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Hidden symmetry, exact relations, and a small parameter in the Kardar-Parisi-Zhang problem with strong coupling

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An exact relation between the Green's function and the dressed third-order vertex Γ was found for the Kardar-Parisi-Zhang (KPZ) model of surface roughening in $(1+d)$ dimensions. This relation, of the Ward-identity type, follows from a hidden symmetry of the problem, which generalizes in some sense the Galilean invariance of the KPZ equation. This relation allows one to conclude that in the region of strong coupling, $\Gamma-\Gamma_0\sim 0.1\Gamma_0$, where Γ_0 is the bare value of the vertex Γ. The identity is generalized for higher-order vertices, enabling us to predict some relations between observable correlation functions.

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The Kardar-Parisi-Zhang (KPZ) equation is written as

$$\partial h/\partial t=\nu_0\nabla^2 h+\lambda(\nabla h)^2+\xi\,, \tag{1}$$

where $h(t,\mathbf{r})$ is a scalar field, λ is an interaction constant, ν_0 is a diffusion coefficient, and $\xi(t,\mathbf{r})$ is a white noise with an effective temperature T. Let us stress that a system described by (1) is far from equilibrium. The KPZ equation describes roughening of an interface in different cases, such as growths of solids [1], two fluid flows [2,3], motion of domain walls [4], or boundaries of clusters [5], etc. This equation is equivalent to the Burgers equation [6–8]. It is also equivalent to the equation for the partition function of directed polymers [9] and of dislocations [10] or vortices [11] in a random potential (in these cases we should take the third coordinate instead of the time t). This variety of physical contexts is associated with the universal character of the KPZ equation representing the long-wavelength dynamics of any field h if it is invariant under $h\to h+\text{const}$ but not invariant under $h\to -h$.

Considering the interface in the three-dimensional (3D) space or the vortex in the 3D lattice the quantity h should be treated as a function of the 2D radius vector $\mathbf{r}$. Then fluctuations of the field h are relevant. It appears that the case of "asymptotic freedom" is realized, that is, the dimensionless coupling constant grows with increasing scale [12]. In this situation one cannot say anything definite about the long-wavelength properties of correlation functions of h on the basis of perturbation methods like renormalization-group equations. Numerics [13–15] show a scaling long-wavelength behavior. From a theoretical point of view it is a surprise since in known exactly solvable models where "asymptotic freedom" is simulated the long-wavelength behavior of correlation functions is not of the scaling type [16,17]. The possibility of the scaling behavior of the correlation functions of h is related to cancellation of ultraviolet divergences in the KPZ model [18,19].

Hidden symmetry of the problem. In this paper we examine statistical properties of solutions of the KPZ equation in terms of the Green's function G and the triple vertex Γ. These are objects describing the linear and nonlinear response of the field h to an infinitesimal "force" f to be added to the right-hand side (rhs) of (1):

$$\begin{aligned} &iG_1\delta_{1,2}=\left\langle\frac{\delta h_1}{\delta f_2}\right\rangle,\\ &G_1\Gamma_{2,3}G_2G_3\delta_{1,2+3}=-\left\langle\frac{\delta^2 h_1}{\delta f_2\delta f_3}\right\rangle, \\ &\delta_{1,2+3}\equiv(2\pi)^{1+d}\delta(\omega_1-\omega_2-\omega_3)\delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\,. \end{aligned} \tag{2}$$

Here d is the dimensionality of the space, subscripts $1,2,3$ designate Fourier harmonics with frequencies ω_j and wave vectors $\mathbf{k}_j$, so that $G_1=G(\omega_1,\mathbf{k}_1)$ and the vertex $\Gamma_{2,3}$ is a function of $\omega_2,\mathbf{k}_2$ and $\omega_3,\mathbf{k}_3$, $\delta_{1,2}$ is defined like $\delta_{1,2+3}$. The average $\langle\,\rangle$ in (2) is performed over the ensemble of random "forces" ξ.

In the limit of a weak interaction one can easily calculate G and Γ directly from the KPZ equation (1):

$$G_0(\omega,\mathbf{k})=1/(\omega+i\nu_0k^2)\,,\quad \Gamma_0(\mathbf{k};\mathbf{q})=-2i\lambda\mathbf{k}\cdot\mathbf{q}\,. \tag{3}$$

Fluctuation corrections lead to the "dressing" of these functions. The dressed values can be represented as

$$G(\omega,\mathbf{k})=1/\left[\omega+i\nu_0k^2-\Sigma(\omega,\mathbf{k})\right], \tag{4}$$

$$\Gamma(\omega,\mathbf{k};\nu,\mathbf{q})=\Gamma_0(\mathbf{k};\mathbf{q})\left[1+\gamma(\omega,\mathbf{k};\nu,\mathbf{q})\right], \tag{5}$$

where Σ is the self-energy function and the dimensionless function γ characterizes the deviation of the dressed vertex Γ from its bare value Γ_0.

