Spectra of turbulence in dilute polymer solutions

A. Fouxon
Physics Department, Weizmann Institute of Science, Rehovot 76100, Israel

V. Lebedev
Landau Institute for Theoretical Physics, Moscow, Kosygina 2, 117940, Russia
and Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545

(Received 17 July 2002; accepted 25 March 2003; published 6 June 2003)

Turbulence in dilute polymer solutions when polymers are strongly stretched by the flow is investigated. We establish power-law spectra of velocity, that are not associated with a flux of a conserved quantity, in two cases. First, such spectrum is formed in the elastic waves range of high Reynolds number turbulence of polymer solutions above the coil–stretch transition. Second, such spectrum is characteristic of the elastic turbulence, where chaotic flow is excited due to elastic instabilities at small Reynolds numbers. © *2003 American Institute of Physics.*
[DOI: 10.1063/1.1577563]

I. INTRODUCTION

In this paper we continue theoretical investigation of turbulence in dilute polymer solutions, started in Refs. 1 and 2. As opposed to Newtonian fluids, such solutions possess additional macroscopic degrees of freedom related to the elasticity of the polymer molecules. Relaxation times of elastic stresses can be comparable with time scales of the flow which means that the relation between the stress and the velocity gradient is nonlocal in time and, consequently, in space. It is a striking property of dilute polymer solutions that minute amounts of polymer can significantly modify the flow. Probably, the most famous example is the drag reduction phenomenon. Addition of long-chain polymers in concentrations as small as, say, 10^{-5} by weight, can induce a substantial reduction of the drag force needed to push a turbulent fluid through a pipe.[3–6] Another example is the elastic turbulence,[7–9] which is a chaotic flow, excited in the dilute polymer solutions at low Reynolds numbers.

The reason why small amounts of polymer can significantly modify properties of the fluid is flexibility of polymer molecules. At equilibrium a polymer molecule coils up into a spongy ball of a radius R_0. The value of R_0 depends on the number of monomers in the molecule, which is usually very large. For a dilute solution with the concentration of the molecules, n, satisfying $nR_0^3 \ll 1$, an influence of equilibrium size molecules on the hydrodynamic properties of the fluid can be neglected. When placed in an inhomogeneous flow, such a molecule is deformed into an elongated structure, which can be characterized by its end-to-end distance R. If the number of monomers in a typical polymer molecule is large, the elongation R can be much larger than R_0. The influence of the molecules on the flow increases with their elongation and may become substantial when $R \gg R_0$.

Deformation of a polymer molecule is determined by two processes, stretching by the velocity gradients and relaxation due to elasticity of the molecule. To understand how the molecule resists the deformation by the flow, one should consider its relaxation. Recent experiments with DNA molecules indicate that the relaxation is linear in the wide region of scales $R_0 \ll R \ll R_{\max}$, where $R_{\max}$ is the maximum molecule extension.[10] In the case of polymers, theoretical arguments and numerics presented in Ref. 11 support the linear relaxation. These results can be understood if we assume that at $R \gg R_0$ the role of excluded volume and hydrodynamic interactions between the monomers are negligible. Then the random walk arguments suggest that the entropy of polymer molecules is quadratic in R in the range $R_0 \ll R \ll R_{\max}$ implying linear relaxation. Whether the polymers are excited by the flow is determined by the softest relaxation mode that corresponds to the dynamics of the elongation R. In the absence of stretching, the relaxation of R is described by the equation $\partial_t R = -R/\tau$, where τ is a relaxation time, which is expected to be R-independent at $R_0 \ll R \ll R_{\max}$. If the end-to-end distance R is of the order of the maximum extension, τ starts to depend on R and the dynamics of the molecule becomes nonlinear.[12] Possible statistical consequences of the nonlinearity have been investigated in Ref. 13.

The behavior of the molecule in an inhomogeneous steady flow depends on the value of the Weissenberg number, Wi, defined as the product of the characteristic velocity gradient and τ. When a polymer molecule is placed in a flow, smooth at the scale R, the velocity difference between the end-points is proportional to R multiplied by the characteristic value of the velocity gradient. At $\mathrm{Wi} \ll 1$ the relaxation is fast as compared to the stretching time and the polymer always relaxes to the equilibrium size, R_0. The behavior of the polymer at $\mathrm{Wi} \gtrsim 1$ depends on the geometry of the flow. For purely elongational flows the molecule gets aligned along the principal stretching direction. If the velocity gradient is larger than the inverse relaxation time, i.e., $\mathrm{Wi} \gtrsim 1$, the elastic response becomes too slow in comparison with the stretching and the molecule gets substantially elongated.[14] The sharp transition from the coiled state to the strongly extended state is called the coil–stretch transition. Rotation can suppress the transition and even damp it completely since the molecule

does not always point in the stretching direction (see, e.g., Ref. 15). For example, no coil–stretch transition occurs in the case of the shear flow, which is a particular combination of the elongational and rotational flows.

In contrast to the steady case, a polymer molecule, moving in a random flow, alternately enters regions of high and low stretching. As the intensity of the flow increases, the effect of the stretching becomes more pronounced. One can generally assert existence of the coil–stretch transition in this case. This has been first demonstrated by Lumley,[15] who considered the situation where the characteristic time of variations of the velocity gradient is much larger than the inverse of the characteristic value of the gradient. He showed that if the amplitude of the velocity gradient fluctuations is large enough, the expectation value of R^2 grows with time, which signifies the coil–stretch transition. We have shown in Refs. 1 and 2 that the coil–stretch transition occurs in any random flow and established a general criterion for the transition. In particular, the transition occurs in the situation where the time of velocity gradient variation is of the order of the inverse of its characteristic value, which is likely to be the case for real flows. The coil–stretch transition in random flows is controlled by the parameter $\lambda_1\tau$, where λ_1 is the average logarithmic divergence rate of nearby Lagrangian trajectories, to be referred to as the principal Lyapunov exponent (which is positive for an incompressible flow[16,17]). The molecules are weakly stretched if $\lambda_1\tau<1$ and strongly stretched otherwise. Therefore, for random flows the parameter $\lambda_1\tau$ plays the role of the Weissenberg number.

As it is well known (see, e.g., Refs. 18 and 19), turbulent flows in Newtonian fluids consist of chaotic eddies from a wide interval of scales, $\eta<r<L$, where L is the integral scale (where the flow is excited) and η is the viscous scale. The energy pumped at the scale L cascades down to the scale η, where it dissipates. The size of the polymer molecules is usually much smaller than the viscous scale. Viscosity makes the flow smooth at scales $r<\eta$, i.e., the velocity can be approximated by linear profiles at these scales. Therefore, if $R<\eta$, then the stretching of molecules is determined by the velocity gradient, which is random in the turbulent flow. The Lyapunov exponent can be estimated as the characteristic value of the velocity gradient, which is determined by the eddies at the viscous scale η. As the Reynolds number grows, the velocity gradient increases, and so does $\lambda_1\tau$. At some value Re_c of the Reynolds number the product $\lambda_1\tau$ reaches the value 1 and the coil–stretch transition occurs.

Several mechanisms can limit the polymer stretching above the coil–stretch transition. The first one is the internal nonlinearity of the elasticity of the polymer molecules. If this mechanism dominates, then above the transition the molecules are stretched up to the elongation of the order of $R_{\max}$. An alternative mechanism has been proposed by Tabor and de Gennes.[20] It is based on the assumption that the elongation of the polymer molecules R becomes larger than the viscous length of turbulence, η, where the elastic force always wins over the stretching at a certain value of the elongation. Below, we assume that $R\ll\eta$, which seems to be reasonable for typical polymer solutions. Another limiting mechanism is the back reaction of the polymers on the flow. It is caused by the collective contribution of coherently deformed polymer molecules into the stress tensor. This elastic part of the stress grows with the molecule elongation. When it becomes of the order of the viscous stresses existing in the flow, the polymers modify the flow around them suppressing the stretching. As a result, a dynamic equilibrium is realized at a characteristic elongation, R_{back}. The total polymer stress is proportional to nR^2, so that R_{back} depends on the polymer concentration n. We assume that the concentration is large enough for the value of R_{back} to be much smaller than $R_{\max}$. Probably, the condition $R_{\mathrm{back}}\ll R_{\max}$ is necessary for existence of a stationary state, because the polymer molecules, stretched up to $R_{\max}$, are intensively destroyed by the flow.

Above the coil–stretch transition the back reaction modifies the small-scale properties of turbulent flows, which leads to the emergence of a new scale, $r_*>\eta$, where energy dissipates mainly due to polymer relaxation. The scale r_* plays the role of a new dissipation scale. Large-scale eddies with the sizes $r>r_*$ do not excite elastic degrees of freedom so the usual inertial energy cascade is realized at these scales. At $\mathrm{Re}\gg\mathrm{Re}_c$ there appears a new region of scales, $\eta_*<r<r_*$, where elastic waves can propagate,[2] which are analogous to the Alfven waves in magnetic hydrodynamics. At the scale η_* viscosity becomes essential, leading to the strong damping of the elastic waves.

In this work we investigate the velocity spectrum in the elastic wave range $kr_*\gg1$ (where $\boldsymbol{k}$ is the wave vector). We show that the spectrum obeys a power law. The ideas of the analysis go back to the works of Townsend[21] and Batchelor.[22] They recognized that fluctuations with scales, smaller than the smoothness scale of the flow (η in the case of usual turbulence and r_* in our case), evolve in the linear velocity profiles. For the passive scalar at large Prandtl numbers Batchelor derived the spectrum with the power-law k^{-1},[22] originating from the exponential character of stretching in the linear flow. Formally, it is explained by zero scaling dimension of the advection term that implies scale invariance. This property is not broken by a linear decay term where the power-law spectrum still holds, as it was shown for a linearly decaying passive scalar by Chertkov.[23] Here we introduce a consistent theoretical scheme for the description of the small-scale fluctuations and show that though the dynamics of the small-scale fluctuations is more complicated, than in the case of the passive scalar (advection and linear decay accompanied by stretching and waves), the power-law in the spectrum still holds. The wave oscillations break the scale-invariance, but their influence on the energy balance is reduced to forcing the equipartition of kinetic and elastic energies of small-scale fluctuations. The power law spectrum terminates at $k\sim(v\tau)^{-1/2}$, where viscosity overcomes stretching. The power-law interval widens as Re grows. Let us stress that this law is not related to a flux of any conserved quantity.

Another situation, where a power-law spectrum of the small-scale fluctuations is observed, is the elastic turbulence, realized in low-Reynolds polymer solutions, if the Weissenberg number Wi is large enough.[7–9] It was shown experimentally in Refs. 7–9, that the coil–stretch transition, occurring at increasing Wi, leads to a chaotic flow even though Re is

small. Its existence is due to hydrodynamic instabilities caused by the presence of the elastic stresses. The velocity spectrum is observed to be power-like in a wide range of scales in this case.[7–9] We demonstrate that, in contrast to the usual hydrodynamic turbulence, in the elastic turbulence the power velocity spectrum is not related to the energy cascade, since the main energy dissipation occurs at the largest scales. The mechanism, leading to this power spectrum is, again, similar to the linearly decaying passive scalar problem.

The structure of the paper is as follows. In Sec. II we introduce a system of equations describing the coupled dynamics of inertial and elastic degrees of freedom. This system is similar to the system of equations describing the magnetohydrodynamics (MHD)[24] with the important difference of a linear decay term in the equation on the ''magnetic field.'' In Sec. III we present results, concerning statistics of a passive scalar with a constant damping, embedded in a random flow. It is a prototype for the subsequent consideration. Section IV is devoted to the description of the principal properties of the large Re turbulence in the presence of polymers and to the derivation of the power-law spectrum in the elastic dissipation range. In Sec. V we establish the power-law spectrum for the elastic turbulence. In Conclusion we summarize our results and discuss possible implications of our work for other subjects. The Appendix is devoted to some details of the Lagrangian statistics.

