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Lectures 29,30.
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In these Lectures we construct the so-called Action-Angle for completely integrable Hamiltonian systems with an additional compactness assumption.

Assume, that we have a Hamiltonian system (M^{2n}, H) with the following properties:

1. The Poisson manifold M^{2n} is smooth and symplectic. It means that the Poisson tensor $h^{ij}(x) = \{x^i, x^j\}$ is non-degenerate at all points of M^{2n}; the inverse tensor $h_{ij}(x)$ is well-defined and smooth. From the Jacoby identity it follows, that the form $\Omega = h_{ij}(x)dx^i \wedge dx^j$ is closed, $d\Omega = 0$. (We do not assume, that Ω is exact). The form Ω is called **symplectic structure**.

2. Our system has n independent integrals of motion in involution: $H^1 = H$, H^2, $\ldots$, H^n; $\{H^i, H^j\} = 0$.

3. Consider the following n-parametric family of n-dimensional submanifolds $T_{\vec{c}} \subset M^{2n}$:
$$\begin{cases} H^1 = c^1 \\ H^2 = c^2 \\ \quad \ldots \\ H^n = c^n \end{cases}$$
(the submanifolds $T_{\vec{c}}$ are the **level sets** for all integrals H^k). We assume, that there exist a domain $U \subset \mathbb{R}^n$ such, that:

   - For all $\vec{c} \in U$ the functions H^k are independent at all points of $T_{\vec{c}}$. It means, that at all points of $T_{\vec{c}}$ the differentials dH^k are linear independent.

     It follows from this condition, that the level sets $T_{\vec{c}}$ (we denote them also by $T_{\vec{H}}$) are non-singular submanifolds of M^{2n}. These submanifolds are invariant with respect to the flows, generated by the Hamiltonians H^k, and the corresponding Hamiltonian vector fields v_k are linear independent at all points of $T_{\vec{c}}$.

- For all $\vec{c} \in U$ the level sets $T_{\vec{c}}$ are compact and connected. (If they are not connected, we can consider a neighborhood of one connected component instead of M^{2n}, therefore the connectness condition is not too essential).

Our aim is to show, that any such Hamiltonian system can be transformed to a very simple canonical form. The corresponding coordinates are called **Action-Angle** variables. We describe them in the Theorem below.

Remark 1 *Let us remind, that the maximal number of independent functions in involution on a* $2n$*-dimensional symplectic manifold is* n*. Any set of* m *independent functions in involution can be extended to a local coordinate system by adding* $2n - m$ *supplementary functions. In such coordinates the tensor* $h^{ij}(x)$ *has* $m \times m$ *zero block. The tensor* $h^{ij}(x)$ *on a symplectic manifold is non-degenerate, therefore the size of the block is not greater, then the dimension of the manifold, divided by 2,* $m \leq n$*.*

Therefore our system has the maximal possible numbers of integrals in involution. If we do not assume, that all integrals of our system are in involution, than the number of independent integrals can be greater then n*. For example, the 3-dimensional Kepler's problem has 5 independent integrals of motion. The phase space of this problem is 6-dimensional, therefore all orbits with negative energy are periodic. But the number of integrals in involution for this system is 3.*

Let us discuss the topology of the level sets $T_{\vec{c}}$. For all $\vec{c} \in U$ we have n commuting everywhere independent flows on $T_{\vec{c}}$. (Let us remind, that a pair of flows on a symplectic manifold, generated by Hamiltonian functions f^1, f^2, commute if and only if $\{f^1, f^2\} = const$). Therefore any connected component of $T_{\vec{c}}$ is the factor of $\mathbb{R}^n$ by a discrete subgroup. It is rather easy to show (see [2]), that any connectet component of the level set $T_{\vec{c}}$ is the manifold $T^m \times \mathbb{R}^{n-m}$, where T^m is the m-dimensional torus, m is the rank of the discrete subgroup. Such a manifold is compact if and only if $m = n$. We assume, the compactness and connectness, therefore the level sets $T_{\vec{c}}$ are n-dimensional tori.

Remark 2 *The level sets for the Kepler problem are compact if* $H < 0$ *and non-compact if* $H > 0$*, where* H *is the energy. The phase space (the manifold*

M^{2n}) for the systems of classical mechanics is usually non-compact. But for many important examples the fixed-energy level sets $H = E_0$ are compact, and it implies the compactness of $T_{\vec{c}}$ (at least for some E_0).

