Confinement of inertial particles in viscous boundary layer of turbulent flow

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We examine space and momentum probability distribution of inertial particles when they are placed in the viscous boundary sublayer of a turbulent flow. We demonstrate that at varying elasticity of the particle collisions with the wall the confinement-deconfinement transition occurs: at $\beta < \beta_c$ the particles are blocked near the wall whereas at $\beta > \beta_c$ they gradually pass into bulk. Here β is elasticity coefficient and $\beta_c = \exp(-\pi/\sqrt{3})$.

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Statistical properties of inertial particles in turbulent flows are subject of numerous investigations (see, e.g., Refs. [1–3]). The problem is of fundamental importance and concerns many practically essential phenomena like behavior of droplets in clouds, dust in atmosphere etc. If the turbulent flow is inhomogeneous then the so-called turbophoresis occurs: inertial particles migrate toward regions of relatively weak turbulence [4–9]. That leads to an inhomogeneous distribution of the particles in space. Particularly, they are accumulated near walls where turbulence is weakened in comparison with bulk.

One can consider another setup: inertial particles initially placed near an impenetrable wall are then washed out into bulk due to turbulent motion. The question is what are dynamical and statistical properties of this transport process. The phenomenon was examined in Refs. [10, 11] for non-inertial particles (passive scalar) in the viscous sublayer of a turbulent flow. Here, we investigate inertial particles placed in the viscous sublayer. Inertia changes drastically the process of particle migration since inertial particles lag the flow that diminishes efficiency of sweeping the particles into bulk.

We demonstrate that the sweeping efficiency depends dramatically on elasticity of the particle collisions with the wall: at small elasticity the particles are confined near the wall whereas at moderate elasticity they are swept gradually by the flow to bulk. There is a critical value of elasticity that is border of the two regimes. The phenomenon is similar to the localization-delocalization transition that was established for inertial particles in an inhomogeneous turbulent flow in Ref. [12]. However, in our case the control parameter is elasticity rather than the inertia degree as in Ref. [12].

The dynamic equation describing an inertial particle motion in a fluid is written as (see, e.g., Ref. [13])

$$\tau \frac{d\mathbf{v}}{dt} + \mathbf{v} = \mathbf{u} + \boldsymbol{\xi}. \qquad (1)$$

Here, $\mathbf{v}$ is the particle velocity, $\mathbf{u}$ is the fluid velocity, $\boldsymbol{\xi}$ is Langevin force (responsible for the Brownian motion of the particle), and τ is the Stokes relaxation time of the particle associated with its inertial properties. One can say that the second term in the left-hand side of Eq. (1) represents "friction" between the particle and the (unmoving) fluid whereas the terms in the right-hand side of Eq. (1) are "forcing" terms. The terms describe the particle drag produced by turbulent and thermal flow fluctuations, respectively.

In the viscous sublayer of a turbulent flow the fluid velocity is smooth in space that is it can be expanded into the Taylor series. Assuming that the wall is flat we find $u_x, u_y \propto z$, $u_z \propto z^2$, where z is separation from the wall. The last relation is explained by the incompressibility condition $\partial_x u_x + \partial_y u_y + \partial_z u_z = 0$. The proportionality laws are valid at $z \ll L$, where L is the thickness of the viscous boundary sublayer (Kolmogorov length). Being smooth in the viscous sublayer, the velocity $\mathbf{u}$ remains a random function of time with the correlation time that is determined by bulk velocity fluctuations of the scale L, we designate their typical value as u_L. The velocity correlation time τ_c in the viscous sublayer is estimated as $\tau_c \sim L/u_L$.

