Intermittency and Turbulence

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We suggest a mechanism for the appearance of intermittency in fully developed turbulence consistent with the pictures recently presented by Kraichnan [Phys. Rev. Lett. **65**, 575 (1990)] and She [Phys. Rev. Lett. **66**, 600 (1991)]. The key features in our model are (i) an inverse cascade associated with the spectral density of an additional finite flux motion invariant, leading to (ii) an attempt to form large-scale structures which (iii) are intrinsically unstable to a broadband spectrum of perturbing modes resulting in a secondary transfer of energy to small dissipative scales in intermittent bursts.

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Kolmogorov [1] proposed a universal theory for small-scale eddies in high-Reynolds-number turbulent flows. It rests on several assumptions: that the transfer of spectral energy to large wave numbers is local over a window of transparency (the inertial range) in wave-number space, and that statistical information is lost in the cascade so that average flow quantities are scale invariant and determined by the mean rate of energy flux ε which is constant in a statistically steady state. If $E(k)$ is the spectral energy $[\overline{u^2}=\int E(k)dk]$ then, for isotropic turbulence, the equation for $E(k)$ is

$$\frac{\partial E(k)}{\partial t}=T(k)-2\nu k^2E(k)+f(k)\,, \qquad (1)$$

where $T(k)$ is the energy transfer integral given by a linear functional of the third-order moment in velocities. It is called a transfer integral because it neither produces nor dissipates energy and it has the property that $\int T(k)dk=0$. The terms $f(k)$ and $-2\nu k^2E(k)$ represent the production and dissipation of energy. The former could be proportional to $E(k)$ if energy is introduced by an instability process. Locality means that there is a transparency window (called the inertial range) in the wave-number spectrum [between the scale k_0^{-1} at which energy is introduced and the dissipation scale $k_d^{-1}=(\nu^3/\varepsilon)^{1/4}$] where $T(k)$ exists. In this window, the production and dissipation terms $f(k)$ and $-2\nu k^2E(k)$ are unimportant. In that case, the balance in (1) has conservation law form $E_t=T(k)=-P'(k)$, where $P(k)$ is the energy flux, positive when the flow of energy is to small scales and large wave numbers. We call $\int_a^b E(k)dk$ $(0\le a<b\le\infty)$ a true constant of the motion if $P(k)=0$ at both $k=a$ and $k=b$, so that the total energy is trapped in the interval (a,b) for all time because of zero flux through the boundaries. However, the presence of viscosity makes these thermodynamic equilibria uninteresting because there is a constant leakage of energy through to the dissipation scales. Therefore, in any interval (a,b) of the transparency window where $a>k_0$, $b<k_d$, $(\partial/\partial t)\int_a^b E(k)dk$ is zero by virtue of the fact that the fluxes $P(k)$ at $k=a$ and $k=b$ are not zero but the same. In this case, we call $\int_a^b E(k)dk$ a finite flux constant of the motion, and it is finite flux constants which are important to us here. Moreover, within the window of transparency, the interval (a,b) is arbitrary, so that then $P(k)$ is a constant and equal to the mean dissipation rate ε throughout the inertial range. From dimensional considerations $E(k)$ (l^3t^{-2}), k (l^{-1}), and ε (l^2t^{-3}) are related by the well-known Kolmogorov law $E(k)=c_2\varepsilon^{2/3}\times k^{-5/3}$, where c_2 is a universal constant. Suitably normalized higher-order moments of velocity differences $u_l=u(\mathbf{x}+\mathbf{l})-u(\mathbf{x})$ (velocity gradients in the limit $l\to 0$) are also universal constants. The relevance of such solutions to turbulence rests on the Kolmogorov assumption that the energy dissipation rate $\varepsilon=-(d/dt)\overline{u^2}=2\nu\int k^2E(k)dk$ does indeed settle down to a steady-state value in which the energy production rate $\int f(k)dk$ is balanced by the dissipation rate. In this Letter, we will suggest that the primary cascade of spectral energy to small scales is necessarily accompanied by an inverse cascade of the density of another finite flux motion invariant to large scales and that the accumulation of this second density there leads to fast instabilities which result in a secondary cascade of energy towards small scales. It is this secondary cascade that gives rise to intermittency. In many respects our ideas are consistent with the works of Kraichnan [2] and She [3]. Kraichnan shows how the deviation from Gaussian behavior in the probability density function (PDF) for velocity gradients (which can be taken as a definition of intermittency) can be explained by following the dynamical and nonlinear evolution of the PDF due to the combined influences of straining and viscous relaxation. She follows Kraichnan but is more specific in attributing the non-Gaussian behavior to local structures with high amplitude fluctuations in the velocity gradient field. By contrast we suggest a physical mechanism for intermittency by identifying a source (the inverse cascade) for building the large-scale structures whose instabilities lead to intermittent events, a source which is present even when these structures are not directly forced or even when the external forcing has been switched off.