The aim of this lecture is to prove the following important theorem:

Theorem 1 *Suppose we have a Hamiltonian system (M^{2n}, H) satisfying the above formulated conditions. Let $\vec{c}_0$ be a point from the domain U. Then there exits a coordinate system $J^1,\dots,J^n$, $\phi^1,\dots$, ϕ_n on a neighborhood of the level set $T_{\vec{c}_0}$ with the following properties:*

1. *The functions J^k, $k = 1,\dots,n$ are constant on the tori $T_{\vec{c}}$. The tori $T_{\vec{c}}$ are the level sets for the Hamiltonians H^j, therefore J^k are functions of $\vec{H} = (H^1,\dots,H^n)$, $J^k = J^k(\vec{H})$.*

2. *The functions ϕ^k are local coordinates on the tori $T_{\vec{c}}$ such, that for all $\vec{c}$ from the neighborhood of the point $\vec{c}_0$ the torus $T_{\vec{c}}$ is the factor of the n-dimensional space $\mathbb{R}^n$ by the standard lattice*
$$\vec{l} = k_1\vec{l}_1 + k_2\vec{l}_2 + \ldots + k_n\vec{l}_n, \quad k_j \in \mathbb{Z},$$
where
$$\begin{array}{c} \vec{l}_1 = (2\pi, 0, 0, \dots, 0) \\ \vec{l}_2 = (0, 2\pi, 0, \dots, 0) \\ \cdots \\ \vec{l}_n = (0, 0, 0, \dots, 2\pi). \end{array} \tag{1}$$
Therefore ϕ^k are the angles along the basic cycles of the tori $T_{\vec{c}}$.

3. *The coordinates J^k, ϕ^l are symplectic coordinates, i.e. their Poisson brackets have the standard form:*
$$\{J^k, J^l\} = \{\phi^k, \phi^l\} = 0, \quad \{J^k, \phi^l\} = \delta^{kl}.$$

4. *The functions J^k can be calculated explicitly using the following formula*
$$J^k(\vec{H}) = -\frac{1}{2\pi}\oint\limits_{a_k} d^{-1}\Omega, \tag{2}$$

where a_k denotes the k-th basic cycle on the torus $T_{\vec{H}}$:

$$\begin{gathered}\phi^1 = const \\ \phi^2 = const \\ \dots \\ 0 \le \phi^k \le 2\pi \\ \dots \\ \phi^n = const.\end{gathered}$$

The variables J^k are called **Action variables***, the variables ϕ^k are called* **Angle variables** *or* **Angles***.*

Remark 3 *To use formula (2) it, we have to define the meaning of the term $d^{-1}\Omega$ in the right-hand side. It is natural to assume, that $d^{-1}\Omega$ denotes a 1-form ω such, that $d\omega = \Omega$. But we do not know a priori, that such a form exists (we do not assume, that the form Ω is exact on the whole manifold M^{2n}, and it may be impossible to construct ω globally). And if it exists, it may be not unique. We discuss these problems below.*

The proof of this Theorem is rather long. Let us do it step by step.

Step 1. We introduce coordinates w^1,...,w^n at all tori in the neighborhood of the level set $T_{\vec{c}_0}$ such, that

$$\{H^k, w^l\} = \delta^{kl}.$$

At each torus $T_{\vec{H}}$ we have n commuting flows

$$\frac{\partial f}{\partial_{t_k}} = \{H^k, f\},$$

generated by the Hamiltonians H^k. They are linearly independent at all points of $T_{\vec{H}}$, therefore we have n commuting everywhere linearly independent vector fields v_1, ..., v_n on an n-dimensional smooth manifold.

Assume that we have chosen a starting point $W_0(\vec{H})$ at each torus $T_{\vec{H}}$; and this point depends smoothly on $\vec{H}$. We can obtain any point W of $T_{\vec{H}}$ in the neighborhood of the point $W_0(\vec{H})$ by the following procedure: We move $W_0(\vec{H})$ along the trajectory of the field v_1 for the time w^1, then we move it along the trajectory of the field v_2 for the time w^2,..., we move it along the trajectory of the field v_n for the time w^n. The flows v_k commute pairwise, therefore the shifts can be applied in an arbitrary order. By definition the coordinates of the point W are $(w^1, \dots, w^n)$. It follows from the construction, that

$$\frac{\partial w^k}{\partial t_l} = \delta^{kl},$$

but

$$\frac{\partial w^k}{\partial t_l} = \{H^l, w^k\}.$$

Therefore

$$\{H^k, w^l\} = \delta^{kl}.$$

The Step 1 is completed.

