Large-scale properties of passive scalar advection

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We consider statistics of the passive scalar on distances much larger than the pumping scale. Such statistics is determined by statistics of Lagrangian contraction, that is by probabilities of initially distant fluid particles coming close. At the Batchelor limit of spatially smooth velocity, the breakdown of scale invariance is established for scalar statistics. © *1999 American Institute of Physics.* [S1070-6631(99)03008-1]

I. INTRODUCTION

If an external pumping supplies the fluctuations of the scalar at some scale L, then the advection by a spatially inhomogeneous velocity field produces scalar fluctuations at all scales, both larger and smaller than L. In an incompressible velocity field, the flux of the scalar variance flows downscales, this direct cascade is quite well understood by now.[1–5] From a general physical viewpoint, it is also of interest to understand the properties of turbulence at scales larger than the pumping scale. If only direct cascade exists, one may expect equilibrium equipartition at large scales with the effective temperature determined by small-scale turbulence.[6,7] The peculiarity of our problem is that we consider scalar fluctuations at the scales that are larger than the scale of excitation yet smaller than the correlation scale of the velocity field, which provides for mixing of the scalar. Although we find simultaneous correlation functions of different orders, it is yet unclear if such a statistics can be described by any thermodynamics-like variational principle.

Since we are interested in the behavior of the passive scalar on large scales, the diffusivity can be neglected, so that the properties of the scalar statistics are solely due to Lagrangian dynamics. In a turbulent flow, the distances between fluid particles generally grow with time. The law of such growth determines the correlation functions of the scalar at distances smaller than L. For example, the pair correlation function $\langle\theta(\mathbf{r})\theta(0)\rangle$ is proportional to the average time two fluid particles spend within the pumping correlation scale. For $r<L$, that is the time when separation grows from r to L. On the contrary, the scalar statistics at scales larger than L is related to the probabilities of initially distant particles to come close. Study of the large-scale statistics thus reveals new information on the properties of Lagrangian dynamics in a random flow. We shall show below that the statistics of Lagrangian contraction critically depends on the spatial smoothness of the velocity field. We shall argue that nonsmooth velocity provides for a scale-invariant statistics of a scalar which is even getting Gaussian at the limiting case of extremely irregular velocity. On the contrary, the statistics is rather peculiar at spatially smooth random flow (the so-called Batchelor limit): it demonstrates strong intermittency and non-Gaussianity at large scales. Another unexpected feature of the scalar statistics in this limit is a total breakdown of scale invariance: not only are the scaling exponents anomalous (i.e., they do not grow linearly with the order of correlation function) but even any given correlation function is not generally scale invariant (that is, the scaling exponents depend on the angles between the vectors connecting the points).

The paper is organized as follows. We introduce the problem and discuss the results that could be understood qualitatively in Sec. II. These results are supported by straightforward calculations within the framework of the Kraichnan model,[2] presented in Secs. III–V. We briefly describe the case of nonsmooth velocity in Sec. VI. We consider arbitrary space dimensionality d. The two-dimensional case deserves separate consideration due to an additional degeneracy.

II. QUALITATIVE DESCRIPTION

The evolution of the passive scalar $\theta(\mathbf{r},t)$ under the action of velocity $\mathbf{v}(\mathbf{r},t)$ and pumping $\phi(\mathbf{r},t)$ is described by

$$\partial_t\theta+\mathbf{v}\nabla\theta=\phi. \tag{2.1}$$

Let us introduce Lagrangian trajectories $\varrho(\mathbf{r},t)$ determined by the equation $\partial_t\varrho=\mathbf{v}(t,\varrho)$ and by the initial condition $\varrho(0,\mathbf{r})=\mathbf{r}$. Next, introducing $\tilde\theta(t,\mathbf{r})=\theta(t,\varrho)$ we rewrite (2.1) as $\partial_t\tilde\theta=\phi$, which gives the formal solution

$$\theta(0,\mathbf{r})=\int_{-\infty}^{0}dt\;\phi(t,\varrho). \tag{2.2}$$

Here we have taken into account that at $t=0$ the functions θ and $\tilde\theta$ coincide.