Optical turbulence.—We illustrate the ideas in the context of the optical turbulence connected with the regularized nonlinear Schrödinger (NLS) equation $i\psi_t+\nabla^2\psi+\alpha\psi^2\psi^*=i\gamma\cdot\psi$, where $\gamma\cdot\psi$ means $\int\gamma(k)A(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}d\mathbf{k}$ with $A(\mathbf{k})$ the Fourier transform of $\psi(\mathbf{x},t)$. The damping function $\gamma(k)$ is positive near $k=0$ and $k=k_d\gg k_0$ and is negative, that is, amplifying, in a narrow window $(k_0-\Delta k,\,k_0+\Delta k)$. In a spatially homogeneous random field we take $i\int(\psi\nabla\psi^* - \psi^*\nabla\psi)d\mathbf{x}$ to be zero and then the two "invariants" are number density $N=\int\psi\psi^* d\mathbf{x}$ and energy $H=\int(|\nabla\psi|^2 - \frac{1}{2}\alpha|\psi|^4)d\mathbf{x}$. When the amplification rate is sufficiently small, the primary fluxes of number and energy density in wave-number space are described by weak turbulence theory, and a kinetic equation [4,5] $\partial N_\omega/\partial t + 2\gamma(\omega)N_\omega = T(n)$ can be written for the number density averaged over angle, $N=\int n_k d\mathbf{k} = \int N_\omega d\omega$, where $n_k\delta(\mathbf{k}-\mathbf{k}') = \langle A(\mathbf{k})A^*(\mathbf{k}')\rangle$ and $N_\omega = \Omega_0 k^{d-1}(dk/d\omega)n_k$ with $\omega = k^2$ and Ω_0 the solid angle in d dimensions. The kinetic equation is closed because $T(n)$ can be approximated by an integral involving triple products of n_k whose form makes clear that the principal transfer mechanism is a four-wave resonant interaction. Moreover, one can write $T=\partial^2 R/\partial\omega^2$, $R=\int_0^\omega(\omega-\omega')T(n)d\omega'$, $Q=\partial R/\partial\omega$, $P=R-\omega\,\partial R/\partial\omega$ and then it is easy to see that $Q(P)$ represents the flux of number density N_ω (energy density $E_\omega = \omega N_\omega$) towards low (high) wave numbers. Equilibrium solutions are of (a) thermodynamic type, $n_k = \tau(s+\omega)^{-1}$, where τ is temperature and s chemical potential for which both fluxes Q and P are zero, (b) pure Kolmogorov type, $n_k = c_1 Q^{1/3}\omega^{-d/2-1/3}$, $n_k = c_2 P^{1/3}\omega^{-d/2}$, which correspond respectively to a constant finite flux $Q(P)$ and zero flux $P(Q)$ towards low (high) wave numbers and are valid for $d \gtrsim 3$, and (c) a combination of (a) and (b), $n_k = \tau[s+\omega+aQ\tau^{-3}\omega^2\ln^2(\omega/\omega_c)]^{-1}$, which describes the equilibrium state for $0<\omega<\omega_0$ ($\omega_0 = k_0^2$) for $d=2$, and has constant finite flux Q. Further, all the usual Kolmogorov assumptions obtain for the primary fluxes of number and energy densities. Interactions are local [$T(n)$ converges for solutions n_k in the neighborhood of the finite flux equilibrium spectra], statistical information is lost (all nonresonant interactions are ignored and the kinetic equation is irreversible), and average flow quantities in the windows $(0,\omega_0)$ and (ω_0,ω_d) are scale invariant and depend only on Q and P, respectively.