Remark 4 *The Poisson brackets of the coordinate functions w^k depend on the choice of the starting point $W_0(\vec{H})$ and may be non-trivial. We shall fix the choice of the starting point at Step 4.*

Step 2. The coordinates w^k are not the angle coordinates on the tori $T_{\vec{H}}$, because in these coordinates the basic periods do not have the canonical form (1). In fact, the tori $T_{\vec{H}}$ are represented as the factor of $\mathbb{R}^n$ by some n-dimensional lattice. For generic systems this lattice depends on $\vec{H}$. Denote the basic vectors of this lattice by $\vec{l}_1(\vec{H})$,..., $\vec{l}_n(\vec{H})$, $\vec{l}_k = (l_k^1, \dots, l_k^n)$. Let us define the coordinates $\tilde{\phi}^1$, ..., $\tilde{\phi}^n$ by assuming, that

$$2\pi w^k = \sum_{j=1}^{n} l_j^k(\vec{H})\tilde{\phi}^j. \tag{3}$$

The functions $\tilde{\phi}^k$ constructed above have the following Poisson brackets:

$$\{H^k, H^l\} = 0, \quad \{H^k, \tilde{\phi}^l\} = b^{kl}(\vec{H}), \quad \{\tilde{\phi}^k, \tilde{\phi}^l\} = a^{kl}(\vec{H}, \vec{\phi}). \tag{4}$$

Step 3. Let us construct the Action variables – the variables J^1,...,J^n such, that

$$\{J^k, \tilde{\phi}^l\} = \delta^{kl}.$$

Of course we can try to prove the existence theorem. But fortunately we have explicit formula (2). Let us prove, that the quantities J^k are well-defined and have the correct Poisson brackets with $\tilde{\phi}^k$.

The first step is to construct the form $d^{-1}\Omega$.

It follows from formula (4), that in the coordinates H^k, $\tilde{\phi}^k$ the symplectic structure Ω has the following form:

$$\Omega = a_{kl}(\vec{H}, \vec{\phi}) dH^k \wedge dH^l + b_{kl}(\vec{H}) d\tilde{\phi}^k \wedge dH^l, \tag{5}$$

where $b_{kl}(\vec{H})$ is the inverse matrix to $b^{kl}(\vec{H})$.

We can always assume, that the domain $U(\vec{c}_0)$ is convex in $\mathbb{R}^n$. Therefore the domain $\vec{H} \in U(\vec{c}_0)$ in M^{2n} can be homotopically contracted to the torus $T_{\vec{c}_0}$. The restriction of Ω to the level sets $T_{\vec{H}}$ is identically zero, therefore there exists at least one form ω such, that $\Omega = d\omega$. (For more detailed discussion see the volume 3 of the book [1]).

Assume, that we have 2 different curves a_k, $\tilde{a}_k$ on $T_{\vec{H}}$ representing the same basic cycle. Then by the Stokes' theorem

$$\oint_{a_k} d^{-1}\Omega - \oint_{\tilde{a}_k} d^{-1}\Omega = \int\int_S \Omega$$

where S denotes a film connecting these curves. The restriction of Ω to $T_{\vec{H}}$ is identically 0, therefore in formula (2) we can integrate over arbitrary contours representing the cycles a_k.

We have proved, that the quantities J^k are constant on each torus $T_{\vec{H}}$, and they are functions of the variables $H^1, \ldots, H^n$ only.

The right-hand side of (2) depends on the choice of the form $d^{-1}\Omega$. But if we have a pair of contours A_1, A_2, connected by a film S, then

$$\oint_{A_1} d^{-1}\Omega - \oint_{A_2} d^{-1}\Omega = \int\int_S \Omega$$

and the difference of the integrals is well-defined. Therefore if we add a closed 1-form to $d^{-1}\Omega$, we simply shift the functions J^k to some constants $J^k \to J^k + d^k$. But this shift does not affect the Poisson brackets, therefore we can use any form ω such, that $d\omega = \Omega$ to define the functions J^k.

