) using the method presented in section 2 Below we report results for $s=2$, $\beta = 10^{-9}$, $m = 6$, and $\Psi(r,0) = \exp(-r^2/16)$. It is convenient to consider space–dimension d as a not obligatory integer value. We varied d over the range $2.5 \leq d \leq 5$ in such a way that the grid of variants covered all possible dynamical regimes When we fixed the dimensionality at $d=3$ and varied s, we found no qualitative change in the picture.

In all variants we observed that the solution reliably approached the mode of the weak self-similar collapse in the inertial interval, to the point that the nonlinear damping came into play

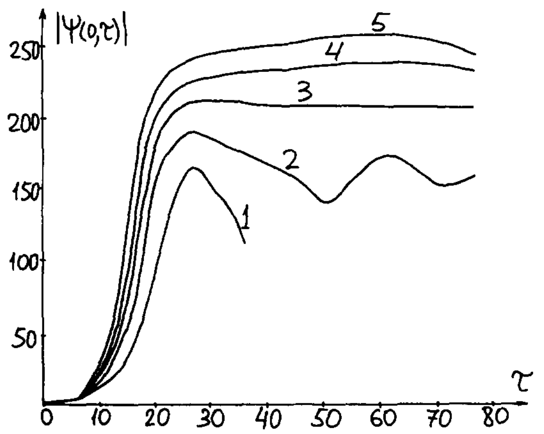

Fig 11 Evolution of the field at the center for $s=2$ and various variants (1) $d=2.5$, (2) $d=3$, (3) $d=3.5$, (4) $d=4$, (5) $d=5$

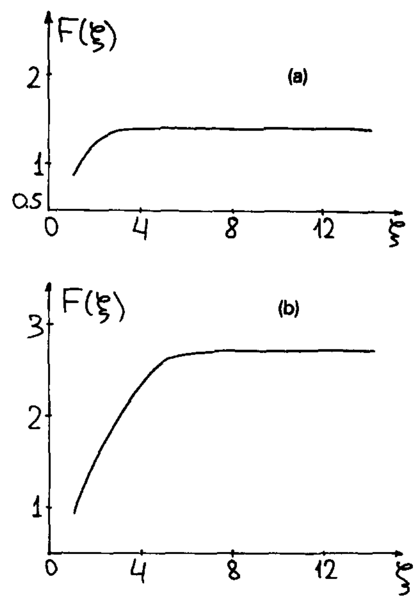

Fig 12 Spatial structure of the solution in the quasistationary state for two variants (a) $d=3.5$, (b) $d=4$

at the level $|\Psi(0,\tau)| \sim 10^4$–10^5. We see from fig. 11, which shows the time evolution of the amplitude at the center, that the behavior of the solution at $d=2.5$ is characteristic of the ordinary scenario of weak collapse. At $d=3$ we observe an oscillatory regime, which we can somewhat arbitrarily regard as a "quasistationary" regime. For the variants $d=3.5$, $d=4$, and $d=5$ the behavior of the amplitude is approximately stationary. The hypothesis of the existence of hot spots is thus effectively confirmed. We also found an agreement between the spatial behavior of the collapse mode which was established and the formulas given above. For example, we see from fig. 12 that the value $F(\xi) = \xi^{\kappa}|\Psi(\xi,\tau)|/|\Psi(0,\tau)|^{s\kappa/2}$ (for the variants shown here $\kappa = 2/s = 1$) becomes essentially independent of the spatial variable outside the effective radius of the nonlinear damping. Finally, the existence of stationary singular solutions (5)–(7) of eq. (2) was verified by our direct numerical integration of the stationary equation with a discontinuous damping coefficient $\beta = \beta_0\theta(r_{min} - r)$, $r_{min} \ll 1$, where θ is the theta function.

So, the analysis and calculations show that at $sd \geq 2s+2$ the supercritical case is superstrong. Collapse is weak only if $4 < sd < 2s+2$, $s>1$. According to the hypothesis proposed in ref. [46], this fact is explained by the weakness of nonlinear effects in region of parameters pointed out and gradual relaxation of the field to the state in which the nonlinearity is overwhelmed by the dispersion.

We shall emphasize that hot spots supplied by a constant energy flux can exist regardless of the degree of nonlinearity only in a case with $d>2$ (realistically, only for three-dimensional physical systems). In cases with $d \leq 2$ there can be (at $sd>4$) only weak collapse, which is what was observed in ref. [37] and an analysis of eq. (2) with $d=1$ and $s=6$, as a model suggested for the three-dimensional problem. It is clear now that this assumption is unjustified and that constancy of the product sd does not guarantee a qualitative similarity in the behavior of the solutions as the dimensionality is reduced. This fact was also pointed out in a recent paper [40]. In this work the stability of superstrong collapse with respect to perturbations destroying its symmetry has also been demonstrated (for $s=2$, $d=3$; see also ref. [45]).

It should be pointed out in the conclusion of this section that we understand the "quasi-stationary" nature of superstrong collapse in the sense that the lifetime of the regime is significantly longer than the time scale of the preceding evolution in the inertial interval.

6. Conclusion

We have developed an adaptive method of numerical investigation of singularities arising in the collapses in the framework of one of the fundamental models of theoretical physics – the nonlinear Schrödinger equation. This method is based on the nonlinear substitution of variables corresponding to the transition into the coordinate system of the collapsing region with simultaneous "straightening" of time and throwing back the peripheral part of the distribution. The obtained record exceedings over the intensity ($P_{max} \sim 10^{18}$) exceed essentially the values reached in most advanced calculations. Using the method developed we have investigated the structure of wave field singularity near the collapse point in detail.

We have demonstrated numerically that the stable regime of weak self-similar collapse is realized in the supercritical case $sd > 4$ before including stabilizing mechanisms. The collapse mode is defined by eqs. (9), (27). The analysis of the postcollaptical state has shown that at $sd \geq 2s + 2$ there are different regimes of superstrong collapse – quasistationary burning in the center supplied by constant energy flux into the singularity. We have constructed corresponding stationary singular solutions (51)–(54). Analytical conclusions are confirmed by computer simulation. Supercritical collapse is weak in the region $4 < sd < 2s + 2$, $s > 1$.

The singularity structure of critical ($sd = 4$) collapse is investigated. The going out of the field to the self-similar asymptotic distribution (34) defined by stationary soliton is shown. The energy captured into the singularity is fixed and equal to the critical one. The approach law to the collapse point is investigated, the quasi-self-similar behavior of time law is confirmed. The approximate formula for collapse velocity is obtained. The method for time dependence definition is obtained. This method is connected with the solution of the auxiliary nonlinear eigenvalue problem.

The results obtained not only can be applied to the problems described by NSE but also are of principal importance from the point of view of general theory of wave collapses, playing a central role in many problems of strong turbulence physics. We must point out a remarkable physical example of superstrong collapse: It is natural to consider the shock wave formation in gas dynamics as a collapse of spatial gradients of velocity field occurring for a finite time. As for the shock wave representing the long-lived zone of energy dissipation it can be considered as a one-dimensional superstrong collapse.

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