In order to examine statistical properties of solutions of the KPZ equation we will utilize a diagram technique of the type first developed by Wyld [20] (see also [21]). As it was shown in [22,23] Wyld's diagrammatic technique is generated by a conventional quantum field theory method starting from an effective action I. This action can be constructed on the basis of nonlinear dynamic equations of a system and for the KPZ equation is

$$I=\int dt\,d\mathbf{r}[p\partial h/\partial t-\lambda p(\nabla h)^2+\nu_0\nabla p\nabla h+iT\nu_0p^2]. \tag{6}$$

Here p is the auxiliary field "conjugated" to the field h. Correlation functions containing the p field are susceptibilities, e.g., the Green's function $G(t_1 - t_2, \mathbf{r}_1 - \mathbf{r}_2)$ determined by (2) is $-\langle h_1 p_2 \rangle = -\int \mathcal{D}h\mathcal{D}p \exp(iI) h_1 p_2$, where $h_1 = h(t_1, \mathbf{r}_1)$, $p_2 = p(t_2, \mathbf{r}_2)$. The nonlinear susceptibility $\Gamma_{1,2}$ determined by (2) can be expressed via the h and p fields with the help of the relation $\langle h_1 p_2 p_3 \rangle = -\langle \delta^2 h_1/\delta f_2 \delta f_3 \rangle$.

Consider the symmetry of the effective action (6) giving a full statistical description of the KPZ problem. The KPZ equation is invariant under Galilean transformations; the infinitesimal one is

$$h' = h - (\boldsymbol{\zeta}\cdot\boldsymbol{\nabla})h - \dot{\boldsymbol{\zeta}}\cdot\mathbf{r}/2\lambda\,, \quad p' = p - \boldsymbol{\zeta}\cdot\boldsymbol{\nabla}p\,, \tag{7}$$

where $\dot{\boldsymbol{\zeta}} \equiv \partial\boldsymbol{\zeta}/\partial t$=const. One can check that (6) is invariant under the transformation (7) up to a term linear in p supplying the nonzero average $\langle h' \rangle = \dot{\boldsymbol{\zeta}}\cdot\mathbf{r}/2\lambda$. It leads to the conventional Ward identity [12]

$$\gamma(0,0;\nu,\mathbf{q}) = -\partial\Sigma(\nu,\mathbf{q})/\partial\nu\,. \tag{8}$$

We claim that the effective action (6) has exactly the same transformation properties under *generalized Galilean transformation* (7) in which $\boldsymbol{\zeta}$ is an arbitrary function of time t but not of coordinates $\mathbf{r}$. One may say that after statistical averaging solutions of the KPZ problem became invariant under passage into reference systems with a time dependent velocity $\mathbf{V}(t) = \dot{\boldsymbol{\zeta}}(t)$. This is similar to the gauge invariance of quantum electrodynamics [24]. The difference is that in quantum electrodynamics a gauge function depends both on t and $\mathbf{r}$ while the KPZ problem possesses a hidden "gauge" symmetry with gauge functions depending on time only. This situation is in some sense analogous to that appearing at the investigation of the nematic-smectic phase transition: de Gennes's model [25] for this phase transition possesses a "gauge" symmetry determined by a function of two (but not three) coordinates [26].

Generalized Ward identity. The proven symmetry enables us to formulate a set of relations between correlation functions. The simplest such relation can be derived by taking a variation of the Green's function $G = -\langle h\,p \rangle$ of the linear part of the transformation (7) (without $\dot{\boldsymbol{\zeta}}$) and equating it to an additional term in G induced by the appearance of the average $\langle h' \rangle$:

$$\frac{\omega}{2\lambda}\lim_{k\to 0}\frac{\partial}{\partial\mathbf{k}}[\Gamma(\omega;\mathbf{k},\nu,\mathbf{q}) - \Gamma_0(\mathbf{k},\mathbf{q})] = i\mathbf{q}(\Sigma(\omega+\nu,\mathbf{q}) - \Sigma(\nu,\mathbf{q}))\,. \tag{9}$$