Both $\mathbf{v}$ and ϕ are assumed to be random functions of time and space. We will examine n-point correlation functions of the passive scalar $F_n(\mathbf{r}_1, \ldots, \mathbf{r}_n) = \langle \theta(\mathbf{r}_1) \ldots \theta(\mathbf{r}_n) \rangle$, averaged over both the statistics of the advecting velocity $\mathbf{v}$ and of the pumping ϕ. Since our main interest here is to study the scalar statistics on large distances and time scales, then without lost of generality we may consider pumping statistics to be close to white Gaussian

$$\langle \phi(t_1,\mathbf{r}_1)\phi(t_2,\mathbf{r}_2)\rangle = \delta(t_1-t_2)\chi(|\mathbf{r}_1-\mathbf{r}_2|). \tag{2.3}$$

Here χ is assumed to decay on a scale L. One can treat a deviation from Gaussianity by introducing the three-point pumping correlation function

$$\langle \phi(t_1,\mathbf{r}_1)\phi(t_2,\mathbf{r}_2)\phi(t_3,\mathbf{r}_3)\rangle = \delta(t_1-t_2)\delta(t_1-t_3)\chi_3(|\mathbf{r}_1-\mathbf{r}_2|,|\mathbf{r}_1-\mathbf{r}_3|,|\mathbf{r}_2-\mathbf{r}_3|), \tag{2.4}$$

where χ_3 is supposed to have the same characteristic length L as χ. Note, that even when χ_3 introduces a small correction to the Gaussian statistics of the source, it produces a new effect, making the odd correlation functions of the scalar nonzero. The correlation functions can be represented as

$$F_{2n} = \int_{-\infty}^{0} dt_1 \cdots \int_{-\infty}^{0} dt_n \langle \chi[R_{12}(t_1)] \cdots \chi[R_{2n-1,2n}(t_n)]\rangle + \cdots, \tag{2.5}$$

$$F_{2n+1} = \int_{-\infty}^{0} dt_1 \cdots \int_{-\infty}^{0} dt_n \langle \chi_3[R_{12}(t_1),R_{13}(t_1),R_{23}(t_1)] \times \chi[R_{45}(t_2)] \cdots \chi[R_{2n,2n+1}(t_n)]\rangle + \cdots, \tag{2.6}$$

where angular brackets mean averaging over the statistics of the velocity and one should perform summation over all sets of the pairs of the points $\mathbf{r}_i$. Using (2.2) we have written the correlation functions in terms of the Lagrangian separations

$$R_{ij}(t) = |\varrho(t,\mathbf{r}_i) - \varrho(t,\mathbf{r}_j)|. \tag{2.7}$$

Most of this paper is concerned with the case where the velocity field can be considered spatially smooth, which means we can write

$$v_\alpha(t,\mathbf{r}_1) - v_\alpha(t,\mathbf{r}_2) = \sigma_{\alpha\beta}(t)[r_{1\beta} - r_{2\beta}]. \tag{2.8}$$

Here $\sigma_{\alpha\beta}$ is the random strain matrix depending only on time. At such a velocity field, the distances $R_{ij}(t)$ grow exponentially, the stretching rate $\lambda(t) = \ln[R(t)/R(0)]$ has Gaussian statistics with nonzero mean $\bar{\lambda}$ and with the dispersion decreasing as $t^{-1/2}$ at time intervals far exceeding the correlation time of $\hat{\sigma}$.

Let us briefly recall the properties of the small-scale scalar statistics as they follow from (2.5) to (2.8). When $r \ll L$, the pair correlation function is proportional to the mean time when $R(t) < L$ so that $F_2(r) = \chi(0)\bar{\lambda}^{-1}\ln(L/r)$ with logarithmic accuracy.[1,2,8] With the same accuracy, the moments with $n \ll \ln(L/r)$ are Gaussian at small scales.[3,5]

The situation is drastically different at $r > L$. Now, nonzero correlation at two distant points appears only when two fluid particles manage to come there that were in the past within the pumping correlation length. We thus have to estimate the probability for the vector $\mathbf{R}(t)$ that was once within the pumping correlation length L to come exactly to the prescribed point $\mathbf{r}$ which is far away. Since the volume is conserved, then all the particles from the pumping volume L^d will evolve in such a way as to be stretched in a narrow strip with the length r. Assuming ergodicity [which requires that the stretching time $\bar{\lambda}^{-1}\ln(r/L)$ is much larger than the strain correlation time], we thus come to the conclusion that the probability that two points separated by r belong to a ''piece'' of scalar originated from within L is given by the volume fraction $(L/r)^d$. That gives the law of the decrease of the two-point scalar correlation: $F_2 \propto r^{-d}$.