However, weak turbulence theory eventually fails. The reason is that the flux of energy density $E_\omega = \omega N_\omega$ towards high wave numbers is necessarily accompanied by a flux of particle number density towards $\omega=0$. This occurs because of the conservation laws. A particle born at $\omega=\omega_0$ carries with it an energy ω_0. When it dies at the damping frequency $\omega_d \gg \omega_0$ it has significantly greater energy. Therefore very few particles born at ω_0 get to $\omega=\omega_d$. Instead, most end up with energies $\omega<\omega_0$. The redistribution of energy is achieved by nonlinear interactions in which a small number of particles N_d ($N_d\omega_d \sim N_0\omega_0$) pick up energy but most, i.e., $N_0 - N_d$, lose energy. What is the fate of the particles that drift to low frequencies? Near $k=\omega=0$, the low amplitude theory fails because the quadratic term in H no longer dominates the quartic and one must resort to a fully nonlinear theory. In the defocusing case, $\alpha=-1$, condensates $\psi=\psi_0\exp(i\alpha|\psi_0|^2 t)$ are built and their growth can only be controlled by damping at $k=0$. Further, the weak turbulence theory of fluctuations about condensates has a different character [5]. In the focusing case $\alpha=+1$, the condensate state is unstable, a saddle point in the phase space of the system. While the condensate itself is never attained, its unstable manifold, which in physical space consists of collapsing filaments [6], is reached, and a secondary flux of number density reverses the direction of the inverse cascade and sends particle number density (but not energy density because $H=0$ for a collapsing filament) back towards high frequencies. No damping at $k=0$ is required. The nature of the secondary flow is entirely different from that of the primary flows. It is simply the manifestation of a collapsing filament in physical space in which number density is squeezed from large to small scales in a highly organized and coherent manner. No statistical information is lost in each event. Statistical considerations are introduced, however, by the intermittent nature of events, the uncertainty in time and space as to when and where they occur. The process is probably governed by Poisson statistics whose parameters depend on the primary flux of particle numbers towards the origin. Because these events involve large amplitude fluctuations, their effect is experienced principally by the tails of the probability density function for $\psi(\mathbf{x},t)$. Their manifestation in the particle number dissipation rate is seen as an intermittent sequence of spikes superposed on a background arising from those particles which reach large frequencies through four-wave mixing. Further, the inverse flux appears to be enhanced when intermittency is present because the incomplete burnout of collapsing filaments leads to the production of new wave-train particles, some of which gain in frequency due to four-wave mixing but most of which lose.

Hydrodynamic turbulence.— Using these ideas, we ask if it is possible that a similar scenario occurs in three-dimensional, isotropic, hydrodynamic turbulence? Define the velocity correlations,

$$\overline{u^2}f(r) = \langle u(\mathbf{x})u(\mathbf{x}+\mathbf{r})\rangle = \int_0^\infty F(k)\cos kr\,dk\,,$$
$$\overline{u^3}h(r) = \langle v^2(\mathbf{x})u(\mathbf{x}+\mathbf{r})\rangle = \int_0^\infty kH(k)\cos kr\,dk\,,$$
$$\overline{u^3} = (\overline{u^2})^{3/2}\,,$$

where u and v are the velocity components parallel and perpendicular to $\mathbf{r}$ and $\overline{u^2}=E$ is two-thirds the kinetic energy. The von-Karman–Howarth equation for $f(r)$, without forcing, is

$$\frac{\partial}{\partial t}\overline{u^2}f(r) + 2\overline{u^3}\frac{1}{r^4}\frac{\partial}{\partial r}r^4 h(r) = 2\nu\overline{u^2}\frac{1}{r^4}\frac{\partial}{\partial r}r^4\frac{\partial f}{\partial r}\,,$$

from which we obtain formally

$$\frac{\partial E}{\partial t} = -2\nu\frac{E}{\lambda^2} = -\varepsilon,\quad \frac{\partial M}{\partial t} = 0,\quad \frac{\partial L}{\partial t} = -2\overline{u^3}c = -\mu\,, \tag{2}$$

where $M=\lim_{r\to\infty} r^3 f(r)$, $L=\int_0^\infty \overline{u^2}r^4 f(r)dr$, $\lim_{r\to\infty} r^4 \times h(r) = c$, and $\lambda = [-f''(0)]^{-1/2}$ is the Taylor microscale. In the absence of viscosity, the first equation expresses conservation of energy. In the presence of forcing at intermediate scales k_0^{-1} and under the assumption that viscosity acts only after the viscous scales k_d^{-1}, the Kolmogorov theory asserts that the turbulence relaxes to a steady state for which ε is constant and, along with the local scale k^{-1}, determines statistical behavior in the wave-number window $(k_0,\ k_d=(\varepsilon\nu^{-3})^{1/4})$. The second equation corresponds to the conservation of (average) squared linear momentum (SLM)

$$M=\lim_{r\to\infty} r^3 f=\int_0^\infty (r^3 f)'dr$$
$$=(4\pi)^{-1}\int \sum_{i=1}^{3}\langle u_i(\mathbf{x})u_i(\mathbf{x}+\mathbf{r})\rangle d\mathbf{r}\,.$$