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We examine the case where the velocity correlation time τ_c in the viscous sublayer is much less than the Stokes time τ. By other words, the Stokes number St, defined as $\mathrm{St} = \tau/\tau_c$, is large. Therefore both, the random velocity $\mathbf{u}$ and the Langevin force $\boldsymbol{\xi}$, can be treated as short correlated in time (as white noises). Then statistical properties of the "forcing" in Eq. (1) are characterized by the following correlation functions

$$\langle u_z(t_1, x, y, z) u_z(t_2, x, y, z)\rangle = 2\mu z^4 \delta(t_1 - t_2), \quad (2)$$

$$\langle \xi_i(t_1) \xi_j(t_2)\rangle = 2\kappa \delta_{ij} \delta(t_1 - t_2), \quad (3)$$

where angular brackets mean averaging over time and κ is the particle diffusion coefficient. One can say that the expression (2) determines the turbulent diffusion, the factor μ can be estimated as $\mu \sim u_L/L^3$. Analogous expressions can be written for the velocity components u_x, u_y.

Comparing the expressions (2) and (3), one finds the scale $r_d = (\kappa/\mu)^{1/4} = L/\mathrm{Pe}^{1/4}$. Here Pe is Péclet number defined as $\mathrm{Pe} = \tau_\kappa/\tau_c$, where $\tau_\kappa = L^2/\kappa$ is the characteristic diffusion time in the viscous layer. The scale r_d determines the thickness of the diffusion boundary sublayer formed near the wall: at $z < r_d$ the particle diffusion dominates whereas at $z > r_d$ the turbulent diffusion does. There is another scale in the problem, $z^* = 1/\sqrt{\mu\tau} = L/\sqrt{\mathrm{St}}$. At $z < z^*$ the "friction" term $\mathbf{v}$ in Eq. (1) dominates over the ballistic term $\tau d\mathbf{v}/dt$ whereas at $z > z^*$ the last one dominates. Particles delay essentially from the fluid motion here. We assume the following set of inequalities: $r_d \lll z^* \lll L$.

There is a closed description of the particle motion along Z-direction in terms of its coordinate z and $v \equiv \equiv v_z = dz/dt$. Since both the flow velocity and the Langevin force are short correlated in time, it is possible to derive from Eq. (1) the Fokker–Planck equation for the probability density of a particle distribution in the z, v-space, $\rho(z, v)$. Then $\int dz\, dv\, \rho = 1$. Alternatively, ρ can be treated as the particle density in the z, v-space, if one considers an ensemble of large number of particles. Then the integral $\int dz\, dv\, \rho$ is equal to the total number N of the particles.

Following the standard procedure (see, e.g., Ref. [14]) one obtains the following Fokker–Planck equation

$$\partial_t \rho = -v \partial_z \rho + \frac{1}{\tau} \partial_v (v\rho) + \frac{\mu}{\tau^2} (z^4 + r_d^4) \partial_v^2 \rho, \quad (4)$$

that follows from the equation (1) and the relations (2), (3). The first term in the right-hand side of Eq. (4) can be called the "advection" one, whereas the second term in the right-hand side of Eq. (4) is produced by the "friction" term in (1).

The Fokker–Planck equation (4) has to be supplemented by boundary conditions. We assume that $\rho(z, v)$ tends to zero fast enough as $v \to \pm\infty$ and $z \to \infty$. Next, a boundary condition at the wall (at $z = 0$) should be posed. We consider the case of partly inelastic scattering, assuming that the particle loses a definite part $1 - \beta$ of its velocity along the Z-axis after each collision with the wall, where β is some factor, $\beta \le 1$. Then

$$\rho(z = 0, v) = \beta^{-2} \rho(z = 0, -v/\beta) \quad \text{for } v > 0. \quad (5)$$

The factor β^{-2} in Eq. (5) is related to the particle number (or probability) conservation: the outcoming flux $\rho\, v\, dv$ for positive v should coincide with the incoming flux for $-v/\beta$.

Integrating over v the Fokker–Planck equation (4), one obtains the local conservation law $\partial_t n = -\partial_z j$. Here, the particle density $n(z)$ and the particle flux $j(z)$ in Z-direction are defined as the integrals

$$n(z) = \int_{-\infty}^{+\infty} dv\ \rho(z, v), \quad (6)$$

$$j(z) = \int_{-\infty}^{+\infty} dv\ v\rho(z, v). \quad (7)$$

Below we demonstrate that regimes with zero and non-zero flux j can be realized. However, even the non-zero flux is relatively small. That means validity of a quasi-stationary approach since then all parameters of the system vary slowly in time and the distribution $\rho(z, v)$ adjusts adiabatically to the parameters.