II. BASIC RELATIONS

We study dynamics of dilute polymer solutions at scales much larger than the intermolecular distance where the polymer solution can be regarded as a continuous medium and described by macroscopic fields. Characteristic times of considered processes are regarded to be comparable with the polymer relaxation time τ. In this case, besides the usual hydrodynamic degrees of freedom, one has to take into account degrees of freedom, related to the polymer elasticity. These degrees of freedom can be described in terms of the polymer stress tensor.[12]

We assume that the flow can be treated as incompressible, that is $\nabla\cdot\boldsymbol{v}=0$, where $\boldsymbol{v}$ is the velocity of the flow. This is justified provided processes at a given scale are slow in comparison with sound oscillations at the same scale. Then the velocity dynamics can be described in terms of the following equation:[12]

$$\partial_t v_i+(\boldsymbol{v}\nabla)v_i+\varrho^{-1}\nabla_i P=\nu\nabla^2 v_i+\nabla_j\Pi_{ij}, \tag{1}$$

which is a generalization of the Navier–Stokes equation to the case of viscoelastic fluids. Here P is the pressure, ν is the kinematic viscosity of the solvent, ϱ is the fluid mass density, and Π_{ij} is the polymer contribution into the stress tensor per unit mass.

Equation (1) has to be supplemented by an equation for the polymer stress tensor Π_{ij}.[12] We assume the following equation:

$$\partial_t\Pi_{ij}+(\boldsymbol{v}\nabla)\Pi_{ij}=\Pi_{kj}\nabla_k v_i+\Pi_{ik}\nabla_k v_j-\frac{2}{\tau}(\Pi_{ij}-\Pi_0\delta_{ij}), \tag{2}$$

where τ is the polymer relaxation time, and Π_0 is related to the thermal fluctuations of the polymer conformations.[25] Let us briefly repeat applicability conditions of Eq. (2), discussed in Ref. 2. Linearity of the decay term in (2) assumes $R\ll R_{\max}$, where R is the typical polymer molecule size. Equation (2) implies that there is a single mode related to the polymer deformations, which is an idealization. A polymer molecule has a lot of deformational degrees of freedom, that have different relaxation times. A number of such degrees of freedom was observed in experiments with DNA.[10] Nevertheless, in the turbulent flows, only the mode with the largest relaxation time is strongly excited, whereas other modes are excited at most weakly. Thus, Eq. (2) should be treated as the equation related to the principal mode.

The concentration of the polymer molecules n enters systems (1) and (2) only via Π_0, $\Pi_0\propto n$ and implicitly via the assumption $R\ll R_{\max}$ since $\Pi\propto nR^2$. If n is inhomogeneous, then the system of equations (1) and (2) has to be supplemented by the equation for the concentration $\partial_t n+\boldsymbol{v}\nabla n=0$ (we neglect small diffusivity of polymer molecules). In this paper we consider the case when polymer molecules are strongly extended. Then $\Pi\gg\Pi_0$, and the term with Π_0 in Eq. (2) can be discarded. In this case any explicit dependence on the concentration of the polymer molecules n drops from the system of equations (1) and (2). Therefore, the dynamics of the polymer solutions with different values of n is identical in this regime as long as n is large enough for the condition $R\ll R_{\max}$ to be satisfied.

A. Lagrangian description

One can establish some properties of the polymer stress tensor Π, using the Lagrangian description of a fluid. It is based on the notion of fluid particles trajectories (Lagrangian trajectories) $\boldsymbol{x}(t,\boldsymbol{r})$, which are determined by the relations

$$\partial_t\boldsymbol{x}=\boldsymbol{v}(t,\boldsymbol{x}),\quad \boldsymbol{x}(t_0,\boldsymbol{r})=\boldsymbol{r}. \tag{3}$$

The point $\boldsymbol{r}$ plays the role of a Lagrangian marker. If Π_0 in Eq. (2) is neglected, then it is possible to write its solution as

$$\Pi(t,\boldsymbol{x})=W(t,t_0)\Pi(t_0,\boldsymbol{r})W^T(t,t_0)e^{-2(t-t_0)/\tau}, \tag{4}$$

where the superscript T denotes a transposed matrix. Here W is the Lagrangian mapping matrix determined by the relations

$$\partial_t W(t,t_0)=\sigma(t)W(t,t_0),\quad W(t_0,t_0)=1, \tag{5}$$

$$\sigma_{ij}(t,\boldsymbol{r})=\nabla_j v_i[t,\boldsymbol{x}(t,\boldsymbol{r})]. \tag{6}$$

Above σ is the tensor of the velocity derivatives along the Lagrangian trajectory $\boldsymbol{x}(t)$ that includes the strain tensor and local rotations. The incompressibility condition $\nabla\cdot\boldsymbol{v}=0$ is formulated in terms of σ as $\mathrm{tr}\sigma=0$. Then a consequence of Eq. (5) is $\det W=1$.

The matrix W describes deformations of infinitesimal fluid volumes. For example, the separation, $\delta\boldsymbol{x}$, between two close fluid particles, moving along the Lagrangian trajectory $\boldsymbol{x}(t)$, evolves according to

$$\delta\boldsymbol{x}(t)=W(t,t')\,\delta\boldsymbol{x}(t'). \tag{7}$$

Therefore, $W_{ij}(t,t_0,\boldsymbol{r})=\partial x_i(t,\boldsymbol{r})/\partial r_j$. Now it is easy to understand a meaning of Eq. (4). The polymers are advected along the Lagrangian trajectories being stretched by the velocity gradient and relaxing to their equilibrium shape due to the polymer elasticity.

We now briefly describe statistical properties of the matrix W; details can be found in the Appendix. We represent the matrix as $W=M\Lambda N$, where M and N are orthogonal matrices, while Λ is a diagonal matrix. At times much larger than the velocity gradients correlation time τ_σ the main eigenvalue $\exp(\rho_1)$ of Λ becomes much larger than the rest, under the condition that the set of the Lyapunov exponents λ_i is nondegenerate. If a statistically steady state is realized, then the observation time is arbitrarily large, and we conclude from Eq. (4) that the matrix Π has to be uniaxial

$$\Pi_{ik}=B_iB_k, \tag{8}$$

as it was noted in Ref. 2. This conclusion is almost self-evident once one goes back to the derivation of Eq. (2), recognizing that at $R\gg R_0$ the contribution of thermal fluctuations into Π is negligible, so that $\Pi_{ij}\propto R_iR_j$ holds. We observe that the vector $\boldsymbol{B}$ characterizes the direction and the strength of the coherent molecule elongations weighted by their contribution into the stress tensor. Note, that $\boldsymbol{B}$ is defined up to sign, in analogy with the director in nematic liquid crystals. It follows from Eqs. (4) and (8) that

$$\boldsymbol{B}(t,\boldsymbol{x})=\exp[-(t-t_0)/\tau]W(t,t_0)\boldsymbol{B}(t_0,\boldsymbol{r}). \tag{9}$$

If $t-t_0\gg\tau$ then W in this relation can be estimated as e^{ρ_1}.

There are some modifications of the W statistics with respect to a Newtonian fluid, that are imposed by the above relations. As it follows from Eq. (9), stationarity of the $\boldsymbol{B}$ statistics implies that $\rho_1(t)-t/\tau$ has a stationary distribution. In particular, we conclude that the principal Lyapunov exponent $\lambda_1=\lim_{t\to\infty}\rho_1/t$ of the flow is equal to $1/\tau$, independently of the Reynolds number. This means that above the coil–stretch transition the characteristic value of the velocity gradient is fixed at the scale $1/\tau$. The above behavior is contrasted to the Newtonian fluids for which λ_1 grows with increasing Re and fluctuations of ρ_1-t/τ grow with time. The absence of the growth of the fluctuations is related to anticorrelations in the temporal dynamics of the component $\tilde{\sigma}_{11}(t)$ of σ_{ij} along $\boldsymbol{B}$. These anticorrelations show themselves in the equality $\int dt\langle\langle\tilde{\sigma}_{11}(0)\tilde{\sigma}_{11}(t)\rangle\rangle=0$ (double brackets designate an irreducible correlation function) and originate in the special interaction of the inertial and elastic degrees of freedom, explained in more detail in Sec. IV.

B. Dynamic equations for dilute polymer solutions

It is convenient to rewrite Eqs. (1) and (2) in terms of the vector $\boldsymbol{B}$ thus getting rid of extra degrees of freedom. Substituting the decomposition (8) into Eq. (2), one obtains

$$\partial_tB_i+(\boldsymbol{v}\nabla)B_i=B_k\nabla_kv_i-B_i/\tau. \tag{10}$$

This equation is similar to the one satisfied by the magnetic field in MHD,[24] with the constant damping instead of the magnetic resistivity. The resemblance is made even stronger by noting that $\boldsymbol{B}$ has to be solenoidal. Indeed, it follows from Eq. (10) that

$$\partial_t\nabla\cdot\boldsymbol{B}+(\boldsymbol{v}\nabla)\nabla\cdot\boldsymbol{B}=-\nabla\cdot\boldsymbol{B}/\tau. \tag{11}$$

Therefore, $\nabla\cdot\boldsymbol{B}$ monotonically decays becoming zero in the (statistically) steady state. Substituting the expression (8) into Eq. (1), and taking into account the constraint $\nabla\cdot\boldsymbol{B}=0$, one obtains

$$\partial_t\boldsymbol{v}+(\boldsymbol{v}\nabla)\boldsymbol{v}=\nu\nabla^2\boldsymbol{v}-\varrho^{-1}\nabla P+(\boldsymbol{B}\nabla)\boldsymbol{B}. \tag{12}$$

Now the analogy of systems (10) and (12) with the system describing MHD[24] at zero magnetic resistivity is almost complete. The only difference is in the damping term in Eq. (10).

The energy density per unit mass is given by the sum of kinetic $v^2/2$ and elastic $B^2/2$ terms. The energy balance equation, following from Eqs. (10) and (12), is

$$\begin{aligned}(\partial_t+\boldsymbol{v}\nabla)(v^2/2+B^2/2)=&(\boldsymbol{B}\nabla)(\boldsymbol{B}\cdot\boldsymbol{v})-\varrho^{-1}\boldsymbol{v}\nabla P\\&+\nu\nabla_i(\boldsymbol{v}\nabla_i\boldsymbol{v})-\nu(\nabla_iv_k)^2\\&-\tau^{-1}B^2.\end{aligned} \tag{13}$$

The energy dissipation is due to the viscous and the polymer relaxation terms. Other terms in Eq. (13) represent energy fluxes (in real space), they can be written as full divergences due to the constraints $\nabla\cdot\boldsymbol{v}=0=\nabla\cdot\boldsymbol{B}$.

III. MECHANISM OF SCALE-INVARIANCE: PASSIVE SCALAR WITH LINEAR DAMPING

Before investigating statistical properties of the polymer solutions, described by Eqs. (10) and (12), we present statistical properties of a passive scalar with constant damping at scales smaller than the smoothness scale of the flow, examined in Ref. 23. This simple case enables one to recognize the origin of a power spectrum for passively advected fields.

The equation for the passive scalar θ in the considered case is

$$\partial_t\theta+\boldsymbol{v}\nabla\theta=-\theta/\tau+\phi, \tag{14}$$

where τ is the passive scalar decay time and ϕ is a forcing term needed to maintain the stationary state. It is assumed to be concentrated at a finite range of wave vectors near k_f. We have omitted the diffusive term which can be neglected in comparison with the constant damping (in some region of scales) provided the diffusion coefficient is small enough. Note that the constant damping of the passive scalar leads to a well-defined steady statistics even in the absence of the passive scalar diffusion.

A flow is smooth on scales smaller than the velocity gradients correlation length l. The smoothness means that a velocity difference between two points can be approximated by a linear profile

$$\delta v_i=\sigma_{ij}\delta r_j, \tag{15}$$

where δr_j is the separation between the points and σ is a function of time. Obviously, $\sigma_{ij}=\nabla_jv_i$. For the usual turbulent velocity the correlation length l is equal to the viscous

scale, $l=\eta$. The smoothness of the turbulent velocity at scales less than η was first exploited by Batchelor,[22] who considered statistical properties of a passive scalar at these scales.