Remark 5 *It is possible to construct the variables J^k without use of the form $d^{-1}\Omega$. We can assume $J^1 = J^2 = \ldots = J^n = 0$ at the point $\vec{H} = \vec{c}_0$ and define the right-hand side of (2) for other values of $\vec{H}$ using the Skokes' theorem. But if we use the canonical symplectic structure $\Omega = \sum dp^j \wedge dq^j$, (this* *situation is rather standard for the systems of classical mechanics), then we can define $d^{-1}\Omega$ explicitly assuming $d^{-1}\Omega = \sum p^j dq^j$. In such situation representation (2) is much more convenient, then the Stokes' theorem approach.*

We have defined the functions J^k. Let us check, that they are locally independent, i.e. the matrix

$$\left(\frac{\partial J^k}{\partial H^l}\right)$$

is non-degenerate at the point $\vec{H} = \vec{c}_0$, and they have canonical Poisson brackets with the variables $\tilde{\phi}^k$.

To calculate the matrix $\left(\frac{\partial J^k}{\partial H^l}\right)$ consider the following film S_{kl}:

$$\begin{cases} H^1 = c^1 \\ H^2 = c^2 \\ \dots \\ c^l \leq H^l = c^l + \epsilon \\ \dots \\ H^n = c^n \end{cases} \qquad \begin{cases} \tilde{\phi}^1 = 0 \\ \tilde{\phi}^2 = 0 \\ \dots \\ 0 \leq \tilde{\phi}^k \leq 2\pi \\ \dots \\ \tilde{\phi}^n = 0 \end{cases}$$

By definition of J^k

$$-\frac{1}{2\pi}\int\!\!\int_{S_{kl}} \Omega = \epsilon \frac{\partial J^k}{\partial H^l}\bigg|_{\vec{H}=\vec{c}} + O(\epsilon^2).$$

This integral can be calculated explicitly using representation (5) for the form Ω:

$$-\frac{1}{2\pi}\int\!\!\int_{S_{kl}} \Omega = -\frac{1}{2\pi} \int\limits_{0 \leq \tilde{\phi}^k \leq 2\pi} \quad \int\limits_{c^l \leq H^l \leq c^l+\epsilon} b_{kl}(\vec{H}) d\tilde{\phi}^k \wedge dH^l =$$

$$= \frac{1}{2\pi} \int\limits_{c^l \leq H^l \leq c^l+\epsilon} \quad \int\limits_{0 \leq \tilde{\phi}^k \leq 2\pi} b_{kl}(\vec{H}) dH^l \wedge d\tilde{\phi}^k = \int\limits_{c^l \leq H^l \leq c^l+\epsilon} b_{kl}(\vec{H}) dH^l =$$

$$= \epsilon b_{kl}(\vec{H})\Big|_{\vec{H}=\vec{c}} + O(\epsilon^2).$$

Therefore

$$\frac{\partial J^k}{\partial H^l}\bigg|_{\vec{H}=\vec{c}} = b_{kl}(\vec{H})\Big|_{\vec{H}=\vec{c}}.$$

The form Ω is non-degenerate therefore the matrix $b_{kl}(\vec{H})$ is non-degenerate. Hence we have proved that the functions J^k, $\tilde{\phi}^k$ are local coordinates in our domain.

The functions J^k depend on the coordinates H^l only, therefore $\{J^k, J^l\} = 0$. In the coordinates J^k, $\tilde{\phi}^k$ the form Ω has the following representation

$$\Omega = A_{kl}(\vec{J}, \vec{\phi}) dJ^k \wedge dJ^l + B_{kl}(\vec{J}) d\tilde{\phi}^k \wedge dJ^l.$$

Integrating the form Ω over the film $\tilde{S}_{kl}$:

$$\left\{ \begin{array}{c} J^1 = d^1 \\ J^2 = d^2 \\ \dots \\ d^l \leq J^l = d^l + \epsilon \\ \dots \\ J^n = d^n \end{array} \right. \qquad \left\{ \begin{array}{c} \tilde{\phi}^1 = 0 \\ \tilde{\phi}^2 = 0 \\ \dots \\ 0 \leq \tilde{\phi}^k \leq 2\pi \\ \dots \\ \tilde{\phi}^n = 0 \end{array} \right.$$

we get the following identity:

$$\delta J^k = B_{kl}(\vec{d}) \delta J^l.$$

The functions J^k are independent, therefore $B_{kj}(\vec{J}) \equiv \delta_{kl}$. It means, that

$$\Omega = A_{kl}(\vec{J}, \vec{\phi}) dJ^k \wedge dJ^l + \sum d\tilde{\phi}^k \wedge dJ^k.$$

Therefore $\{J^k, \tilde{\phi}^l\} = \delta^{kl}$.