Let us explain qualitatively the distribution of the inertial particles in the viscous sublayer. Majority of particles are accumulated in the diffusion sublayer, at $z \lesssim r_d$, where they undergo the Brownian motion. Some particles are distributed in the layer $r_d \lesssim z \lesssim z^*$ where a z-dependent Maxwellian distribution is formed. The most fast particles move at larger z, where the particles form strongly non-equilibrium distribution. The fast particles undergo random acceleration due to the fluid velocity fluctuations. However, the fluctuations force the particles to move toward the wall from time to time. Collisions with the wall lead to losing particles' energy. The balance between the acceleration and the losses determines the result. If β is small then the losses dominate and the particle velocity diminishes in average. Then the flux is zero. If β is moderate then the random acceleration dominates and the particle velocity increases in average. Then fast particles can reach z larger than L escaping to bulk. In this case the particle flux to large z is non-zero. Below we show, that the two regimes are separated by the critical value of β, $\beta_c = \exp(-\pi/\sqrt{3})$.

Further we pass to units of measurements where $\mu = \tau = 1$. Then $z^* = 1$, $L \sim \sqrt{\mathrm{St}} \gg 1$, $r_d \sim$

$\sim \sqrt{\mathrm{St}}/\mathrm{Pe}^{1/4} \ll 1$. The last inequality enables us to neglect the term with r_d in Eq. (4) for $z \gg r_d$. Due to the adiabaticity we neglect also the term with the time derivative in Eq. (4). Then we turn to the following stationary equation

$$v\partial_z \rho = \partial_v(v\rho) + z^4 \partial_v^2 \rho, \tag{8}$$

that determines the particle distribution in our setup.

Let us analyze the case $z \ll 1$. For $v \ll z$ it is possible to neglect "advection" term in Eq. (8). Then it is reduced to

$$\partial_v(v\rho) + z^4 \partial_v^2 \rho = 0. \tag{9}$$

We see that the typical v is of order z^2 that justifies our attention to the limit $v \ll z$. The Eq. (9) has an "local equilibrium" (quasi-Maxwellian) solution

$$\rho \propto \frac{1}{z^6} \exp\left(-\frac{v^2}{2z^4}\right), \tag{10}$$

where the factor at the exponent can be established if to return to Eq. (8) and to use the perturbation theory [15, 16].

The solution (10) is even in v and, consequently, does carry no flux. Note that the particle distribution in the region $z \ll 1$ depends weakly on the boundary conditions since their velocities relax effectively due to the leading role of "friction" in the region. Integrating the expression (10) over v, one finds $n \propto z^{-4}$. The z-dependence of the particle density n shows that the integral $N = \int dz\ n$, giving the total number of particles, diverges at small z. Thus, majority of particles are localized in the diffusive sublayer of the thickness r_d. Cutting the integral $\int dz\ n$ at $z \sim r_d$, we obtain $N \sim r_d^{-3} n(z=1)$. The concentration $n(1)$ is an upper estimation for the particle flux j from the wall. Therefore $N \gg j$ (due to $r \ll 1$) that justifies our adiabatic approach.

The particles in the region $1 \ll z \ll \sqrt{\mathrm{St}}$ are far from the "local equilibrium". For the case $v \gg z$ the equation is reduced to

$$v\partial_z \rho = z^4 \partial_v^2 \rho, \tag{11}$$

where we neglected the "friction" term in comparison with the "advection" one. Eq. (11) admits a self-similar substitution

$$\rho = z^{-5a} h(\zeta), \quad \zeta = \frac{5}{9}\frac{v^3}{z^5}, \tag{12}$$

where a is some scaling index. Then Eq. (11) gives

$$\zeta \partial_\zeta^2 h + (2/3 + \zeta)\, \partial_\zeta h + a h = 0. \tag{13}$$

Multiplying the equation by $\zeta^{-1/3}$, integrating over ζ and assuming that h tends to zero fast enough at $\zeta \to \pm\infty$, we obtain

$$(a - 2/3) \int dv\ v\rho = 0.$$

The condition means zero flux, $j = 0$, if $a \neq 2/3$.