In the absence of forcing, it is a true motion constant as there is no leakage at small or large scales. Its invariance was first noted by Saffman [7] who argues that in general M is not likely to be zero and supports his argument by showing that if a turbulence field is generated by a distribution of random impulsive forces with convergent integral moments of cumulants, then M is nonzero and constant. This is in contrast to the work of Batchelor and Proudman [8] who, assuming that the turbulent field has initially convergent integral moments of the velocity distribution, found $M=0$ and the Loitsyanskii "invariant" L [representing the average of the squared angular momentum (SAM) $=(4\pi)^{-1}\int r^2\langle u(\mathbf{x})u(\mathbf{x}+\mathbf{r})\rangle d\mathbf{r}$] to exist although, due to large range pressure correlations, the third-order velocity correlation $h(r)$ does not decay sufficiently fast so that L is constant. Rather, they found that $h(r)\to cr^{-4}$ as $r\to\infty$ which leads to (2). Nevertheless, we shall argue that when L exists, it is a finite flux motion invariant in exactly the same way E is and that its loss occurs at low wave numbers near $k=0$. We assume that if the fluid is stirred at intermediate scales k_0^{-1}, the energy density $E(k)$ [$E=\int E(k)dk$] flows to high wave numbers at constant rate ε and we then *prove* that the density of SAM $J(k)$ [$L=\int J(k)dk$] must flow to low wave numbers and suggest that the flux rate will be μ. We shall argue the same for $M(k)$ [$M=\int M(k)dk$], the density of SLM.

We now formally define spectral densities for L and M. Consider

$$j(r)=\int_r^\infty \overline{u^2}r^4 f(r)dr=\int_0^\infty J(k)\cos kr\,dk$$

and

$$m(r)=\int_r^\infty \overline{u^2}(r^3f)'dr=M-\overline{u^2}r^3f(r)$$
$$=\int_0^\infty M(k)\cos kr\,dk$$

so that $j(0)=L=\int_0^\infty J(k)dk$ and $m(0)=M=\int_0^\infty M(k)dk$. A little analysis will show that $kJ(k)$ and $kM(k)$ are the Fourier integral sine transforms of $\overline{u^2}r^4f(r)$ and $\overline{u^2}(r^3f)'$. The Fourier integral cosine transforms of these two quantities are $F''''(k)$ and $kF'''(k)-3F''(k)=3[k^{-1}E(k)]'$, where $3E(k)=k^2F''(k)-kF'(k)$ [9]. Hence (P is Cauchy principal value),

$$F''''(k)=\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{k'J(k')dk'}{k'-k},$$
$$kJ(k)=-\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{F''''(k')dk'}{k'-k}, \tag{3}$$

$$[3k^{-1}E(k)]'=\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{k'M(k')dk'}{k'-k},$$
$$kM(k)=-\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{3[k'^{-1}E(k')]'}{k'-k}dk'. \tag{4}$$

From (3) we obtain, for small k, that $E(k)\sim(2L/9\pi)k^4$, and for large k, $kJ(k)\sim 24E/\pi k^5$ so that $\int_{k_0}^\infty J(k)dk=24E/5\pi k_0^5<\infty$. From (4), $E(k)=(M/6\pi)k^2$ for small k, i.e., $E(k)$ is thermodynamic, and for large k, $M(k)=-6E/\pi k^4$ so that $\int_{k_0}^\infty M(k)dk=-2E/\pi k_0^3<\infty$. Therefore, since the interval (k_0,∞) can only contain a finite amount of L and M and since, as we show in the next paragraph, there is no leakage of either quantity through $k=\infty$, if either $J(k)$ or $M(k)$ is produced at a finite rate near k_0, the density of each must increase in the low wave-number range $k<k_0$. Indeed, recent experiments of Douady, Couder, and Brachet [10] who, using a new bubble visualization technique, observe that short-lived high-vorticity filaments appear to form spontaneously and disintegrate through helical instabilities which stir large eddies, are consistent with our picture of an inverse cascade of $J(k)$.