The Step 3 is completed.

Step 4. We have constructed the coordinates J^k, $\tilde{\phi}^k$ such, that $\{J^k, J^l\} = 0$, $\{J^k, \tilde{\phi}^l\} = \delta^{kl}$. But they are not canonical, because the Poisson brackets $\{\tilde{\phi}^k, \tilde{\phi}^l\}$ may be non-zero. Therefore we have to "correct" our coordinates.

Lemma 1 *The Poisson brackets $A^{kl}(\vec{J}, \vec{\phi}) = \{\tilde{\phi}^k, \tilde{\phi}^l\}$ do not depend on $\vec{\phi}$: $A^{kl}(\vec{J}, \vec{\phi}) = A^{kl}(\vec{J})$.*

This Lemma follows from the Jacoby identity for the functions $\tilde{\phi}^k$, $\tilde{\phi}^l$, J^m:

$$0 = \{\{\tilde{\phi}^k, \tilde{\phi}^l\} J^m\} + \{\{\tilde{\phi}^l, J^m\} \tilde{\phi}^k\} + \{\{J^m, \tilde{\phi}^k\} \tilde{\phi}^l\} =$$

$$= \{A^{kl}(\vec{J}, \vec{\phi}), J^m\} - \{\delta^{lm}, \tilde{\phi}^k\} + \{\delta^{mk}, \tilde{\phi}^l\} = -\frac{\partial A^{kl}(\vec{J}, \vec{\phi})}{\partial \tilde{\phi}^m} + 0 + 0.$$

Therefore

$$\frac{\partial A^{kl}(\vec{J},\vec{\phi})}{\partial\tilde{\phi}^m}=0$$

for all m. The Lemma is proved.

Let us calculate now the Jacoby identity for the functions $\tilde{\phi}^k$, $\tilde{\phi}^l$, $\tilde{\phi}^m$:

$$0=\{\{\tilde{\phi}^k,\tilde{\phi}^l\}\tilde{\phi}^m\}+\{\{\tilde{\phi}^l,\tilde{\phi}^m\}\tilde{\phi}^k\}+\{\{\tilde{\phi}^m,\tilde{\phi}^k\}\tilde{\phi}^l\}=$$

$$=\frac{\partial A^{kl}(\vec{J})}{\partial J^m}+\frac{\partial A^{lm}(\vec{J})}{\partial J^k}+\frac{\partial A^{mk}(\vec{J})}{\partial J^l}.$$

We see that the form

$$\omega_1=\sum k\ lA^{kl}(\vec{J})dJ^k\wedge dJ^l$$

is closed. By Poincare Lemma any closed form in a convex domain in $\mathbb{R}^n$ is exact, therefore there exists a collection of functions $f^1(\vec{J})$, $\ldots$, $f^n(\vec{J})$ such, that

$$A^{kl}(\vec{J})=\frac{\partial f^k(\vec{J})}{\partial J^l}-\frac{\partial f^l(\vec{J})}{\partial J^k}.$$

Now we are ready to define the Angle variables. Assume, that

$$\phi^k=\tilde{\phi}^k-f^k(\vec{J}).$$

A simple direct calculation shows

$$\{\phi^k,\phi^l\}=\{\tilde{\phi}^k-f^k(\vec{J}),\tilde{\phi}^l-f^l(\vec{J})\}=$$

$$=\{\tilde{\phi}^k,\tilde{\phi}^l\}-\{f^k(\vec{J}),\tilde{\phi}^l\}-\{\tilde{\phi}^k,f^l(\vec{J})\}+\{f^k(\vec{J}),f^l(\vec{J})\}=$$

$$=A^{kl}(\vec{J})-\frac{\partial f^k(\vec{J})}{\partial J^l}+\frac{\partial f^l(\vec{J})}{\partial J^k}=0.$$

$$\{J^k,\phi^l\}=\{J^k,\tilde{\phi}^l\}-\{J^k,f^l(\vec{J})\}=\delta^{kl}+0=\delta^{kl}.$$

Step 4 is completed.

The main theorem is proved.

References

[1] Dubrovin, Novikov, Fomenko. Modern Geometry. V.1-3. Springer-Verlag.

[2] V.I.Arnold. Mathematical methods of calssical mechanics. Springer-Verlag, 1978.