The linearity of the velocity difference leads to a power law for the passive scalar spectrum $E(k)$, which is defined as $\langle\theta(\boldsymbol{k})\theta(\boldsymbol{k}')\rangle=(2\pi)^3\delta(\boldsymbol{k}+\boldsymbol{k}')E(k)$, where $\theta(\boldsymbol{k})$ is Fourier transform of $\theta(\boldsymbol{r})$ with the wave vector $\boldsymbol{k}$, angular brackets designate averaging over the statistics of $\boldsymbol{v}$, and we assume homogeneity and isotropy of the statistics. Indeed, the time of the energy transfer from k to $2k$ at $kl\gg1$ is scale-independent due to the linearity of the velocity profile. On the other hand, during the spectral transfer time the energy decay is also k-independent (since the damping term is scale-independent). As a result the spectral function $E(k)$ satisfies a relation $E(2k)=CE(k)$ (with a constant $C<1$). The solution of this equation is a power-law $E(k)\propto k^{-\alpha}$ with $2^{\alpha}C=1$.

Now we put the above consideration into a more rigorous frame. We consider the passive scalar spectrum $E(k)$ at $kl\gg1$. The evolution of wave packets with such wave vectors is determined by the velocity gradient σ. Let us consider the evolution during a time t_0 and express $\theta(t_0)$ via $\theta(0)$. A value of $\theta(t_0)$ near a point $\boldsymbol{r}_1$ is determined by an evolution of θ in the vicinity of the Lagrangian trajectory $\boldsymbol{x}(t,\boldsymbol{r}_1)$. To examine this evolution, one may perform the Taylor expansion of the velocity $\boldsymbol{v}$ in Eq. (14) up to the first order in $\boldsymbol{r}-\boldsymbol{x}$ since the homogeneous advection does not affect equal-time correlation functions due to the Galilean invariance. Then one obtains

$$\partial_t\theta+[\boldsymbol{u}+\sigma\cdot(\boldsymbol{r}-\boldsymbol{x})]\nabla\theta=-\theta/\tau+\phi. \tag{16}$$

Here $\boldsymbol{u}(t)=\boldsymbol{v}(t,\boldsymbol{x})$ and $\sigma=\sigma(t,\boldsymbol{x})$ are the velocity and the velocity gradients matrix along the Lagrangian trajectory $\boldsymbol{x}$. Fourier transform θ_k of the field $\tilde{\theta}$ measured in the moving frame $\tilde{\theta}(t,\boldsymbol{r})\equiv\theta(t,\boldsymbol{r}+\boldsymbol{x})$ satisfies

$$\partial_t\theta_k-\left(\boldsymbol{k}\sigma\cdot\frac{\partial}{\partial\boldsymbol{k}}\right)\theta_k=-\frac{\theta_k}{\tau}+\phi_k\exp[i\boldsymbol{k}\cdot\boldsymbol{x}]. \tag{17}$$

Further we confine ourselves to wave vectors $k\gg k_f$, that is much larger than those on which the pumping ϕ is supported. In this case θ_k is determined by the convection from smaller wave vectors and the forcing term can be neglected. Then Eq. (17) can be solved explicitly, and we find

$$\theta(t,\boldsymbol{k})=e^{-t/\tau}\theta(0,\boldsymbol{k}W), \tag{18}$$

where $W=W(t,0)$ is the Lagrangian mapping matrix, see Sec. II A. Returning to the real space, we obtain

$$\theta(t,\boldsymbol{r}+\boldsymbol{x})=e^{-t/\tau}\int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}e^{i\mathrm{k}[\mathrm{r}+W\boldsymbol{x}(0)]}\theta_0(\boldsymbol{k}W), \tag{19}$$

where $\theta_0(\boldsymbol{k})$ is the Fourier transform of $\theta(0,\boldsymbol{r})$. The above formula is valid for $|W^{-1}\boldsymbol{r}|\ll l$, the condition means that the passive scalar coming to a point $\boldsymbol{r}+\boldsymbol{r}_1$ was all the time in the l-vicinity of $\boldsymbol{x}$ allowing the Taylor expansion for the velocity.

Let us now consider the pair correlation function of the passive scalar $f(r)\equiv\langle\theta(t_0,\boldsymbol{r}_1)\theta(t_0,\boldsymbol{r}_1+\boldsymbol{r})\rangle$, defined as the spatial average over $\boldsymbol{r}_1$. We assume that the averages $\langle\theta\rangle$ and $\langle\phi\rangle$ are zero (which can always be achieved by a shift of θ and ϕ by a constant). The product of the fields is given by Eq. (19) [remind that $\boldsymbol{x}(t_0)=\boldsymbol{r}_1$] and depends on $\boldsymbol{r}_1$ via the argument of W. The average over space (over $\boldsymbol{r}_1$) is equivalent to the average over the velocity statistics, or over the velocity gradients statistics along the Lagrangian trajectories. If $\lambda_1 t_0\gg1$ then the average over the interval $0<t<t_0$ and negative times can be done independently. Indeed, $\sigma(t)$ has a Lagrangian correlation time λ_1^{-1}. Thus velocity at negative times is correlated with σ only at $|t|\sim\lambda_1^{-1}$ while $W(t_0)$ is not sensitive to the value of σ there, due to $\lambda_1^{-1}\ll t_0$ (see the Appendix). Therefore, we can write

$$f(r)=e^{-2t_0/\tau}\int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\exp(i\boldsymbol{k}\boldsymbol{r})\langle E[\boldsymbol{k}W(t_0)]\rangle, \tag{20}$$

where $E(k)$ is the spectrum function introduced above. Noting that $E(k)$ equals the Fourier transform of the pair-correlation function we obtain the following stationarity condition for the spectrum (we substitute t_0 by t)

$$E(\boldsymbol{k})=\langle\exp(-2t/\tau)E(\boldsymbol{k}W)\rangle, \tag{21}$$

where $W=W(t,0)$. Equation (21) is applicable at $kl\gg1$, as follows from the condition $r\ll l$ in Eq. (19).

The relation (21) has a simple meaning. The wave vectors of small-scale fluctuations of the passive scalar evolve according to $\boldsymbol{k}(t)=\boldsymbol{k}(0)W^{-1}(t)$ as was shown by Kraichnan.[26] Thus the energy of a fluctuation with the wave vector $\boldsymbol{k}$ is equal to its energy time t ago at the wave vector $\boldsymbol{k}W(t,0)$ multiplied by the decaying factor $\exp(-2t/\tau)$. Note that we could equally well start directly from (17) to derive Eq. (21). One can show that in the spatially homogeneous situation one can always introduce such equation for the investigation of the spectrum at $kl\gg1$.

Equation (21) is the quantification of the heuristic arguments given in the beginning of the section taking into account that the energy transfer time is by itself a random quantity. Its solution is a power law $E(k)\propto k^{-\alpha}$. Substituting the expression into Eq. (21), one gets the relation

$$\exp(2t/\tau)=\langle|\boldsymbol{k}W/k|^{-\alpha}\rangle, \tag{22}$$

which determines the exponent α. At $\lambda_1 t\gg1$ the moments of $|\boldsymbol{k}W/k|$ behave exponentially with time. Indeed, they are roughly equal to the product of $\lambda_1 t$ independent identically distributed random variables. Besides at these times the moments are independent of $\boldsymbol{k}/k$ due to the isotropization of $W(t_0)$ described in the Appendix. As a result the above equation has a unique physical solution examined in more detail in Appendix 3, where the inequality $\alpha>3$ is established. The inequality has simple meaning that the spectrum has to decay faster than the Batchelor spectrum k^{-3} (k^{-1} in the spherical normalization) holding at infinite τ.

Let us now extend the above results. The power law spectrum persists, even if the relaxation time τ is $\boldsymbol{k}$-dependent, but scales as zero power of k, that is if τ depends on the direction of $\boldsymbol{k}$ only. This dependence can be regular (which makes the spectrum anisotropic) or random. Another remark is that addition of an oscillating term (with ω_k) into the equation for θ_k

$$\partial_t \theta_k - \left(\boldsymbol{k}\boldsymbol{\sigma} \cdot \frac{\partial}{\partial \boldsymbol{k}} \right) \theta_k = -\theta_k/\tau - i\omega_k(t)\theta_k, \tag{23}$$

does not change its spectrum even though the oscillating term breaks the scale-invariance. Indeed, let us pass from θ to $\tilde{\theta}$, which is $\tilde{\theta} = \exp(i\varphi_k)\theta$, with the phase, satisfying

$$\partial_t \varphi_k - \left(\boldsymbol{k}\boldsymbol{\sigma} \cdot \frac{\partial}{\partial \boldsymbol{k}} \right) \varphi_k = \omega_k. \tag{24}$$

Then for $\tilde{\theta}$ we return to Eq. (19), which leads to the power spectrum. It remains to note that the spectra of θ and $\tilde{\theta}$ coincide. In other words, oscillating terms conserve energy and are, consequently, irrelevant for the energy balance.

Below we generalize the simple picture, presented in this section, to the polymer solutions.

IV. HIGH REYNOLDS FLOWS

Here we consider turbulence in dilute polymer solutions, when the Reynolds number exceeds the critical value Re_c, corresponding to the coil–stretch transition. Then the polymer molecules are strongly elongated. Two different cases are possible, depending on the concentration of the polymer molecules n. If it is very small, the elastic stresses are small in comparison with the viscous stresses. Then the polymers are stretched to their maximal elongation, $R_{\max}$, and the properties of the fluid do not differ significantly from those of the pure solvent. Below we consider the second, more interesting, case, when the concentration of the polymers n is large enough, so that elastic stresses can be larger than the viscous stresses. Then the polymer back reaction substantially modifies the flow.

Whereas in the pure solvent typical velocity gradients grow unlimited as the Reynolds number increases, in polymer solutions above the coil–stretch transition the balance of inertial and elastic degrees of freedom fixes the characteristic value of the velocity gradient at $1/\tau$. Indeed, if the instantaneous velocity gradient exceeds $1/\tau$, it extends the polymers, so that the elastic stress grows and damps the gradient. On the other hand, if the velocity gradient is smaller than $1/\tau$, the molecules contract and their influence on the flow diminishes. Then the velocity gradients tend to grow up to the value characteristic of the pure solvent, which is larger than $1/\tau$. Thus the velocity gradients fluctuate near $1/\tau$, that explains the statistically steady state realized above the coil–stretch transition. Let us derive the condition for the existence of this steady state, related to the existence of the maximal size $R_{\max}$ of the polymer molecules. In the vicinity of the coil–stretch transition $\nabla v \sim 1/\tau$ so that $B^2_{\mathrm{back}} \sim v\nabla v \sim v/\tau$ as it follows from Eq. (12). This leads to the condition for the existence of the back reaction regime $B^2_{\max} \gg v/\tau$, where $B^2_{\max}$ is the maximal value of the elastic stress tensor achieved at $R \sim R_{\max}$. Using estimates for the microscopic parameters, proposed in Ref. 25, one can rewrite this condition as $n \gg (R_0 R^2_{\max})^{-1}$. Below we will assume that there exists an interval in Re such that $B_{\max}$ exceeds the value of B prescribed by the flow. The latter increases as Re grows so that the condition will break down at certain Re. After this happens either polymer degradation occurs or polymers start to behave as rigid bodies with size $R_{\max}$. In the latter case the fluid becomes Newtonian again with renormalized viscosity.