The peculiarity of the smooth velocity field (2.8) is that it preserves straight lines under advection. That makes it easy to determine r dependence of the correlation function of arbitrary order if all the points lie on a line. In this case, the history of stretching is the same for all the distances. Looking backward in time we may say that when the largest distance between points was within L then all other distances were as well. Therefore, the n-point correlation function for collinear geometry is determined by the largest distance: $F_n \propto r^{-d}$. This is true also when different pairs of points lie on parallel lines. Note that the exponent is n independent, which corresponds to a strong intermittency and an extreme anomalous scaling. The fact that for collinear geometry $F_{2n} \gg F_2^n$ is due to strong correlation of the points along the line.

When we consider an arbitrary geometry, the opposite takes place, namely the stretching of different noncollinear vectors is generally anticorrelated because of incompressibility and volume conservation. Indeed, for a smooth velocity field there exists a number of invariants, preserved by the flow. A d-volume $\epsilon_{\alpha_1\alpha_2\cdots\alpha_d}\rho_1^{\alpha_1}\cdots\rho_d^{\alpha_d}$ is conserved for any d Lagrangian trajectories $\boldsymbol{\rho}_i(t)$. In particular, for $d=2$ there are area conservation laws $\epsilon_{\alpha\beta}\rho_1^\alpha\rho_2^\beta$ for any two vectors relating three points. Let us now consider a two-dimensional flow where the anticorrelation due to area conservation can be easily understood and the scaling for noncollinear geometry can be readily appreciated. Since the area of any triangle is conserved, the three points that form a triangle with an area s much larger than L^2 will never come within the pumping correlation length. Therefore, the triple correlation function

$$F_3(r_{12},r_{13},r_{23}) = \int_{-\infty}^{0} dt \langle \chi_3[R_{12}(t_1),R_{13}(t_1),R_{23}(t_1)]\rangle \tag{2.9}$$

is determined by the asymptotic behavior of χ_3 at $r_{ij} \gg L$, which is very small. For example, if χ_3 decays exponentially then $F_3 \propto \exp(-s/L^2)$. On the other hand, for collinear geometry $F_3 \propto r^{-2}$. We thus see that F_3 as a function of the angle ϑ between the vectors $\mathbf{r}_{12}$ and $\mathbf{r}_{13}$ has a sharp maximum at zero angle and decreases within the interval $\vartheta \simeq L^2/r^2 \ll 1$.

Similar considerations apply for the fourth-order correlation function

$$F_4=\int_{-\infty}^{0}dt_1\int_{-\infty}^{0}dt_2\langle\chi[R_{12}(t_1)]\chi[R_{34}(t_2)]\rangle+\cdots, \tag{2.10}$$

where dots stay for all possible permutations of points. Let us consider the contribution from the first term. Again, since the area $|\mathbf{R}_{12}\times\mathbf{R}_{34}|$ is conserved, the answer is crucially dependent on the relation between $|\mathbf{r}_{34}\times\mathbf{r}_{12}|$ and L^2. When $|\mathbf{r}_{34}\times\mathbf{r}_{12}|\ll L^2$ we have a collinear answer $F_4\propto r^{-2}$. Let us now consider the case of noncollinear geometry and find the probability of an event that during evolution R_{12} became of the order L, and then, at some other moment of time, R_{34} reached L (only such events will contribute to F_4). Note that, unlike the case of the third-order function, now there is a reducible part in pumping, which makes F_4 nonzero (decaying as power of r_{ij}) even when $|\mathbf{r}_{34}\times\mathbf{r}_{12}|\gg L^2$. The probability that R_{12} came to L is L^2/r_{12}^2. Due to area conservation, there is an anticorrelation between R_{12} and R_{34}: if $R_{12}\sim L$, then $R_{34}\sim r_{12}r_{34}/L$. So the probability for R_{34} to come back to L is $L^2/(r_{12}r_{34}/L)^2=L^4/r_{12}^2r_{34}^2$. Therefore, the total probability can be estimated as L^6/r^6, which is much smaller than the naive Gaussian estimation L^4/r^4, while the collinear answer L^2/r^2 is much larger than Gaussian.