The expression (12) for the self-similar variable ζ shows that the typical v at a given z can be estimated as $v \sim z^{5/3}$. Thus, the region $v \lesssim z$ where Eq. (11) is inapplicable is relatively narrow at $z \gg 1$. Therefore one can consider a solution of Eq. (11) at all ζ (all v) regarding that the solution and its derivative at $\zeta = 0$ are continuous.

Eq. (13) is the confluent hypergeometric equation, its solutions are the Kummer function $M(a, 2/3, -\zeta)$ and the Tricomi function $U(a, 2/3, -\zeta)$. Since the function $M(a, 2/3, -\zeta)$ diverges exponentially at large negative ζ, one should choose $h = U(a, 2/3, -\zeta)$ at negative ζ. At positive ζ the function h is a linear combination of $M(a, 2/3, -\zeta)$ and $U(a, 2/3, -\zeta)$. Equating the values of the function h and of its derivative over v at $\zeta = +0$ and $\zeta = -0$, one finds

$$h = \frac{2}{\sqrt{3}} \mathrm{Im}\, U(a, 2/3, -\zeta) + \frac{\Gamma(1/3)}{\Gamma(a + 1/3)} M(a, 2/3, -\zeta),$$

if $\zeta > 0$. Exploiting the asymptotic behavior of M and U at large values of their argument, one finds

$$h \approx |\zeta|^{-a} \quad \text{if} \quad \zeta < 0,$$
$$h \approx \frac{2}{\sqrt{3}} \left\{ \sin(\pi a) + \sin\left[\left(\frac{2}{3} - a\right)\pi\right] \right\} \zeta^{-a}, \ \zeta > 0. \tag{14}$$

If we consider the region $v \gg 1$, $z \lesssim 1$ then the main term in Eq. (8) is the "advection" term $v\partial_z \rho$. Therefore ρ is z-independent in this region, that corresponds to ballistic motion of fast particles. Hence, the laws (14) can be drawn to $z \to 0$ to obtain

$$\rho(v) = \frac{2}{\sqrt{3}} \left\{ \sin(\pi a) + \sin\left[\left(\frac{2}{3} - a\right)\pi\right] \right\} \rho(-v) \tag{15}$$

at $z = 0$ for $v \gg 1$. Comparing Eq. (15) with the boundary condition (5) one obtains the relation

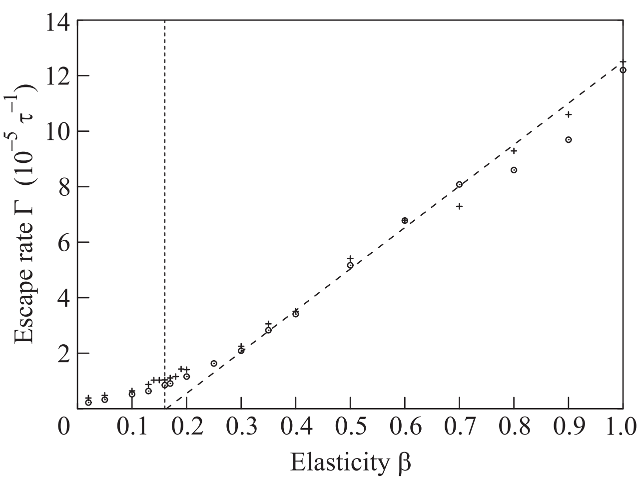
$$\frac{\sin(\pi a) - \sin[\pi(a - 2/3)]}{\sin(2\pi/3)} = \beta^{3(a-2/3)}. \tag{16}$$

The Eq. (16) has a solution $a = 2/3$ at any β, that corresponds to non-zero flux from of the wall to bulk. Another solution starts from $a = 5/6$ at $\beta = 0$, then a decreases as β grows and corresponds to zero flux. At $\beta \to \beta_c$, where

$$\beta_c = \exp\left(-\frac{\pi}{\sqrt{3}}\right) \approx 0.163, \tag{17}$$

the solution of Eq. (16) gives $a = 2/3$. When β grows further, exceeding β_c, the considered branch gives $a < < 2/3$. In this case the flux $\int dv\, v\rho$ becomes infinite, that is the solution is unphysical. We conclude that at $\beta = \beta_c$ a transition occurs corresponding to switching from zero-flux solution to non-zero flux solution. In terms of observable quantities, the transition is the confinement-deconfinement transition for the inertial particles near a wall.