We assume $V\tau \ll L$, where V is the velocity at the turbulence integral scale. Then the gradient related to the large eddies is smaller than τ^{-1}. Therefore, the large eddies do not excite polymers, which means that the elastic stress tensor is not correlated at these scales. Since only coherent excitations of the elastic stress tensor can influence the flow, we conclude that the elasticity is negligible for large eddies. The interaction of inertial and elastic degrees of freedom becomes essential at a scale r_*, where velocity gradients are of the order of $1/\tau$. Here the energy starts to dissipate due to the polymer relaxation, that is the inertial cascade terminates at $r \sim r_*$. Since the velocity gradients fluctuate near the value $1/\tau$, reached at r_*, at $r < r_*$ velocity difference scales linearly with the distance that is r_* is the smoothness scale of the flow. Near the coil–stretch transition characteristic velocity gradient is determined by the viscous scale and is of the order of $1/\tau$, hence $r_* \sim \eta$. As the Reynolds number increases, velocity fluctuations increase, so that the scale r_* grows which is very different from the Newtonian fluids where the smoothness scale η decreases with Re. As the energy input increases the viscous energy dissipation rate, $v(\nabla \boldsymbol{v})^2$, remains of the order of v/τ^2. Therefore, far above the transition the principal part of the energy is dissipated by the polymer relaxation. Then the viscous term in Eq. (13) can be neglected and we obtain

$$\langle B^2 \rangle = \epsilon\tau, \tag{25}$$

where ϵ is the energy injection rate per unit mass, estimated as V^3/L. The relation (25) means that a typical value of B grows as the energy input increases, and, consequently, the elastic stress tensor does.

The above quantities can be estimated, using the Kolmogorov-type reasoning. Then we obtain from Eq. (25) that $B \sim \sqrt{\epsilon\tau}$. Next, as follows from Eq. (12), at the scale r_* we have $v \sim B$. Equating then the characteristic velocity gradient v/r_* to $1/\tau$, one obtains $r_* \sim (\epsilon\tau^3)^{1/2}$. Note that this agrees with the direct Kolmogorov theory estimate of r_* based on $\delta v(r_*) \sim (\epsilon r_*)^{1/3} \sim r_*/\tau$. Near the coil–stretch transition the viscous and elastic dissipation terms in the energy balance equation (13) are of the same order. Estimating ϵ by the viscous dissipation term v/τ^2 one finds $\mathrm{Re}_c \sim [L^2/(v\tau)]^{2/3}$ for the value of the Reynolds number at the transition. The same answer can be found by equating r_* and the Kolmogorov-41 estimate $(v^3/\epsilon)^{1/4}$ for the viscous scale η.

Near the onset of drag reduction (for Re close to Re_c) not all the polymer molecules are strongly stretched, see Ref. 2. Therefore, it is impossible to neglect the term with Π_0 in Eq. (2), and the uniaxial expression (8) for the stress tensor is not valid. Since Π_0 is proportional to the polymer concentration, properties of the polymer solution near the onset are sensitive to the concentration. Particularly, the onset itself has to be dependent of the concentration, in accordance with experiment.[3–6] And only far above the onset, where it is possible to neglect the term with Π_0 in Eq. (2), the system passes to a universal behavior corresponding to the maxi-

mum drag reduction asymptote.[2] Note also, that near the onset properties of the system could be sensitive to a possible inhomogeneity of the polymer space distribution (again, because of the term with Π_0 in the constitutive equation), though the inhomogeneity seems to be effectively smeared by the turbulent diffusion. But in the regime corresponding to the maximum drag reduction asymptote the properties of the solution are insensitive to the polymer concentration. Particularly, a possible inhomogeneity of the polymer concentration does not influence dynamics of the system.

Below we investigate the case $\mathrm{Re}\gg\mathrm{Re}_c$ that elucidates most clearly the role of the polymer elasticity. Since the condition implies that the viscous term is negligible at the scale r_*, a new interval of scales, where viscosity is negligible but elasticity is not, has to exist below r_*. The analogy with the magnetic hydrodynamics, noted above, helps us to understand dynamics of fluctuations in this interval. These small-scale fluctuations, which occur on the background of stresses excited at $r\sim r_*$, are elastic waves similar to the Alfvén waves propagating in the presence of a large-scale magnetic field in plasma.[24,27] The dispersion relation for the waves is $\omega=\boldsymbol{B}\boldsymbol{k}$, where ω is the wave frequency and $\boldsymbol{k}$ is its wave vector. Therefore, the group velocity of these waves is $\boldsymbol{B}$ which can be estimated in accordance with Eq. (25) as $\sqrt{\epsilon\tau}$. The wave velocity fluctuates, but the fluctuations occur at times $\sim\tau$ and are slower than the wave oscillations at $kr_*\gg1$, showing that the waves are well-defined. There exist two mechanisms of the elastic waves attenuation: polymer relaxation and viscous dissipation. The first mechanism leads to the scale-independent attenuation $\sim\tau^{-1}$, which is smaller than the frequency, at $kr_*\gg1$. The second mechanism produces the attenuation $\sim\nu k^2$, which is much smaller than the frequency for $k\eta_*\ll1$ where $\eta_*=\nu(\epsilon\tau)^{-1/2}$. Thus the elastic waves attenuate weakly in the interval $r_*^{-1}\lesssim k\lesssim\eta_*^{-1}$. This interval can be called the elastic waves range.

The dynamics at scales $r\ll r_*$ is also characterized by the stretching that takes place at a time scale τ and is slower than the wave's oscillations. It is this dynamics that determines the velocity spectrum at $kr_*\gg1$, since the wave oscillations do not influence the spectrum, like in the example presented at the end of Sec. III. As a result, we come to a power spectrum, which is examined in the next subsection.

A. Power-law spectrum in the elastic waves range

As we explained, statistical stationarity implies, that the velocity gradients fluctuate near $1/\tau$, the value characteristic of the scales $r\sim r_*$. Therefore, velocity gradients have to decrease with diminishing scale r at $r<r_*$, which can be formulated as $\nabla v'\ll\nabla v$. Here $\boldsymbol{v}'$ is the small-scale component of the velocity containing only harmonics with wave vectors satisfying $kr_*\gg1$. The existence of elastic waves at these scales leads to equipartition of kinetic and elastic energies (see Ref. 27 and the proof below) so that $\nabla B'\ll\nabla B$ holds too. As a result the influence of $\boldsymbol{v}'$ and $\boldsymbol{B}'$ on motions at scales $\sim r_*$ is negligible, that is $\boldsymbol{v}'$ and $\boldsymbol{B}'$ can be treated as passively advected and stretched by $\boldsymbol{v}$ and $\boldsymbol{B}$. Equations for $\boldsymbol{v}'$ and $\boldsymbol{B}'$ can be found by linearizing Eqs. (10) and (12)

$$\begin{aligned}&\partial_t\boldsymbol{B}'+(\boldsymbol{v}\nabla)\boldsymbol{B}'+(\boldsymbol{v}'\nabla)\boldsymbol{B}=(\boldsymbol{B}\nabla)\boldsymbol{v}'+(\boldsymbol{B}'\nabla)\boldsymbol{v}-\boldsymbol{B}'/\tau,\\&\partial_t\boldsymbol{v}'+(\boldsymbol{v}\nabla)\boldsymbol{v}'+(\boldsymbol{v}'\nabla)\boldsymbol{v}\\&\quad=-\nabla p'+(\boldsymbol{B}\nabla)\boldsymbol{B}'+(\boldsymbol{B}'\nabla)\boldsymbol{B}+\nu\nabla^2\boldsymbol{v}',\end{aligned}\tag{26}$$

where $p'=P'/\varrho$.

The inequalities $\nabla v'\ll\nabla v$, $\nabla B'\ll\nabla B$ imply that at $r\lesssim r_*$ the differences $\delta\boldsymbol{v}$ and $\delta\boldsymbol{B}$, taken at points, separated less, than r_*, scale linearly with the separation according to

$$\delta v_i=\sigma_{ij}\delta r_j,\quad \delta B_i=\gamma_{ij}\delta r_j,\tag{27}$$

which is a generalization of Eq. (15). Both matrices $\sigma_{ik}=\nabla_k v_i$ and $\gamma_{ik}=\nabla_k B_i$ have typical values $1/\tau$ and correlation times of the order of τ. To investigate the statistics of the small-scale components we may use the same scheme, as was developed in Sec. III for the passive scalar, expanding Eqs. (26) near a Lagrangian trajectory, like in Eq. (16). As we explained in Sec. III, the velocity should be expanded to the first order (because of the Galilean invariance). The need to expand $\boldsymbol{B}$ to the first order too follows from the fact that zeroth order term produces Alfvén waves that do not affect energy balance of the waves of the same type as explained at the end of Sec. III. Passing to Fourier components (of the functions of the argument $\boldsymbol{r}-\boldsymbol{x}$), we obtain the equations

$$\begin{aligned}&\partial_t\boldsymbol{v}_k'-\left(\boldsymbol{k}\sigma\cdot\frac{\partial}{\partial\boldsymbol{k}}\right)\boldsymbol{v}_k'+\left(\boldsymbol{k}\gamma\cdot\frac{\partial}{\partial\boldsymbol{k}}\right)\boldsymbol{B}_k'+\sigma\boldsymbol{v}_k'\\&\quad=-i\boldsymbol{k}p_k'+i(\boldsymbol{B}\cdot\boldsymbol{k})\boldsymbol{B}_k'+\gamma\boldsymbol{B}_k'-\nu k^2\boldsymbol{v}_k',\end{aligned}\tag{28}$$

$$\begin{aligned}&\partial_t\boldsymbol{B}_k'-\left(\boldsymbol{k}\sigma\cdot\frac{\partial}{\partial\boldsymbol{k}}\right)\boldsymbol{B}_k'+\left(\boldsymbol{k}\gamma\cdot\frac{\partial}{\partial\boldsymbol{k}}\right)\boldsymbol{v}_k'+\gamma\boldsymbol{v}_k'\\&\quad=i(\boldsymbol{B}\cdot\boldsymbol{k})\boldsymbol{v}_k'+\sigma\boldsymbol{B}_k'-\frac{\boldsymbol{B}_k'}{\tau},\end{aligned}\tag{29}$$

analogous to Eq. (17). The quantities σ, γ, and $\boldsymbol{B}$ in Eqs. (28) and (29) are measured in the Lagrangian frame (they are functions of time and the Lagrangian marker). Correlation functions of the fields $\boldsymbol{v}'$ and $\boldsymbol{B}'$ are defined as averages over volume (or, what is the same, over different Lagrangian trajectories), that is over a statistics of σ, γ, and $\boldsymbol{B}$. There is a new ingredient in comparison with the consideration of Sec. III, which is the stretching terms like $\sigma\boldsymbol{v}_k'$. However, these terms preserve zero scaling dimension of the time-evolution operator and, consequently, they are not expected to destroy the power character of the velocity spectrum $E(k)$.