That consideration can be readily generalized for an arbitrary number of noncollinear pairs. We expect that $F_{2n}\propto(L/r)^{\Delta_{2n}}$. In accordance with (2.5) the separations R_{ij} should be diminished in the evolution process up to L to produce a nonzero contribution to the integral. Suppose that R_{12} is diminished up to L. Such process (explained in the consideration of the pair correlation function) gives the probability $(L/r_{12})^2$. Next, due to the conservation law of the triangular areas, all other R_{ij} will increase by the factor r_{12}/L. Then we should diminish, say, R_{34} from $r_{34}r_{12}/L$ down to L. Such process gives the probability $L^2/(r_{12}r_{34})$. Due to the conservation law of the triangular areas other R_{ij} will be larger than their initial values by the factor r_{34}/L at the moment. Repeating the process we come to the factor $(L^2/r^2)^{2n-1}$ for the nth order correlation function. Therefore

$$\Delta_{2n}=4n-2. \tag{2.11}$$

The above analysis can be generalized for arbitrary geometry. Suppose that among the separations $\mathbf{r}_{ij}$ are parallel vectors (more precisely, with angles less than L^2/r^2). Let us divide $\mathbf{r}_i$ into sets consisting of pairs of points with parallel separations $\mathbf{r}_{ij}$. All points of such set behave as a single separation at the Lagrangian evolution. Therefore instead of n we should substitute into (2.11) the (minimal) number of sets. The estimates obtained above will be supported by rigorous calculations in Sec. IV.

Unfortunately, not much can be argued qualitatively about the scaling at $d>2$. The crucial point for our considerations in $d=2$ was the conservation of the area. It allowed us to get the correct answers even without calculations. In other terms, it is related to the fact that there is a single Lyapunov exponent at two dimensions. When $d>2$ we have only the conservation of the d-dimensional volumes and hence more freedom in the dynamics. Consider, for instance, the three-point correlation function for noncollinear geometry. Unlike $d=2$ we cannot assert that it is zero, since now the area of the triangle is not fixed and can change during the evolution. Nevertheless, the anticorrelation between different Lagrangian trajectories exists, and therefore the answer for the exponent Δ_3 should be larger than $2d$, which is the estimate one would get without the anticorrelation. In the following sections we find that $\Delta_3=d+(d-1)\sqrt{d/(d-2)}$. This is determined by the hierarchy of Lyapunov exponents giving the stretching rates at different directions — Sec. V C. Note that in the limit of large d the anticorrelation should disappear and the answer tends to $2d$. The four-point correlation function is also determined by a joint evolution of two distances and $\Delta_4=\Delta_3$.

III. ANALYTIC CALCULATIONS

We do all the calculations assuming the strain to be delta correlated in time

$$\langle\sigma_{\alpha\beta}(t_1)\sigma_{\mu\nu}(t_2)\rangle =D[(d+1)\delta_{\alpha\mu}\delta_{\beta\nu}-\delta_{\alpha\nu}\delta_{\beta\mu}-\delta_{\alpha\beta}\delta_{\mu\nu}]\delta(t_1-t_2). \tag{3.1}$$

The tensorial structure in (3.1) is due to isotropy and the incompressibility condition $\operatorname{div}\mathbf{v}=\operatorname{tr}\hat{\sigma}=0$. Zero correlation time of the strain allows one to derive closed equations for the correlation functions of the scalar[2]:

$$D\hat{\mathcal{L}}F_{2n}(\mathbf{r}_k)=-\sum_{ij}\chi(|\mathbf{r}_i-\mathbf{r}_j|)F_{2n-2}(t,\mathbf{r}_{k'}), \tag{3.2}$$

$$D\hat{\mathcal{L}}F_{2n+1}(\mathbf{r}_k)=-\sum_{ij}\chi(|\mathbf{r}_i-\mathbf{r}_j|)F_{2n-1}(t,\mathbf{r}_{k'})-\sum_{ijm}\chi_3 \times(|\mathbf{r}_i-\mathbf{r}_j|,|\mathbf{r}_i-\mathbf{r}_m|,|\mathbf{r}_j-\mathbf{r}_m|)F_{2n-2}(t,\mathbf{r}_{k''}),$$

where $\mathbf{r}_{k'}$ is the set $\mathbf{r}_k$ with $\mathbf{r}_i$ and $\mathbf{r}_j$ excluded and $\mathbf{r}_{k''}$ is the set $\mathbf{r}_k$ with $\mathbf{r}_i$, $\mathbf{r}_j$, and $\mathbf{r}_m$ excluded. The dimensionless operator $\hat{\mathcal{L}}$ is written as follows:

$$\hat{\mathcal{L}}=\sum_{ij}\left[\frac{d+1}{2}r_{ij}^2\delta^{\alpha\beta}-r_{ij}^{\alpha}r_{ij}^{\beta}\right]\nabla_i^{\alpha}\nabla_j^{\beta}, \tag{3.4}$$

Eqs. (3.2) and (3.3) are rather complicated partial differential equations. We start our analysis from the pair correlation function.