Here the definition (4) was utilized. Using now the definition (5) we derive the following identity:

$$\omega\gamma(\omega,0;\nu,\mathbf{q}) = \Sigma(\nu,\mathbf{q}) - \Sigma(\omega+\nu,\mathbf{q})\,, \tag{10}$$

which is valid for any dimension d of the space. In the limit $\omega \to 0$ this relation gives a well-known Ward identity (8). Therefore we may say that (10) is a manifestation of the generalized Galilean invariance of the problem. For a finite value of ω the relation (10) has a form close to one of the Ward identities in quantum electrodynamics [24]. The difference is that in quantum electrodynamics there are both arbitrary ω and $\mathbf{k}$ in the Ward identity [24], while in (10) $\mathbf{k} = \mathbf{0}$ and only ω is arbitrary.

It is obvious that the invariance of the action (6) under the transformation (7) enables us to derive also relations of the type (9) for many-point correlation functions. We present here an example concerning a one-particle irreducible vertex of the fourth order $\Gamma^{(4)}$ defined as

$$i\langle \delta^3 h_1/\delta f_2\delta f_3\delta f_4\rangle = -\langle h_1 p_2 p_3 p_4\rangle = \delta_{1,2+3+4}\Big[G_1\Gamma^{(4)}_{2,3,4}G_2G_3G_4 + G_1\Gamma_{2,3+4}G_2G_{3+4}\Gamma_{3,4}G_3G_4 + G_1\Gamma_{3,4+2}G_3G_{4+2}\Gamma_{4,2}G_4G_2 + G_1\Gamma_{4,2+3}G_4G_{2+3}\Gamma_{2,3}G_2G_3\Big]\,, \tag{11}$$

where a subscript such as 2+3 designates a function of arguments $\omega_2+\omega_3$ and $\mathbf{q}_2+\mathbf{q}_3$ and $\delta_{1,2+3+4}$ is defined similarly to $\delta_{1,2+3}$ in (2). The identity for $\Gamma^{(4)}$ is

$$\frac{\omega}{2\lambda}\lim_{k\to 0}\frac{\partial}{\partial\mathbf{k}}\Gamma^{(4)}(\omega,\mathbf{k};\nu_2,\mathbf{q}_2;\nu_3,\mathbf{q}_3) = i\mathbf{q}_2(\Gamma(\omega+\nu_2,\mathbf{q}_2;\nu_3,\mathbf{q}_3) - \Gamma(\nu_2,\mathbf{q}_2;\nu_3,\mathbf{q}_3)) + i\mathbf{q}_3(\Gamma(\nu_2,\mathbf{q}_2;\nu_3+\omega,\mathbf{q}_3) - \Gamma(\nu_2,\mathbf{q}_2;\nu_3,\mathbf{q}_3))\,. \tag{12}$$

One can continue to derive the series of relations started from (9), (12) for higher-order vertices, the vertices of n- and $(n+1)$-order entering each such identity.

We have considered the nonlinear susceptibilities determining the response of $\langle h \rangle$ to an external force f. Analogously, nonlinear susceptibilities determining the response of $\langle hh \rangle$, $\langle hhh \rangle$, etc. to f can also be treated. We can formulate the identities of (9), (12) type for each sequence of these susceptibilities. The above procedure can be applied also to other nonlinear statistical problems described by an equation with a nonlinearity of the convective type (recall that the KPZ problem reduces to the noisy Burgers equation having just the convection type of nonlinearity). An example to which our consideration may be usefully extended is the Euler equation.

Dressed vertex in long-wavelength limit. In the long-wavelength limit the interaction in the KPZ equation becomes strong and the dressed values of G and Γ differ essentially from their "bare" values (3). Let us stress that it is impossible to find the dressed values with the help of a perturbation series since in this case we encounter a problem with a coupling constant growing with increasing scale [12] (in the quantum field theory such a situation is called "asymptotic freedom"). Nevertheless there are analytical and computational arguments in favor of the existence of scaling behavior of the correlation functions in the region of strong coupling [13,18,19]. This enables us to assert that in this region

$$G(\omega,\mathbf{k}) = g\left(\Omega_k\right)/\mu_k\,, \quad g(\Omega) \equiv 1/\left[\Omega - \sigma(\Omega)\right]. \qquad (13)$$

Here g and σ are dimensionless functions of the dimensionless argument $\Omega_k = \omega/\mu_k$, $\mu_k \equiv \mu(k) \propto k^z$, z being the dynamic scaling exponent of the KPZ problem, $z = 3/2$ for $d = 1$ and $z \simeq 1.6$ for $d = 2$ [12,13]. Isotropy of the solution is assumed. Then (10) will be

$$\gamma(\omega, 0; \nu, \mathbf{q}) = \left[\sigma(\Omega) - \sigma(\Omega + \Omega')\right]/\Omega'\,, \qquad (14)$$

where $\Omega = \nu/\mu_q$ and $\Omega' = \omega/\mu_q$.