We may write equations for the spectral densities $F(k)$, $M(k)$, and $J(k)$ directly from their definitions and the von-Karman–Howarth equation with added forcing. Each contains a transfer integral, dissipation, and forcing terms. It is clear that in the absence of forcing, the contribution to $(\partial/\partial t)\int_0^\infty F(k)dk$ comes from dissipation and high wave numbers, $(\partial/\partial t)\int_0^\infty M(k)dk$ is identically zero, and $(\partial/\partial t)\int_0^\infty J(k)dk=-\int_0^\infty T_3(k)dk$, where

$$T_3(k)=(4/\pi)\overline{u^3}\int_0^\infty [c-r^4h(r)]\cos kr\,dr\,.$$

The decay of L comes from small wave numbers and is due to the large-scale behavior of the third-order correlation because of long-range pressure forces. The contribution from the dissipation term is zero. For example, if we take $r^4h(r)\sim c[1-r_1^2(r_1^2+r^2)^{-1}+\cdots]$, then $T_3(k)=2c\overline{u^3}\delta(k)$ as $r_1\to\infty$. If $J(k)$ is increased at $k=k_0$ and lost at $k=0$, we might expect the squared angular momentum flux rate $\mu=2\overline{u^3}c$ plays a similar role in the window of transparency $(k_1=r_1^{-1},\ k_0)$ as ε, the dissipation rate, plays in (k_0,k_d), although this assertion is not crucial to our argument that $J(k)$ flows to small wave numbers. If the Kolmogorov hypotheses hold in this region, then the usual dimensional considerations give $J(k)=c_1\mu^{2/3}k^{-10/3}$ and $E(k)\sim k^{5/3}$.

No matter whether M or L is invariant, we have shown that, if produced at a constant rate at k_0, the corresponding spectral density will flow to small wave numbers. The question then is: What is the fate of these "particles"? Do they condense into large-scale structures as in the cases of defocusing NLS or two-dimensional hydrodynamics where mean-squared vorticity density flows to small scales and energy to large scales where it builds large vortices? Or do they behave as in the case of the focusing NLS where instead of building condensates, they nucleate collapsing filaments which return the energy to high wave numbers? Our conjecture is that the inverse cascade of $J(k)$ should lead to the formation of large vortical structures just as the inverse cascade of particle number in NLS should lead to condensates. But in the focusing case, we have seen that because these condensates are unstable, they never get a chance to form. Instead, as soon as the particle number density reaches scales large enough to nucleate collapsing filaments, the latter are formed and the inverse cascade is reversed. So just as in optical turbulence, where, although the condensate state is never reached, its unstable manifold plays an important role in the dynamics, in three-dimensional hydrodynamics we should look at the instabilities of large vortical structures although these structures themselves will never get the chance to form. Bayly [11] has shown that elliptical vortices are unstable to a subharmonic resonance between the inertial wave $e^{i\mathbf{k}(t)\cdot\mathbf{x}}$ with frequency $\omega = 2\Omega\cos\theta$, where Ω is the rotation speed of the vortex and $\cos\theta = \mathbf{\Omega}\cdot\mathbf{k}/\Omega k$. The subharmonic resonance occurs at $\theta = \frac{1}{3}\pi$, the window of instability $|\theta - \frac{1}{3}\pi|$ depends on the amount of ellipticity α $[\mathbf{u} = (\Omega(-1-\alpha)y, \Omega(1-\alpha)x, 0)]$ in the original vortex, and the rate of growth is proportional to α and independent of the wave number k. Therefore the amount of energy which is inserted directly in short waves is largest. While one would expect that the net effect of the instability is to restore an isolated elliptical vortex to a circular shape, the lowest energy configuration for a given angular momentum, the constant flux of SAM to low wave numbers keeps producing distorting fields and the resulting instabilities continue to feed high wave numbers. We suggest that this secondary flow of energy density has the required behavior to account for intermittency. Moreover, this conjecture could be directly tested with careful numerical simulations. We have verified numerically [5] in the case of optical turbulence with $\alpha = +1$ that intermittency can be suppressed by applying damping for all $k < k_0$. Likewise, intermittency in hydrodynamic turbulence will be suppressed if there are no sources available to build and maintain (either through direct forcing or by an inverse cascade) those large-scale structures which lead to high-k instabilities.

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