From now on we neglect the viscous term in (28) which is justifiable for not too large wave vectors (a criterion is determined below). Using the incompressibility condition, we express the pressure $p_k'=2i[\boldsymbol{k}\sigma\boldsymbol{v}'-\boldsymbol{k}\gamma\boldsymbol{B}']/k^2$. The description is significantly simplified in terms of the Elsasser variables $\boldsymbol{g}_\pm=\boldsymbol{v}_k'\pm\boldsymbol{B}_k'$, which satisfy

$$\begin{aligned}\partial_t\boldsymbol{g}_\pm-\left(\boldsymbol{k}\sigma_\pm\cdot\frac{\partial}{\partial\boldsymbol{k}}\right)\boldsymbol{g}_\pm&=\pm i(\boldsymbol{B}\cdot\boldsymbol{k})\boldsymbol{g}_\pm-\frac{\boldsymbol{g}_\pm}{2\tau}+\frac{\boldsymbol{k}}{k}\left(\frac{\boldsymbol{k}}{k}\sigma_\pm\boldsymbol{g}_\pm\right)\\&\quad+\left(\frac{1}{2\tau}-\sigma_\mp\right)\boldsymbol{g}_\mp+\frac{\boldsymbol{k}}{k}\left(\frac{\boldsymbol{k}}{k}\sigma_\mp\boldsymbol{g}_\mp\right),\end{aligned}\tag{30}$$

where $\sigma_\pm=\sigma\mp\gamma$. Of course, Eqs. (30) are compatible with the conditions $\boldsymbol{k}\cdot\boldsymbol{g}_\pm=0$ following from the solenoidality of $\boldsymbol{v}$ and $\boldsymbol{B}$. The right-hand side of (30) contains the Alfvén term $i\boldsymbol{B}\cdot\boldsymbol{k}$, describing the wave oscillations. As we explained at the end of Sec. III, it is convenient to eliminate the oscillations, introducing the corresponding phase and amplitude $\boldsymbol{g}_\pm=\boldsymbol{a}_\pm\exp[i\varphi_\pm]$ with $\varphi_\pm$ satisfying

$$\partial_t\varphi_\pm-\left(\boldsymbol{k}\sigma_\pm\cdot\frac{\partial}{\partial\boldsymbol{k}}\right)\varphi_\pm=\pm(\boldsymbol{B}\cdot\boldsymbol{k}), \tag{31}$$

$$\varphi_\pm=\pm\int_0^t \mathrm{d}t'\boldsymbol{B}(t')W_\pm^{-1,T}(t')W_\pm^T(t)\boldsymbol{k}, \tag{32}$$

where $\partial_t W_\pm=\sigma_\pm W_\pm$, $W_\pm(0)=1$. The equations for the amplitudes $a_\pm$ are

$$\begin{aligned}\partial_t\boldsymbol{a}_\pm&-\left(\boldsymbol{k}\sigma_\pm\cdot\frac{\partial}{\partial\boldsymbol{k}}\right)\boldsymbol{a}_\pm\\&=\frac{\boldsymbol{k}(\boldsymbol{k}\sigma_\pm\boldsymbol{a}_\pm)}{k^2}-\frac{\boldsymbol{a}_\pm}{2\tau}+\left[\left(\frac{1}{2\tau}-\sigma_\mp\right)\boldsymbol{a}_\mp\right.\\&\quad\left.+\frac{\boldsymbol{k}(\boldsymbol{k}\sigma_\mp\boldsymbol{a}_\mp)}{k^2}\right]\exp(\mp i\phi),\end{aligned} \tag{33}$$

$$\phi=\int_0^t \mathrm{d}t'\boldsymbol{k}[W_+(t)W_+^{-1}(t')+W_-(t)W_-^{-1}(t')]\boldsymbol{B}(t').$$

The above equations are, again, compatible with the constraints $\boldsymbol{k}\cdot\boldsymbol{a}_\pm=0$.

We observe that a characteristic time of the variations of the amplitudes is τ while the phase in the last term varies by 2π during the characteristic time $(kB)^{-1}\ll\tau$ (the last inequality coincides with the previously derived condition for the existence of waves). Indeed, the exponent appearing in the last line of Eq. (33) at $t\gg\tau$ can be estimated as $B(t)\tau ke^{t/\tau}$, where $ke^{t/\tau}$ is the current value of the wave vector, increasing due to the stretching process. Averaging (33) over times much larger than $(kB)^{-1}$ but much smaller than τ (the procedure is nothing but the Bogolubov–Krylov averaging method), we find the following amplitude equation:

$$\partial_t\boldsymbol{a}_\pm-\left(\boldsymbol{k}\sigma_\pm\cdot\frac{\partial}{\partial\boldsymbol{k}}\right)\boldsymbol{a}_\pm=\frac{\boldsymbol{k}}{k}\left(\frac{\boldsymbol{k}}{k}\sigma_\pm\boldsymbol{a}_\pm\right)-\frac{\boldsymbol{a}_\pm}{2\tau}. \tag{34}$$

We observe that the equations for $\boldsymbol{a}_+$ and $\boldsymbol{a}_-$ decouple. This is in accordance with the qualitative considerations of Kraichnan[27] who argued that the interaction of the waves, described by the amplitudes a_+ and a_-, is weak because their propagation directions are reverse.

Equation (34) has the same structure as the equation for the linearly decaying scalar, considered in Sec. III. The difference is in its vectorial nature and in the presence of the term directed along $\boldsymbol{k}$ that comes from the solenoidality condition $\boldsymbol{k}\boldsymbol{a}_\pm=0$. A formal solution of Eq. (34) can be written as

$$\boldsymbol{a}_\pm(t,\boldsymbol{k})=e^{-t/2\tau}M_\pm[t,\boldsymbol{q}_\pm]\boldsymbol{a}_\pm[0,\boldsymbol{q}_\pm], \tag{35}$$

$$\partial_t M_\pm(t,\boldsymbol{k})=(f_\pm^{-2}\boldsymbol{f}_\pm)(\boldsymbol{f}_\pm\sigma_\pm M_\pm)^T, \tag{36}$$

where $\boldsymbol{q}_\pm=\boldsymbol{k}W_\pm(t)$, $\boldsymbol{f}_\pm=\boldsymbol{k}W_\pm^{-1}(t)$, and the initial condition for the matrices $M_\pm$ is $M_{ik}(t=0,\boldsymbol{k})=\delta_{ik}-k_ik_k/k^2$. The term k_ik_k/k^2 in the initial condition for $M_\pm$ vanishes after contraction with solenoidal field $\boldsymbol{a}(t,\boldsymbol{k})$, that leads to the correct initial condition for $\boldsymbol{a}_\pm$. Note that $\boldsymbol{f}_\pm M_\pm(t,\boldsymbol{k})=0$. Indeed, $\boldsymbol{k}M_\pm(0,\boldsymbol{k})=0$ and $\partial_t[\boldsymbol{f}_\pm M_\pm(t,\boldsymbol{k})]=0$, as follows from Eq. (36) and the equations $\partial_t\boldsymbol{f}_\pm=-\boldsymbol{f}_\pm\sigma_\pm$.

Remind that $\boldsymbol{B}$ is defined up to sign. Therefore, all the statistical properties of the solution have to be invariant under the transformation $\boldsymbol{B}\to-\boldsymbol{B}$. This transformation interchanges $\boldsymbol{g}_+$ and $\boldsymbol{g}_-$. Therefore, statistical properties of $\boldsymbol{g}_+$ and $\boldsymbol{g}_-$ are identical. Particularly, the spectra of $\boldsymbol{g}_+$ and $\boldsymbol{g}_-$ coincide, that is $\langle\boldsymbol{g}_{\pm i}(\boldsymbol{k})\boldsymbol{g}_{\pm j}(\boldsymbol{k}')\rangle=(2\pi)^3\delta(\boldsymbol{k}+\boldsymbol{k}')(\delta_{ij}-k_ik_j/k^2)E(k)$. Thus, without any loss of generality, one can consider $\boldsymbol{g}_+$ solely. At calculating the correlation function of $\boldsymbol{g}_+$, entering the definition of E, we can exploit an independence of $W_+(t)$ and $M_+(t)$ of $\boldsymbol{a}_+(0,\boldsymbol{k})$ that holds at t much larger than the correlation time τ of σ_+ (this is again in complete analogy with the consideration of Sec. III). Then the stationarity condition for $E(\boldsymbol{k})$ reads

$$\begin{aligned}&2E(\boldsymbol{k})=\exp(-t/\tau)\langle Z(t,\boldsymbol{q}_+)E(\boldsymbol{q}_+)\rangle,\\&Z(t,\boldsymbol{k})=\mathrm{tr}M_+^T(t,\boldsymbol{k})M_+(t,\boldsymbol{k})-\frac{\boldsymbol{k}M_+^T(t,\boldsymbol{k})M_+(t,\boldsymbol{k})\boldsymbol{k}}{k^2},\end{aligned} \tag{37}$$

where we used $M(t,-\boldsymbol{k})=M(t,\boldsymbol{k})$. Note that $Z=2$. Indeed, using Eq. (36) and $\boldsymbol{f}_\pm M_\pm(t,\boldsymbol{k})=0$ one easily shows that the time derivative of $M^T(t,\boldsymbol{k})M(t,\boldsymbol{k})$ vanishes so that $M^T(t,\boldsymbol{k})M(t,\boldsymbol{k})=\delta_{ij}-k_ik_j/k^2$ and $Z(t,\boldsymbol{k})=2$. Thus Eq. (37) simplifies to

$$E(\boldsymbol{k})=\exp(-t/\tau)\langle E(\boldsymbol{q}_+)\rangle, \tag{38}$$

almost identical to Eq. (21) established in Sec. III. Similarly, one can formulate an equation for the cross-correlation spectrum function $E'(\boldsymbol{k})$ defined by $\langle(g_+)_i(\boldsymbol{k})(g_-)_j(\boldsymbol{k}')\rangle=\delta(\boldsymbol{k}+\boldsymbol{k}')(\delta_{ij}-k_ik_j/k^2)E'(k)$. The fast oscillating phase does not cancel in this equation leading to the inequality $E'(k)\ll E(k)$. It leads to the conclusion that the spectra of $\boldsymbol{v}$ and $\boldsymbol{B}$ coincide and are equal to $E(k)/2$ each. This proves the equipartition of the elastic and the kinetic energies claimed above.

In analogy with the consideration of Sec. III one can establish that the solution of (38) is power-like $E(k)\propto k^{-\alpha}$ where α is determined implicitly by

$$1=\exp(-t/\tau)\left\langle\frac{1}{|\boldsymbol{k}W_+(t)/k|^\alpha}\right\rangle. \tag{39}$$

Again, $\alpha>3$ (see Appendix 3). In fact a stronger inequality follows from the stationarity condition. Namely, the spectrum has to decay faster than k^{-5} (k^{-3} in the spherical normalization). Otherwise $\langle(\nabla\boldsymbol{v})^2\rangle=\int E(k)k^2\mathrm{d}\boldsymbol{k}$ is determined by scales smaller than r_*, violating the condition that the gradients have to be saturated at the value $1/\tau$ reached at r_*. The condition $\alpha>5$ coincides with the applicability condition of the above consideration that uses $\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{r})-\boldsymbol{v}(\boldsymbol{x})\approx\sigma\boldsymbol{r}$ for $r\ll r_*$. Indeed, then $\int E(k)k^2\mathrm{d}\boldsymbol{k}$ is determined by $kr_*\lesssim1$ and $\langle(\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{r})-\boldsymbol{v}(\boldsymbol{r})-\sigma\boldsymbol{r})^2\rangle\ll\langle(\sigma\boldsymbol{r})^2\rangle$ for $r\ll r_*$.

It is natural to ask, whether α is a universal number, independent of Re. The above analysis shows that α is determined by the statistics of σ and γ. Within the framework

of the Kolmogorov theory the statistics is independent of the inertial interval length and can be characterized by a single parameter $\lambda_1 = 1/\tau$. We conclude that the dimensionless quantity α is a universal number in the Kolmogorov theory. In fact, due to intermittency the statistics of velocity gradients depends on the length of the inertial interval and, consequently, on the Reynolds number. The current understanding of intermittency does not allow us to estimate α for a given Re, yet some qualitative assertions can be formulated. Intermittency enhances the probability of large gradients which leads to faster transfer of energy to large wave vectors. Therefore the velocity spectrum becomes flatter as the intermittency increases. As we established, the length of the inertial interval decreases with the growth of the Reynolds number for polymer solutions (since the lower boundary of the inertial interval r_* increases). Consequently, the intermittency decreases as Re grows. We conclude that α should be a monotonically increasing function of Re.

Now we establish the region of scales where the power spectrum exists. Its lower boundary is related to the viscous dissipation, which grows with increasing k, destroying the power spectrum at large wave vectors. Comparing the viscous term $\upsilon k^2 \boldsymbol{v}'$, the last one in Eq. (28), with, say, the stretching term $\sigma \boldsymbol{v}'$, we find, that the viscous term wins at the scale $\sim \sqrt{\upsilon\tau}$. As a result the power-law terminates at $k \sim (\upsilon\tau)^{-1/2}$. For larger wave vectors the velocity spectrum diminishes faster than a power of k, that is the power spectrum occurs in the interval $r_*^{-1} \ll k \ll (\upsilon\tau)^{-1/2}$. Note that at $\mathrm{Re} \gg \mathrm{Re}_c$ we have $\sqrt{\upsilon\tau} \gg \eta_*$ so that the power spectrum occurs in the interval, where the elastic waves are well-defined.