Let us formulate known properties of the Green's function $G(\omega,\mathbf{k})$ for the KPZ problem in the long-wavelength limit. First of all there is the frequency sum rule

$$\int_{-\infty}^{\infty} \mathrm{Im}\{G(\omega,\mathbf{k})\}\,d\omega = -\pi\,, \qquad (15)$$

which is a consequence of the analyticity of G in the upper half-plane. We can also assert that there exists a regular expansion of G in ω with a radius of convergence of the order of μ_k. Second the Green's function of $G(\omega,\mathbf{k})$ in the region $\omega \gg \mu_k$ possesses an asymptotic behavior

$$\mathrm{Re}\{G(\omega,\mathbf{k})\} = \omega^{-1}\,, \quad \mathrm{Im}\{G(\omega,\mathbf{k})\} \propto k^2\omega^{-2(1+\delta)}\,. \quad (16)$$

The factor k^2 follows from simple physics: in the limit $\omega \gg \mu_k$ the main contribution to the self-energy function $\Sigma(\omega,\mathbf{k})$ is induced by the interaction of fluctuations of scale $1/k$ with short-wavelength fluctuations with $k' \propto \omega^{1/z}$, that is, $k' \gg k$. This is the so-called "turbulent viscosity" leading to Im $\Sigma \propto k^2$. In the region $\omega \gg \mu_k$ $\mathrm{Im}\{G(\omega,\mathbf{k})\} \simeq \mathrm{Im}\ \Sigma/\omega^2$. Therefore in this region the function $\mathrm{Im}\{G(\omega,\mathbf{k})\}$ has to be proportional to k^2. One can establish exponent δ by comparing (16) and (13):

$$\delta = 1/z - 1/2\,. \qquad (17)$$

For (1+1) dimensions where $z = \frac{3}{2}$ the exponent $\delta = \frac{1}{6} \simeq 0.167$; for (1+2) dimensions $z \simeq 1.6$ and $\delta \simeq 0.125$.

One may suggest the simplest interpolation expression for G satisfying these properties

$$g(\Omega) = \frac{ic(\delta)}{\pi}\int_{-\infty}^{\infty} \frac{d\Omega'}{(\Omega - \Omega' + i0)(1+\Omega'^2)^{1+\delta}}\,. \qquad (18)$$

In the approximation (18) g has only one cut along the imaginary axis starting from the point $\Omega = -i$. The distance to a singularity of g from the point $\Omega = 0$ is taken to be equal to unity, which may be done with an appropriate choice of a factor in μ_k. The factor $c(\delta)$ is determined by (15): $c(\delta) = \Gamma(\frac{1}{2})/\left[\Gamma(\frac{1}{2}+\delta)\Gamma(1-\delta)\right]$, where $\Gamma(x)$ is the gamma function. Clearly $c(0) = 1$, for small δ the factor $c(\delta)$ is close to $1 + 1.38\delta$, $c(\frac{1}{6}) \simeq 1.160$. The model function (18) agrees with the results of numerical experiments [18] for the Kuramoto-Sivashinsky (KS) equation in $(1+1)$ dimensions. The statistical behavior of the KPZ and KS equations in the long-wavelength limit (region of strong coupling) is the same. Thus one can conclude that the function (18) is a good approximation for the KPZ problem too. There are no reasons to think that function (18) does not represent the behavior of the Green's function for the KPZ model in $(1+2)$ dimensions.

We will use (18) in order to estimate a fluctuation correction to the bare vertex Γ_0 (3) determined by the factor γ introduced in (5) with the help of identity (14). Clearly in the case $\delta = 0$ the formula (18) gives $g(\Omega) = 1/(\Omega + i)$. This means that $\sigma(\Omega) = -i$, hence for $\delta = 0$, according to identity (14), the factor γ is equal to zero. One may suspect that for small δ the value of γ has to be proportional to δ and (if we are lucky) the numerical factor in front of δ will not be too large.