To check our predictions we performed numerical simulations of the particle dynamics determined by the equation (1) and the relations (2), (3). We take a large number of particles (of the order of 10^3) initially located at $z \sim r_d$ with zero velocity and examine their trajectories. In our simulations, $\tau = 10$, $\mu = 0.1$, and $\kappa = 10^{-5}$. For a typical realization of the random processes, the particles for a long time remain near the wall, at $z \lesssim r_d$, and then at some moment they go coherently away from the wall. The particles crossing $z = z_{\max}$ where treated as escaping to bulk. The quantity $z_{\max}$ can be treated is the width of the viscous layer. The mean rate of the particle escape from the region $z < z_{\max}$ at $z_{\max} = 100$, 200 are plotted in Figure for different β. The rate is



Mean rate of particle escape from region $z < L$ at $L = 100$ (pluses) and $L = 200$ (circles) at different β

practically independent of $z_{\max}$ at $\beta \gtrsim 0.2$ and is less for $L = 200$ than for $L = 100$ at $\beta \lesssim 0.2$. That is in agreement with our analytic predictions.

It is also of interest to consider the case $\mathrm{Pe}^{1/4} \lesssim \sqrt{\mathrm{St}}$, that corresponds to $r_d \gtrsim z^*$ (where we assume $r_d \ll L$, as before). In the limit, the Maxwellian distribution (10) is valid nowhere and the scale z^* has no physical sense. Nevertheless, the self-similar form of the solution (12) remains valid after redefinition $z^5 \to z^5 + 5zr_d^4$. Moreover, our scheme can easily be generalized for the Fokker–Planck equation (4) where the factor $z^4 + r_d^4$ is substituted by an arbitrary function $\chi(z)$ growing faster than z^2 as z increases. In this case the particle confinement has to be observed at $\beta < \beta_c$ with the same critical value (17). The case $\chi(z) \propto z^2$ is marginal, then the transition is driven by the elasticity and/or inertia degree. The case $\chi \propto z^2$ and $\beta = 1$ was investigated in Ref. [12]. In the limit of strong inertia the particle's Lyapunov exponent changes its sign at the critical value (17), see Ref. [17]. The case where $\chi(z)$ is growing slower than z^2 as z increases (the case corresponds, say, to the turbulent boundary layer) needs a special analysis.

The crucial feature of the inertial particles dynamics that is revealed by numerics is that escapes of the particles from the viscous boundary layer of a turbulent flow to bulk are related to rare events. That implies a strong intermittency characteristic of turbulence in general. From the other hand, that leads to the hope that the events and their probability can be examined in the framework of the instantonic technique (saddle-point approximation) [18]. That implies a possibility to determine analytically probability of such events and also their statistics, leading to predictions, say, for high-order correlation functions of the particle flux to infinity. That is a subject of future investigations.

To conclude, we examined the inertial particle statistics in the viscous layer of a turbulent flow. The system is relaxed to the quasi-stationary regime where majority of particles are concentrated inside the diffusion boundary sublayer. The particles outside the layer can be divided into relatively slow ones that have the "local equilibrium" distribution (10) and relatively fast that are characterized by the self-similar distribution (12). Just the fast particles are responsible for a particle flux to bulk. This flux is zero if the elasticity coefficient β of particle collisions with the wall is smaller than the critical value (17). That is the confinement regime. At larger β the flux becomes non-zero that leads to graduate escaping the particles to bulk. Our scheme admits a wide generalization for other types of flows, the critical value (17) remaining the same.

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