TWO-DIMENSIONAL SOLITONS OF THE KADOMTSEV-PETVIASHVILI EQUATION AND THEIR INTERACTION

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Explicit analytic formulae for two-dimensional solitons are given. It is proved that, unlike one-dimensional solitons, two-dimensional ones do not interact at all.

Non-linear quasi-one dimensional waves (with y much larger than x) in a weakly dispersive medium are described by the Kadomtsev-Petviashivili equation [1]:

$$\partial(u_t + 6uu_x + u_{xxx})/\partial x = -3\alpha^2 \, \partial^2 u/\partial y^2 \,. \qquad (1)$$

The sign of the parameter $-\alpha^2$ coincides with that of the dispersion parameter $\partial^2\omega/\partial k^2$.

It was shown that eq. (1) could also be formulated for the inverse scattering problem [2, 3].

It was already noticed [1] that plane solitons are unstable under transverse perturbations in a positive dispersion medium (exact solution of the stability problem is given in ref. [4]). This observation led to the speculation that there are stable two-dimensional solitons localized in the x-y-plane [5] and their profiles are numerically found [5].

In this paper we present analytic expressions for two-dimensional soliton solutions in terms of rational functions of x and y, as well as for arbitrary systems of solitons. In our study of this problem we are largely encouraged by ref. [6], where singular rational solutions of the KdV equation are found.

We construct our exact solutions as follows [2]. The equation

$$K(x,x',y,t) + F(x,x',y,t) + \int_{-\infty}^{x} K(x,x'',y,t)\,F(x'',x',y,t)\,\mathrm{d}x'' = 0 \,, \qquad (2)$$

where F is an arbitrary solution of the system of equations

$$\alpha\frac{\partial F}{\partial y} + \frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial x'^2} = 0 \,, \quad \frac{\partial F}{\partial t} + 4\frac{\partial^3 F}{\partial x^3} + 4\frac{\partial^3 F}{\partial x'^3} = 0, \qquad (3)$$

implies that function

$$u(x,y,t) = -2\,\partial K(x,x,y,t)/\partial x \qquad (4)$$

obeys eq. (1). In particular, putting $\alpha = 1$ and choosing for F the form

$$F = \sum_{n=1}^{N} c_n(y,t)\exp(p_n x + q_n x') \,,$$

where

$$c_n(y,t) = c_n(0)\exp[(-p_n^2 + q_n^2)y - 4(p_n^3 + q_n^3)t] \,,$$

we arrive at the degenerate kernel of eq. (2). With eq. (4) we obtain

$$u = 2\frac{\partial^2}{\partial x^2}\ln\det A \,, \qquad (5)$$

where A is an $N \times N$ matrix

$$A_{nm} = \delta_{nm} + c_n(y,t)\,\mathrm{e}^{(p_n+q_n)x}/p_n + q_m \,. \qquad (6)$$

This solution has first been found in ref. [2].. It describes the N-plane interaction and does not decrease in directions in the x-y-plane.

Formulae (5) and (6) holds for arbitrary complex-valued variables and parameters involved. Note that after substitution $y \to \mathrm{i}y$ one gets from the solution of eq. (1) with $\alpha^2 = 1$ a solution of eq. (1) with $\alpha^2 = -1$.

Convenient variables are $p_n + q_n = \kappa_n$, $p_n - q_n = \nu_n$ and $c_n(0) = -a_n\kappa_n$. In these variables A takes the form

$$A_{nm} = \exp\left(-\frac{\kappa_n - \kappa_m}{2}x + \frac{\nu_n - \nu_m}{2}\right)\cdot B_{nm}\ ,$$

where

$$B_{nm} = \delta_{nm} - \frac{2a_n\kappa_n}{\nu_n - \nu_m + \kappa_n + \kappa_m}$$

$$\times \exp[\kappa_n(x - \nu_n y - 3(\nu_n^2 + \kappa_n^2)t]\ .$$

Evidently, $\det A = \det B$.

Let us now take the limit for $\kappa_n \to 0$, expanding $a_n = 1 - \xi_n\kappa_n + O(\kappa_n^2)$. This results in $\det B = \Pi(-\kappa_n)\det\widetilde{B}$, where $\widetilde{B}$ is a matrix of the form (after substitution $y \to iy$)

$$\widetilde{B}_{nm} = \delta_{nm}(x - i\nu_n y - \xi_n - 3\nu_n^2 t) + (1 - \delta_{nm})\frac{2}{\nu_n - \nu_m}\ . \qquad (7)$$

Thus, the function

$$u = 2\,\partial^2 \ln \det \widetilde{B}/\partial x^2 \qquad (8)$$

is a rational solution of eq. (1) with $\alpha^2 = -1$.

In the general case this function is complex-valued and singular. However, if for $N = 2$ we put $\nu_2 = -\bar{\nu}_1$, $\xi_2 = \bar{\xi}_1$ we arrive at

$$\det\widetilde{B} = 4(\nu_1 + \bar{\nu}_1)^{-2} + |x - i\nu_1 y - \xi_1 - 3\nu_1^2 t|^2\ ,$$

i.e., u is a nonsingular and real function with good behaviour at infinity, $u \sim (x^2 + y^2)^{-1}$. It corresponds to the two-dimensional soliton with velocity $\mathbf{v} = (v_x, v_y)$, $v_x = 3|\nu_1|^2$, $v_y = -6\,\mathrm{Im}\,\nu_1$.

In general the case the soliton is nonsingular, provided that $N = 2k$, and $\nu_{n+k} = -\bar{\nu}_n$, $\xi_{n+k} = \bar{\xi}_n$, i.e., matrix $\widetilde{B}$ is of the block structure:

$$\widetilde{B} = \begin{pmatrix} \psi & \chi \\ -\bar{\chi} & \psi^+ \end{pmatrix},$$

where ψ is the matrix of the form given by eq. (7), n, $m \leqslant k$ and where $\chi_{nm} = 2/(\nu_n + \bar{\nu}_m)$. Since this form of $\widetilde{B}$ implies that $\det\widetilde{B} > 0$, the corresponding functions are nonsingular.

The so constructed solutions describe collisions of k two-dimensional solitons. The in- and out-fields at $x = 3|\nu_i|^2 t + x_0$, $y = -6\,\mathrm{Im}\,\nu_i\, t + y_0$ provide evidence of the above fact. These states are given by superpositions of isolated solitons. A fascinating feature of our solutions is that the corresponding phase shifts, familiar from the scattering of one-dimensional solitons, are exactly zero. Thus, *two-dimensional solitons do not interact at all.*

References

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