Comment on "Exact Results on Scaling Exponents in the 2D Enstrophy Cascade"

In an interesting recent Letter [1], Eyink established some inequalities imposed on short-distance exponents ζ_p of the vorticity field in 2D, $\langle[\omega(r) - \omega(0)]^p\rangle \propto r^{\zeta_p}$. This is important because the 2D Navier-Stokes equation may allow for many statistical steady solutions providing for a direct vorticity cascade. The analysis of decaying turbulence presented in [1] is thus interesting because by preparing different initial data one may hope to observe different exponents at the small-scale limit. However, the discussion of forced turbulence in [1] seems to be outdated as far as the universality of the short-scale asymptotics is concerned. In our paper [2] (published before submission of [1]), the new yet rather simple physical argument is presented which shows that the asymptotics are universal. Moreover, not only are all ζ_p shown to be zero (i.e., the regime is logarithmic), also the new formalism that has been developed gives the powers of logarithms for all correlation functions [2]. Let us explain here why a general large-scale pumping produces a universal small-scale logarithmic regime.

The idea of the proof is to show that the logarithmic regime is the only solution structurally stable with respect to the pumping variations. The absence of less steep spectra (with $\zeta_2 < 0$) trivially follows from the fact that such solutions would be local, i.e., their exponents would be possible to obtain from dimensional analysis which gives $\zeta_2 = 0$ nevertheless. Therefore, all the spectra suggested as an alternative to Kraichnan's logarithmic regime [3] have $\zeta_p > 0$ [4]. Let us imagine for a moment that the pumping at k_0 produces the spectrum with $\zeta_p > 0$ in the inertial interval $k \gg k_0$. In the spirit of the stability theory of Kolmogorov spectra [5], let us add infinitesimal pumping at some k_1 in the inertial interval with the only condition that small yet nonzero flux of squared vorticity is produced. The small perturbation $\delta\omega$ obeys the equation $\partial\delta\omega/\partial t + (v\nabla)\delta\omega + (\delta v\nabla)\omega = \nu\Delta\delta\omega$. Here δv is the velocity perturbation related to $\delta\omega$. The ratio of the third term to the second one could be estimated as $(k_0/k_1)^{\zeta_2}/\zeta_2$ which reflects the fact that the vorticity at the main spectrum is concentrated at k_0. Since $\zeta_2 > 0$, then the third term could be neglected so that $\delta\omega$ behaves as a passive scalar convected by the main turbulence. If $\zeta_p > 0$, then the main contribution into the scalar cascade is given by stretching due to large-scale strain, i.e., the well-known Batchelor regime takes place. The correlation functions of the scalar are logarithmic in this case. Let us emphasize that this statement is not based on any uncontrollable approximations because the scalar correlation functions are shown in [2] to be logarithmic for arbitrary velocity correlation time (see [6] for more details), thus generalizing the classical results of Batchelor for frozen velocity [7] and those of Kraichnan for fast velocity [8]. The fact that the vorticity perturbation is simply convected rather than grows in 2D guarantees that at any realization there are no fast instabilities that may destroy the validity of linear approximation before the logarithmic tail appears. The relative share of the perturbation in any vorticity correlation function thus grows downscales. That means that the hypothetical solution with $\zeta_p > 0$ is structurally unstable with respect to the pumping variation. Note that the possibility of having solutions steeper than Kraichnan's, yet with the main strain due to small scales discussed in Eyink's Reply [9], seems to be rather slim yet probably worth exploring (the above Batchelor argument could not be applied to such an exotic case yet the general approach with the structural instability should be applicable: There should exist some correlation function of the perturbation that decreases slower than that of the main spectrum). The stable solution could be found among the logarithmic regimes with $\zeta_p = 0, \forall p$, where $\langle[\omega(r) - \omega(0)]^p\rangle \propto (\ln r)^{\beta_p}$. By discovering a hidden small parameter (ratio of stretching time to correlation time of the stretching), the controllable formalism has been developed in [2] for the description of a logarithmic regime. By requiring self-consistency of the solution, we show that $\beta_{2p} = 2p/3$ (for $p = 1$, that has been conjectured earlier by Kraichnan [3]). Such logarithmically renormalized Kraichnan's spectrum is neutrally stable with respect to the pumping variations (see Appendix A in [2]) and it represents the universal small-scale asymptotics of the steady forced turbulence.

This work was supported by the Rashi Foundation and by the Minerva Center for Nonlinear Physics. We are grateful to G. Eyink for the useful discussion.

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Received 30 May 1995
PACS numbers: 47.27.–i

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 0031-9007/96/76(11)/1974(1)$10.00