Based on representation (18) one can compute $d\sigma/d\Omega$ for $\Omega = 0$ and different δ. This allows us to find that

$$\gamma_0 \equiv \gamma(0,0;0,\mathbf{q}) = 1 - \frac{\Gamma(\frac{1}{2}+\delta)\Gamma(\frac{3}{2}+\delta)}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})\Gamma^2(1+\delta)}\,. \qquad (19)$$

It follows from (19) that for small δ the factor $\gamma_0 \simeq 0.77\delta$ and $\gamma_0 \simeq 0.064$ for $\delta = 0.1$, $\gamma_0 \simeq 0.096$ for $\delta = \frac{1}{6} \simeq 0.167$. So the correction factor γ_0 is smaller than 0.1 for the KPZ model in (1+1) dimensions with $\delta \simeq 0.167$ and in (1+2) dimensions with $\delta \simeq 0.125$. Nevertheless the correction factor γ in (5) is a function of two frequencies and two wave vectors. It is interesting to know whether these parameters remain small in the actual region of variation of the variables. The results of calculations of the function (14) for $d = 1$ and $\delta = \frac{1}{6}$ are presented in Fig. 1. One can see that we continue to be lucky and that the function (14) for nonzero Ω and Ω' is even smaller than for $\Omega = \Omega' = 0$. In some sense it has to be so because for large Ω there is $\sigma(\Omega) \propto \Omega^{-2\delta}$ and the factor γ has to decay. For $\Omega' \ll \Omega$ $\gamma(\omega, 0; \nu, \mathbf{q}) \simeq \delta/\Omega^{(1+2\delta)}$. Our calculations show that (14) decays continuously with Ω and Ω'.

Of course we do not know the correction factor $\gamma(\omega,\mathbf{k};\nu,\mathbf{q})$ at arbitrary $\mathbf{k}$; it is a dimensionless function of k/q. But for large values of k/q we return to the same function (up to transmutation of arguments) and to the same estimations. It is reasonable to believe that

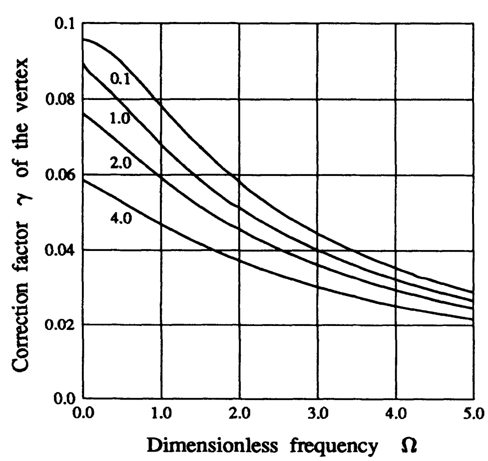

FIG. 1. The correction factor $\gamma(\omega, 0; \nu, \mathbf{q})$ as a function of $\Omega = \nu/\mu_q$ at different $\Omega' = \omega/\mu_q$. The labels on the graph lines correspond to values of Ω'.

in the intermediate region $k/q \sim 1$ the function γ does not essentially differ from its values at $k = 0$ or $q = 0$ and is of the order of 0.1. Therefore we may conclude that the KPZ problem in the region of strong coupling has a small numerical parameter $\propto \delta$ which allows one to use the one-loop approximation in dimensions (1+1) and (1+2) for calculation of the Green's function, correlation function, etc. with accuracy better than 0.1. This is an explanation of the good agreement between calculations of $\text{Im}G(\omega, \mathbf{k})$ in the one-loop approximation for the KPZ model in (1+2) dimensions [27] and the numerics [13]. We believe that the assertion made in [27] that the (self-consistent) one-loop approximation is *exact* for the KPZ problem in (1+2) dimensions is incorrect.

Conclusion. The hidden symmetry of equations with the nonlinearity of the convective type was found. The consequence of this symmetry is the set of new identities of the Ward type. Basing on these identities we analyzed the dressed vertex in the KPZ problem in the region of strong coupling. We found for dimensions $(1+1)$ and $(1+2)$ a small numerical parameter which allows one to calculate correlation functions in the one-loop approximation with accuracy better than 0.1. One may use the Ward-type identities to analyze the behavior of high-order correlation functions for the KPZ and related problems (e.g., our considerations can be extended to the conventional Euler equation).

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