V. ELASTIC TURBULENCE

We pass to the case of low Reynolds numbers. Then a random (chaotic) flow can be excited due to elastic instabilities, if the Weissenberg number Wi is larger than unity. This is the situation of the recently discovered "elastic turbulence."[7–9]

We investigate the case $\mathrm{Re} \ll 1$ where the substantial derivative in Eq. (12) can be neglected. Then systems (10) and (12) become

$$\rho^{-1}\nabla P = (\boldsymbol{B}\nabla)\boldsymbol{B} + \upsilon\nabla^2\boldsymbol{v}, \quad \nabla\cdot\boldsymbol{v} = 0, \tag{40}$$

$$\partial_t \boldsymbol{B} + (\boldsymbol{v}\nabla)\boldsymbol{B} = (\boldsymbol{B}\nabla)\boldsymbol{v} - \boldsymbol{B}/\tau, \quad \nabla\cdot\boldsymbol{B} = 0. \tag{41}$$

The inequality $\mathrm{Re} \ll 1$ implies, that the kinetic energy of the solution can be neglected in comparison with the elastic one. The dissipation of the elastic energy is, however, due to both energy dissipation mechanisms (solvent viscosity and polymer relaxation):

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}\boldsymbol{r}\frac{B^2}{2} = -\frac{1}{\tau}\int \mathrm{d}\boldsymbol{r}\frac{B^2}{2} - \upsilon\int \mathrm{d}\boldsymbol{r}(\nabla_i v_k)^2, \tag{42}$$

as follows from Eqs. (40) and (41) and is in accordance with Eq. (13).

The system of equations (40) and (41) has to be complemented by the boundary conditions for the velocity, which in the absence of polymers would lead to $\mathrm{Wi} > 1$ so that the equilibrium state of polymers is unstable. The instability eventually leads to a chaotic, statistically steady state maintained by the nonlinear dynamics of $\boldsymbol{B}$, see Eq. (41). Stationarity of the statistics, again, implies $\lambda_1 = 1/\tau$ and stationarity of the $\rho_1 - t/\tau$ statistics, as it stems from Eq. (41). It follows that the velocity gradients are of the order of $1/\tau$ in the bulk. Therefore, a boundary layer has to be formed, where the velocity gradient, exceeding $1/\tau$, at the boundary, drops to the value $1/\tau$ in the bulk. In the boundary layer the flow is mainly shear and the polymers are weakly stretched. The existence of the boundary layer is observed also experimentally (A. Groisman, private communication).

The above picture means, that instabilities lead to velocity fluctuations with scales determined by the size of the system and a characteristic gradient $1/\tau$. The following estimates for the values of these large-scale fluctuations hold:

$$B^2 \sim \upsilon\sigma \sim \frac{\upsilon}{\tau}, \quad v^2 \sim \frac{L^2}{\tau^2}, \quad \frac{v^2}{B^2} \sim \frac{L^2}{\upsilon\tau} \sim \mathrm{Re} \ll 1, \tag{43}$$

where L is the linear system size. The estimates (43) imply that both dissipative terms in Eq. (42) are of the same order. The correlation time of $\boldsymbol{B}$ and $\boldsymbol{v}$ is determined by the typical value of the stretching and is of the order of τ. The large-scale velocity fluctuations produce smaller scale fluctuations of $\boldsymbol{B}$ that in turn induce small-scale fluctuations of velocity. The velocity gradients become smaller when the scale decreases, since the large-scale velocity gradient is of the order of $1/\tau$ and the total velocity gradient is fixed at this value by the stationarity condition. This is in complete analogy with the consideration of the previous section.

Now we are going to consider statistical properties of the small-scale fields $\boldsymbol{v}'$ and $\boldsymbol{B}'$. The fields evolve passively in the large-scale fields $\boldsymbol{v}$ and $\boldsymbol{B}$. However, there is a major qualitative difference from the high-Reynolds case which is in the role of the large-scale component of $\boldsymbol{B}$. In the high-Reynolds number case (see Sec. IV) the terms with the large-scale field $\boldsymbol{B}$ in the equations for the small-scale fields conserve energy and, consequently, do not enter the equation for the spectrum (37). That is why the correct description of the dynamics required account of the small gradient $\nabla_i \boldsymbol{B}$ on the background of $\boldsymbol{B}$. In the elastic turbulence case the terms with $\boldsymbol{B}$ are dissipative and, consequently, one can forget about the gradient terms, as subleading ones. Thus the equations for the small-scale components of the fields, following from Eqs. (40) and (41), are

$$\nabla p' = (\boldsymbol{B}\nabla)\boldsymbol{B}' + \upsilon\nabla^2\boldsymbol{v}', \tag{44}$$

$$\partial_t \boldsymbol{B}' + (\boldsymbol{v}\nabla)\boldsymbol{B}' = (\boldsymbol{B}'\nabla)\boldsymbol{v} + (\boldsymbol{B}\nabla)\boldsymbol{v}' - \boldsymbol{B}'/\tau. \tag{45}$$

Let us stress that the above equations assume only $B \gg B'$ (not $\nabla B \gg \nabla B'$, which is in fact wrong for the elastic turbulence).

To analyze the above equations for the small-scale components we use the same scheme as in the previous sections. As we explained above there is no need to account for the spatial variation of $\boldsymbol{B}$ so that (for the Fourier components) Eqs. (44) and (45) take the form

$$\boldsymbol{v}' = \frac{i(\boldsymbol{k}\cdot\boldsymbol{B})}{\upsilon k^2}\boldsymbol{B}', \quad p' = 0, \tag{46}$$

$$\partial_t \boldsymbol{B}' - \left(\boldsymbol{k}\sigma \cdot \frac{\partial}{\partial \boldsymbol{k}} \right) \boldsymbol{B}' = \sigma \boldsymbol{B}' - \frac{\boldsymbol{B}'}{\tau} - \frac{(\boldsymbol{k}\cdot\boldsymbol{B})^2}{v k^2} \boldsymbol{B}'. \tag{47}$$

The constraints $\boldsymbol{k}\cdot\boldsymbol{B}'=0$, $\boldsymbol{k}\cdot\boldsymbol{v}'=0$ that follow from $\nabla\cdot\boldsymbol{v}=0=\nabla\cdot\boldsymbol{B}$ are consistent with Eqs. (46) and (47). The last term in the equation for $\boldsymbol{B}'$ is due to the viscous dissipation, the term is of the same order as the elastic dissipation term $-\boldsymbol{B}'/\tau$ as follows from (43). We observe that, again, the time-evolution operator for $\boldsymbol{B}'$ has zero scaling dimension and (in accordance with the discussion at the end of Sec. III) one expects that the spectrum $F(k)$ of $\boldsymbol{B}$ obeys a power-law. To demonstrate this power behavior we use a formal solution of Eq. (47)

$$\boldsymbol{B}'(t,\boldsymbol{k}) = W\boldsymbol{B}'(0,\boldsymbol{k}W)\exp\left[-\frac{t}{\tau} - \int_0^t \mathrm{d}t'\,\xi(t') \right],$$
$$v\,\xi(t') = [\boldsymbol{B}(t')\boldsymbol{n}(t',t,\boldsymbol{k})]^2, \quad \partial_{t'}\boldsymbol{n} = -\boldsymbol{n}\sigma + \boldsymbol{n}(\boldsymbol{n}\sigma\boldsymbol{n}), \tag{48}$$

where $W=W(t,0)$, and $\boldsymbol{n}(t',t,\boldsymbol{k})$ is determined by the above equation with the final condition $\boldsymbol{n}_k(t,t,\boldsymbol{k})=\boldsymbol{k}/k$.

Let us analyze the stress spectrum function $F(k)$: $\langle B_i'(\boldsymbol{k})B_j'(\boldsymbol{k}')\rangle = \delta(\boldsymbol{k}+\boldsymbol{k}')F(k)(\delta_{ij}-k_ik_j/k^2)$. Assuming that t is much larger than the correlation time τ of $\boldsymbol{B}$ and σ, we may average independently over the velocity statistics at negative and positive times, as we did in the previous sections. Then we find

$$2F(\boldsymbol{k}) = \left\langle Z \exp\left[-2\int_0^t \mathrm{d}t'\,\xi(t') \right] F(\boldsymbol{k}W) \right\rangle, \tag{49}$$

$$Z = \exp(-2t/\tau)\left[\mathrm{tr}W^TW - \frac{\boldsymbol{k}[WW^T]^2\boldsymbol{k}}{\boldsymbol{k}WW^T\boldsymbol{k}} \right]. \tag{50}$$

The coefficient of F in the right-hand side of Eq. (49) is independent of the absolute value k of $\boldsymbol{k}$. Therefore, Eq. (49) admits a power solution $F(k)\propto k^{-\beta}$, where β has to be determined from the equation

$$2 = \left\langle \frac{Z\exp[-2\int_0^t \mathrm{d}t'\,\xi(t')]}{|\boldsymbol{k}W(t)/k|^\beta} \right\rangle. \tag{51}$$

Note that at $t\gg\tau$ the average in Eq. (51) is determined by the events with $\boldsymbol{k}$ directed along the eigen vector of the matrix WW^T, corresponding to the smallest eigen value (in this case the denominator in the expression achieves a minimum). Then the second term in the right-hand side of Eq. (50) can be neglected, and we obtain $Z\approx\exp(2\rho_1-2t/\tau)$ (therefore, due to the stationarity of the ρ_1-t/τ probability distribution, a statistics of Z is time-independent). The positive noise $\xi(t)$ can be considered as stationary with the correlation time τ. Indeed, it follows from Eq. (48) that $\boldsymbol{n}(t')$ forgets its final direction (and thus also becomes $\boldsymbol{k}$-independent) at $t-t'\gg\tau$. Thus the situation is similar to the one analyzed in Secs. III and IV A. Again, we can prove the inequality $\beta>3$, see Appendix 3.

We are now in a position to establish the spectrum of the velocity $E(k)$. Indeed, it follows from Eq. (46) that $E(k)\propto k^{-\beta-2}$ and the spherically normalized spectra obey

$$E_{sph}(k)\sim v^2L(kL)^{-\beta}, \quad F_{sph}(k)\sim B^2L(kL)^{2-\beta}. \tag{52}$$

The estimates for the values v^2 and B^2 are written in Eq. (43). We observe that our scheme that assumes $\nabla v' \ll \nabla v$ and $B' \ll B$ is self-consistent due to the inequality $\beta>3$ proved above. It agrees with the experiment, where β is 3.3–3.5.[7,8]

Finally, let us discuss the question concerning the validity region of the power spectrum. Probably, it is determined by the finite diffusivity κ of the polymer molecules which is described by adding the term $\kappa\nabla^2\Pi_{ij}$ to the right-hand side of Eq. (2). Comparing this term with, say, the relaxation term with τ, one concludes that the power spectrum terminates at $k_{\mathrm{dif}}\sim\sqrt{\kappa\tau}$. At smaller scales the velocity spectrum diminishes much faster due to diffusivity.

VI. CONCLUSION

We have investigated properties of turbulence in dilute polymer solutions in the cases where polymer molecules are strongly stretched. We established power-law distributions of kinetic and elastic energies over scales in some regions, where these power-laws are not related to an energy or other conserved quantity cascade (in contrast to the usual turbulence). In fact, excitation of elastic degrees of freedom at any scale leads to energy dissipation since the elastic dissipation is scale-independent. However, precisely this scale-independence can lead to a scale-invariance in the dissipative intervals, where the flow can be treated as smooth. Small-scale fluctuations are relatively weak and evolve passively in the smooth flow. As a result, the evolution of fine-scale fluctuations depends trivially on the scale and power-law spectra are formed. Let us now describe the cases where the above general ideas are applicable.

The first case, we examined, is the high Reynolds number flow above the coil–stretch transition, when elastic degrees of freedom are activated. Strong interaction between the elasticity and the flow modifies the latter below the scale r_* (at this scale the velocity gradients are of the order of $1/\tau$), which is the new energy dissipation scale. This scale is of the order of the Kolmogorov scale at the transition and becomes larger as Re is increased. At $r\gtrsim r_*$ the properties of turbulence are the same as in Newtonian fluids. The energy cascade downscales from the pumping scale and dissipates due to polymer relaxation at $r\sim r_*$. The flow is smooth at $r\lesssim r_*$ with the principal Lyapunov exponent λ_1 fixed at the value $1/\tau$ by the elastic back reaction. Fluctuations in the interval of scales $v(\epsilon\tau)^{-1/2}\lesssim r\lesssim r_*$ are elastic waves. That leads to the equipartition of the kinetic and of the elastic energies, that is the velocity spectrum $E(k)$ and the elastic spectrum $F(k)$ coincide at these scales. The smoothness of the flow at $r\lesssim r_*$ leads to the conclusion that these spectra are power-like and in the spherical normalization decrease faster than k^{-3} at $kr_*\gtrsim 1$. The power spectra terminate at the scale $(v\tau)^{1/2}$, where the viscous dissipation overcomes stretching.

It is well known that the hydrodynamic turbulence is characterized by strong intermittency (see, e.g., Ref. 18). That means, strictly speaking, that the estimation $r_*\sim(\epsilon\tau^3)^{1/2}$, obtained in the framework of the Kolmogorov phenomenology, has to be corrected. Nevertheless, say, some

first structure functions are satisfactory described in terms of the phenomenology. That is why we believe that this estimation is reasonable. Next, due to the intermittency one can imagine a picture, where the crossover scale, separating the Kolmogorov cascade and the elastic waves region, strongly fluctuates in space and time. Nevertheless, the boundary scale fluctuates around r_* and the fluctuations seem to be characterized by this single scale r_*. Let us stress that the strong fluctuations do not contradict to out scheme, which implies such fluctuations. Moreover, the power spectrum, we predict, is formed as a result of such fluctuations.

Note, that in our theory the elastic and the kinetic energies are balanced at the scale r_*. Thus, we disagree with de Gennes[34] who claimed that there is an additional scale r_{**}, $r_{**} \ll r_*$, where the polymer stresses balance the Reynolds stresses. As de Gennes suggested, polymers are essentially distorted at $r_{**} < r < r_*$ but their feedback on the flow is negligible in the range, and therefore, the Kolmogorov cascade remains unaltered by polymer additives down to r_{**}. This is possible only if the equation for the elastic stress is nonlinear, since this nonlinearity can stop the polymers elongation before the feedback becomes essential even though the Lyapunov exponent is larger than the inverse polymer relaxation time. Contrary, we accept the linear equation for the polymer stress (2), which is motivated in the book, Ref. 12 (see also our paper, Ref. 2). In this case the only mechanism which can stop the polymer elongation is just the feedback. That is why in our theory the elastic stresses balance the Reynolds stresses at r_*.

The second case, we examined, is the elastic turbulence regime.[7–9] It is a chaotic state which is realized at small Reynolds numbers Re. The velocity gradient imposed on the system by the boundary conditions exceeds $1/\tau$ which activates polymer degrees of freedom leading to hydrodynamic instability and chaotization. Again, the power spectra $F(k)$ and $E(k)$ are power-like in this case. However there are no elastic waves that would lead to equipartition. The main energy is carried by the polymers: $E(k) \sim \mathrm{Re}(kL)^{-2}F(k)$, where L is the size of the system. The velocity spectrum $E(k)$ decays faster than k^{-3} in the spherical normalization, which corresponds to the experimental data.[7–9]

The above-described mechanism of forming power-law spectrum for small-scale fluctuations in a chaotic flow seems rather general to be realized for other systems. We expect it to occur in certain regimes in magnetohydrodynamics, flows in liquid crystals and low-dimensional flows on a substrate.

ACKNOWLEDGMENTS

We thank E. Balkovsky, M. Chertkov, G. Falkovich, A. Groisman, I. Kolokolov, and V. Steinberg for valuable discussions. A.F. was supported by the grants of ISF and Minerva foundations.

APPENDIX: LONG-TIME LAGRANGIAN STATISTICS

Let us briefly review the long-time statistical properties of the Lagrangian mapping matrix W, determined by Eqs. (5) and (6). We consider $W(t_+,t_-)$ at $t_+ > t_-$ and assume that $t_+ - t_-$ is much larger than the Lagrangian correlation time τ_σ of the velocity derivatives matrix (6), where one expects a universal statistics.[28]

If the velocity statistics is homogeneous in time, the probability distribution of $W(t_+,t_-)$ depends on the difference $t_+ - t_-$ only. Equation (5) implies that at $t_+ - t_- \gg \tau_\sigma$ the matrix W is a product of a large number of independent matrices. This is the main reason for the universality of the W statistics.

It is convenient to decompose the matrix W as

$$W(t_+,t_-) = M\Lambda N, \tag{A1}$$

where Λ is a diagonal matrix, and M and N are orthogonal matrices.[29] We denote the diagonal elements of Λ as e^{ρ_1}, e^{ρ_2}, and e^{ρ_3}, and assume that they are ordered: $\rho_1 > \rho_2 > \rho_3$. As a consequence of the constraint $\det W = 1$ we have $\rho_1 + \rho_2 + \rho_3 = 0$. Equation (5) can be rewritten in terms of ρ_i, and the matrices M and N. The equations for ρ_i are

$$\partial \rho_i / \partial t_+ = \tilde{\sigma}_{ii}, \tag{A2}$$

where $\tilde{\sigma} = M^T \sigma M$ and no summation over the repeating index i is implied. The matrices M and N satisfy $\partial_t N = \Omega_1 N$ and $\partial_t M = M\Omega_2$, where

$$(\Omega_1)_{ik} = \frac{\tilde{\sigma}_{ik} + \tilde{\sigma}_{ki}}{2\sinh(\rho_i - \rho_k)}, \quad (\Omega_2)_{ik} = \frac{\tilde{\sigma}_{ik} e^{2\rho_k} + \tilde{\sigma}_{ki} e^{2\rho_i}}{e^{2\rho_k} - e^{2\rho_i}},$$

for $i \neq k$ and $(\Omega_1)_{ik} = (\Omega_2)_{ik} = 0$ for $i = k$. It is possible to show that the eigenvalues of W repel each other, so that the inequalities $e^{\rho_1} \gg e^{\rho_2} \gg e^{\rho_3}$ are satisfied at $t_+ - t_- \gg \tau_\sigma$.[30] Then the matrix Ω_1 tends to zero exponentially fast, i.e., N is determined by times of the order of τ_σ in the vicinity of t_-. The matrix Ω_2 becomes ρ-independent at $t_+ - t_- \gg \tau_\sigma$ and the evolution of M is decoupled from that of ρ_i. Then the value of M is determined by the time of the order of τ_σ at $t \approx t_+$, i.e. at $t_+ - t_- \gg \tau_\sigma$ it becomes t_--independent.

The solution of Eq. (A2) is

$$\rho_i = \int_{t_-}^{t_+} \mathrm{d}t' \tilde{\sigma}_{ii}(t'), \tag{A3}$$

where the right-hand side of Eq. (A3) is an integral of a random process independent of ρ_i. Equation (A3) shows that the variables ρ_i fluctuate around their average values $\lambda_i(t_+ - t_-)$. Here the constants λ_i are equal to $\langle \tilde{\sigma}_{ii} \rangle$. They are called the Lyapunov exponents of the flow. Generally, the spectrum of the Lyapunov exponents is nondegenerate: $\lambda_1 > \lambda_2 > \lambda_3$, which is a necessary condition for the formalism to be self-consistent. The incompressibility condition ensures the identity $\lambda_1 + \lambda_2 + \lambda_3 = 0$, which implies $\lambda_1 > 0$ and $\lambda_3 < 0$. Using the relation (7) one can show that λ_1 is indeed the average logarithmic divergence rate of two nearby Lagrangian trajectories:

$$\langle \mathrm{d} \ln |\delta \boldsymbol{x}| / \mathrm{d}t \rangle = \lambda_1.$$

Similarly, $\lambda_1 + \lambda_2 = -\lambda_3$ is the average logarithmic rate of the area growth.

Note that at $t_+ - t_- \gg \tau_\sigma$ the statistics of M, Λ and N are independent. Indeed, the values of ρ_i are accumulated during the whole evolution time $t_+ - t_-$ [see (A3)] and are not sen-

sitive both to the interval $(t_-, t_- + \tau_\sigma)$ determining N and interval $(t_+ - \tau_\sigma, t_+)$ determining M. Both matrices M and N are distributed isotropically because of the assumed isotropy of the velocity statistics (this is the isotropization of W referred in the main text).

1. Lagrangian statistics in usual turbulent or random flows

Here we consider the Lagrangian statistics in the case of a usual random (turbulent or chaotic) flow, when the velocity has finite correlation time and no constraints are imposed on the flow. Then the quantity ρ_i can be treated as a sum of a large number of independent random variables, provided $t_+ - t_- \gg \tau_\sigma$. It is known from the statistical mechanics (see, e.g., Ref. 31) that the distribution of such quantities is given by the exponent of an extensive function. In our case the probability distribution function (PDF) of ρ_i is

$$\mathcal{P}(t,\rho_1,\rho_2,\rho_3) \propto \frac{1}{t}\exp\left[-tS\left(\frac{\rho_1-\lambda_1 t}{t}, \frac{\rho_3-\lambda_3 t}{t}\right)\right] \times \delta(\rho_1+\rho_2+\rho_3), \tag{A4}$$

where $t = t_+ - t_-$ and $\rho_1 > \rho_2 > \rho_3$ is implied.[30] The main exponential factor of the PDF has a self-similar form described by the function S, which can be called entropy function (see Refs. 30, 32 and 33). It is positive convex and has a minimum at zero values of its arguments. The precise form of S is determined by details of the velocity statistics. The PDF has a sharp maximum at $\rho_i = \lambda_i t$. In its vicinity the function S has a quadratic expansion, i.e., the distribution of ρ is Gaussian. However, if one is interested in the expectation values of exponential functions of ρ_i, they are determined by the wings of the PDF where the Gaussian approximation is invalid. This entails the use of the whole entropy function.

To average the functions of ρ_1 only, one can introduce the reduced probability distribution function

$$\mathcal{P}(t,\rho_1) \propto \frac{1}{\sqrt{t}}\exp\left[-tS_1\left(\frac{\rho_1-\lambda_1 t}{t}\right)\right], \tag{A5}$$

which is an integral of $\mathcal{P}(t,\rho_1,\rho_2,\rho_3)$ over ρ_2 and ρ_3. At small x the function $S_1(x)$ can be written as

$$S_1(x) \approx x^2/(2\Delta). \tag{A6}$$

Here $\Delta = \int dt \langle\langle \tilde{\sigma}_{11}(t)\tilde{\sigma}_{11}(0)\rangle\rangle$ (where double brackets designate irreducible correlation function) determines the dispersion of ρ_1: $\langle(\rho_1 - \lambda_1 t)^2\rangle \approx t\Delta$. Expansion (A6) is sufficient to describe typical fluctuations of ρ_1, whereas the whole function S is needed to describe rare events.

2. Special properties of the long-time Lagrangian statistics above the coil–stretch transition

Above the coil–stretch transition the Lagrangian statistics acquires new qualitative features caused by the polymer back reaction. They can be inferred from Eq. (9) which leads to

$$\ln|B(t,\boldsymbol{x})| - \ln|B(t_0,\boldsymbol{r})| \approx \rho_1(t,t_0) - \frac{t-t_0}{\tau}. \tag{A7}$$

The expression (A7) is correct, provided $t - t_0 \gg \tau$ (since the polymer relaxation time τ determines also the velocity gradients correlation time). The left-hand side of the equation (A7) has stationary statistics which leads to dramatic changes in the statistics of ρ_1. Upon averaging one finds $\langle\rho_1\rangle = t/\tau$ which means $\lambda_1 = \langle\tilde{\sigma}_{11}\rangle = 1/\tau$ as it was already explained in the main text. Stationarity of the dispersion of ρ_1 leads to the conclusion that $\int_{-t/2}^{t/2} dt' \langle\langle \tilde{\sigma}_{11}(0)\tilde{\sigma}_{11}(t')\rangle\rangle \propto t^{-1}$ in the limit $t \to \infty$. This is related to the anticorrelation property of $\tilde{\sigma}_{11}$, mentioned in the main text. That means that the dispersion Δ, defined by Eq. (A6), vanishes above the coil–stretch transition. In fact, this vanishing is not abrupt and occurs within $1/\ln(R_{\text{back}}/R_0)$ vicinity of Re_c. Finally one concludes that $\langle\langle \tilde{\sigma}_{11}(0)\tilde{\sigma}_{11}(t)\rangle\rangle \propto t^{-2}$ at large t which is again very different from a Newtonian fluid where exponential decay is observed, see Ref. 28.

3. Inequality for the exponents α, β

Here we establish the inequality for the exponent α, characterizing the passive scalar spectrum, see Sec. III, and the inequality for the exponent β appearing in the elastic turbulence problem, see Sec. V. In both cases we investigate the solution Δ of the equation of the type

$$\left\langle |\boldsymbol{k}W|^\Delta \exp\left[-\int_0^t dt' y(t')\right]\right\rangle \sim k^\Delta, \tag{A8}$$

where t is much larger than the correlation time of a random positive noise $y(t)$ and σ. At these times the behavior of the moments is exponential and one can define $\langle \exp[-\int_0^t dt' y(t')]|\boldsymbol{k}W|^\delta\rangle \sim k^\delta \exp[\tilde{\gamma}(\delta)t]$ where $\tilde{\gamma}(\delta)$ is a convex function due to Hölder inequality. This function is strictly smaller than another convex function $\gamma(\delta)$ defined by $\langle|\boldsymbol{k}W|^\delta\rangle \sim k^\delta \exp[\gamma(\delta)t]$.

The function $\gamma(\delta)$ has a universal behavior which we describe now. It is convenient to write

$$\langle|\boldsymbol{k}W|^\delta\rangle = \int d\boldsymbol{k}' k'^\delta \langle\delta(\boldsymbol{k} - \boldsymbol{k}' W^{-1}(t))\rangle, \tag{A9}$$

making it explicit that the wave vectors evolve according to $\boldsymbol{k}(t) = \boldsymbol{k}(0)W^{-1}(t)$.[26] Introducing $\boldsymbol{k}' W^{-1}(t) = k' \exp[\rho(t)]\boldsymbol{n}$, where $\boldsymbol{n}$ is a unit vector, one finds

$$\rho(t) = \int_0^t dt' \zeta(t'), \quad \frac{d\boldsymbol{n}}{dt} = \boldsymbol{n}\sigma + \boldsymbol{n}\zeta, \quad \zeta \equiv \boldsymbol{n}\sigma\boldsymbol{n}.$$

One observes that ζ is independent of $\rho(0)$ which leads to the conclusion that at $t \gg \tau_c(\sigma)$, where $\tau_c(\sigma)$ is the correlation time of σ, the probability distribution of ρ is described by an entropy function S.

It can be shown that $\langle\rho(t)\rangle = |\lambda_3|t$, where $|\lambda_3|$ is the lowest in the hierarchy of the Lyapunov exponents of the flow. This fact is intuitively clear as λ_3 determines contraction in the real space and thus stretching in k-space. We note that $\rho(t)$ is determined by the whole interval $(0,t)$ while $\boldsymbol{n}(t)$ only by λ_1^{-1} vicinity of t. As a result, ρ and $\boldsymbol{n}$ are independent at $\lambda_1 t \gg 1$. Since $\boldsymbol{n}$ is isotropically distributed over the unit sphere one finds $\langle\delta[\boldsymbol{k} - \boldsymbol{k}' W^{-1}(t)]\rangle = \langle\delta(\rho - \ln(k/k'))\exp[-3\rho]\rangle/(4\pi k'^3)$. Substituting this into (A9) and performing the integral one finds

$$\int \frac{d\rho}{N}\exp\left[-(\delta+3)\rho - tS\left(\frac{\rho-|\lambda_3|t}{t}\right)\right] \cong \exp[\gamma(\delta)t],$$

where $N \propto \sqrt{t}$ is the normalization factor insignificant for the following considerations. It follows that besides the zero at the origin following from the definition, the function $\gamma(\delta)$ vanishes at $\delta=-3$. This is a general consequence of the isotropy employed above, see also Ref. 17. Besides, we observe that $\gamma'(-3)=\lambda_3<0$. Combining these properties with the convexity of $\gamma(\delta)$ we conclude that $\gamma(\delta)$ is negative for $-3<\delta<0$ and positive otherwise.

To prove the inequality on Δ appearing in Eq. (A8) one notes that both $\gamma(\delta)$ and $\tilde{\gamma}(\delta)$ tend to ∞ as $|\delta|\to\infty$. Then it follows from $\tilde{\gamma}(\delta)<\gamma(\delta)$ that there are two solutions of Eq. (A8): One positive and one smaller than -3. Substituting $y(t)=2/\tau$ and recognizing that the exponent α appearing in (22) must be positive we conclude that $\alpha>3$. Analogously, substituting $y(t)=2\xi(t)$ we conclude that the solution of Eq. (51) satisfies $\beta>3$.

Finally, let us give an example of calculating α in a limiting case. The exponent is determined by the equation

$$\int \frac{d\rho}{N}\exp\left[(\alpha-3)\rho - tS\left(\frac{\rho-|\lambda_3|t}{t}\right)\right] = \exp\left[\frac{2t}{\tau}\right]. \quad \text{(A10)}$$

Note that $\alpha-3$ vanishes in the limit $|\lambda_3|\tau\to\infty$ since in this limit the linear decay term is negligible in Eq. (14) and Batchelor k^{-3} spectrum must result. Therefore at large $|\lambda_3|\tau$ the integral (A10) is determined by the maximum of the probability concentrated at $\rho=|\lambda_3|t$ which leads to

$$E(k)\sim k^{-3-2(|\lambda_3|\tau)^{-1}}, \quad |\lambda_3|\tau \gg 1. \quad \text{(A11)}$$

For a general value of $|\lambda_3|\tau$ the exponent α is determined by the concrete form of the entropy function.

[1] E. Balkovsky, A. Fouxon, and V. Lebedev, "Turbulent dynamics of polymer solutions," Phys. Rev. Lett. **84**, 4765 (2000).
[2] E. Balkovsky, A. Fouxon, and V. Lebedev, "Turbulence of polymer solutions," Phys. Rev. E **64**, 056301 (2001).
[3] P. S. Virk, "Drag reduction fundamentals," AIChE J. **21**, 625 (1975).
[4] W. McComb, *The Physics of Fluid Turbulence* (Clarendon, Oxford, 1990).
[5] A. Gyr and H.-W. Bewersdorf, *Drag Reduction in Turbulent Flows by Additives* (Kluwer, London, 1995).
[6] K. R. Sreenivasan and C. M. White, "The onset of drag reduction by dilute polymer additives, and the maximum drag reduction asymptote," J. Fluid Mech. **409**, 149 (2000).
[7] A. Groisman and V. Steinberg, "Elastic turbulence in a polymer solution flow," Nature (London) **405**, 53 (2000).
[8] A. Groisman and V. Steinberg, "Stretching of polymers in a random three-dimensional flow," Phys. Rev. Lett. **86**, 934 (2001).
[9] A. Groisman and V. Steinberg, "Efficient mixing at low Reynolds numbers using polymer additives," Nature (London) **410**, 905 (2001).
[10] T. T. Perkins, S. R. Quake, D. E. Smith, and S. Chu, "Relaxation of a single DNA molecule observed by optical spectroscopy," Science **264**, 822 (1994); S. R. Quake, H. Babcock, and S. Chu, "The dynamics of partially extended single molecules of DNA," Nature (London) **388**, 151 (1997).
[11] J. W. Hatfield and S. R. Quake, "Dynamic properties of an extended polymer in solution," Phys. Rev. Lett. **82**, 3548 (1999).
[12] R. B. Bird, C. F. Curtiss, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Liquids*, 2nd ed. (Wiley, New York, 1987), Vol. 2.
[13] M. Chertkov, "Polymer stretching by turbulence," Phys. Rev. Lett. **84**, 4761 (2000).
[14] J. L. Lumley, "Drag reduction in turbulent flow by polymer additives," J. Polym. Sci., Part D: Macromol. Rev. **7**, 263 (1973).
[15] J. L. Lumley, "On the solution of equations describing small scale deformation," Symp. Math. **9**, 315 (1972).
[16] H. Furstenberg, "Noncommuting random products," Trans. Am. Math. Soc. **108**, 377 (1963).
[17] Ya. B. Zeldovich, A. A. Ruzmaikin, S. A. Molchanov, and D. D. Sokolov, "Kinematic dynamo problem in a linear velocity field," J. Fluid Mech. **144**, 1 (1984).
[18] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, New York, 1995), Chap. 8.
[19] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics, Vols. 1 and 2* (MIT Press, Cambridge, MA, 1975).
[20] M. Tabor and P.-G. de Gennes, "A cascade theory of drag reduction," Europhys. Lett. **2**, 519 (1986); P.-G. de Gennes, "Towards a scaling theory of drag reduction," Physica A **140**, 9 (1986).
[21] A. A. Townsend, "On the fine-scale structure of turbulence," Proc. R. Soc. London, Ser. A **208**, 534 (1951).
[22] G. K. Batchelor, "Small-scale variation of convected quantities like temperature in turbulent fluid," J. Fluid Mech. **5**, 113 (1959).
[23] M. Chertkov, "On how a joint interaction of two innocent partners (smooth advection and linear damping) produces a strong intermittency," Phys. Fluids **10**, 3017 (1998); "Passive advection in nonlinear medium," *ibid.* **11**, 2257 (1999).
[24] H. K. Moffatt, *Magnetic Field Generation in Electrically Conducting Fluids* (Cambridge University Press, Cambridge, 1978).
[25] E. J. Hinch, "Mechanical models of dilute polymer solutions in strong flows," Phys. Fluids **20**, S22 (1977).
[26] R. H. Kraichnan, "Convection of a passive scalar by a quasi-uniform random straining field," J. Fluid Mech. **64**, 737 (1974).
[27] R. Kraichnan, "Inertial range spectrum of hydromagnetic turbulence," Phys. Fluids **8**, 1385 (1965).
[28] S. B. Pope, "Lagrangian pdf methods for turbulent flows," Annu. Rev. Fluid Mech. **26**, 23 (1994).
[29] I. Goldhirsch, P-L. Sulem, and S. A. Orszag, "Stability and Lyapunov stability of dynamical systems," Physica D **27**, 311 (1987).
[30] E. Balkovsky and A. Fouxon, "Universal long-time properties of Lagrangian statistics in the batchelor regime and their application to the passive scalar problem," Phys. Rev. E **60**, 4164 (1999).
[31] L. D. Landau and E. M. Lifshitz, *Statistical Physics*, Part 1 (Pergamon, New York, 1980).
[32] M. Chertkov, G. Falkovich, I. Kolokolov, and M. Vergassola, "Small-scale turbulent dynamo," Phys. Rev. Lett. **83**, 4065 (1999).
[33] R. Ellis, *Entropy, Large Deviations and Statistical Mechanics* (Springer-Verlag, Berlin, 1985).
[34] P.-G. de Gennes, *Introduction to Polymer Dynamics* (Cambridge University Press, Cambridge, 1990).