A. Pair correlation function

Due to isotropy and translational invariance, (3.2) for the pair correlation function can be written as

$$\frac{(d-1)D}{2}r^{1-d}\partial_r(r^{d+1}\partial_rF_2)=-\chi(r). \tag{3.5}$$

One can easily find a solution of Eq. (3.5), satisfying the correct boundary conditions

$$F_2(r)=\frac{2}{(d-1)D}\int_r^{\infty}\frac{dx}{x^{d+1}}\int_0^x dy\,\chi(y)y^{d+1} =\frac{2}{d(d-1)D}\left[r^{-d}\int_0^r dy\,\chi(y)y^{d-1}+\int_r^{\infty}\frac{dy}{y}\chi(y)\right]. \tag{3.6}$$

At $r \gg L$, the function $\chi(r)$ is assumed to decay fast enough (say, exponentially), and it is possible to neglect the terms related to the tail of $\chi(r)$ so that

$$F_2(r) = \frac{2\bar{\chi}}{d(d-1)Dr^d}. \tag{3.7}$$

Here

$$\bar{\chi} = \int_0^\infty dy\, \chi(y) y^{d-1}$$

is proportional to the zeroth Fourier harmonics of $\chi(r)$. Estimating $\bar{\chi} \simeq \chi(0)L^d$ we get $F_2 \sim \chi(0)(L/r)^d$. It is important that $\bar{\chi}$ exists and is nonzero, otherwise the answer is different.

B. Collinear geometry

Here we consider a $2n$th order correlation function of the passive scalar regarding that all points $\mathbf{r}_1, \dots, \mathbf{r}_{2n}$ lie on the same line. Then as follows from (2.8) during the evolution ϱ_i will remain on a line. The direction of the line can be characterized by a random unit vector $\mathbf{m}(t)$ with the statistics determined by

$$\begin{aligned} &\partial_t m_\alpha = \sigma_{\alpha\beta} m_\beta - m_\alpha \zeta, \\ &\zeta = \sigma_{\gamma\beta} m_\beta m_\gamma, \end{aligned} \tag{3.8}$$

following from (2.8). For the collinear geometry,

$$R_{ij}(\tau) = |\mathbf{r}_i - \mathbf{r}_j| \exp\left\{ \int_\tau^0 dt\, \zeta(t) \right\}. \tag{3.9}$$

The statistics of the field ζ is determined by (3.1) and (3.8), which leads to[9,10]

$$\langle \zeta \rangle = \frac{d(d-1)}{2} D, \tag{3.10}$$

$$\langle \zeta(t_1)\zeta(t_2) \rangle = D(d-1)\delta(t_1 - t_2). \tag{3.11}$$

Using expressions (3.9)–(3.11) we can obtain the closed equation for the function $F_{2n}(t,\mathbf{r}_k)$:

$$\begin{aligned} &\frac{(d-1)}{2} D \left[d \sum r_{ij} \frac{\partial}{\partial r_{ij}} + \left(\sum r_{ij} \frac{\partial}{\partial r_{ij}} \right)^2 \right] F_{2n} \\ &= -\sum_{ij} \chi(|\mathbf{r}_i - \mathbf{r}_j|) F_{2n-2}(t, \mathbf{r}_{k'}), \end{aligned} \tag{3.12}$$

where $\mathbf{r}_{k'}$ is the set $\mathbf{r}_k$ with $\mathbf{r}_i$ and $\mathbf{r}_j$ excluded.

Let us parametrize the points $\mathbf{r}_i$ like

$$\mathbf{r}_i = \mathbf{r}_1 + e^\xi \mathbf{n} l_i, \tag{3.13}$$

where $\mathbf{n}$ is a unit vector and l_i are some coefficients. Then, Eq. (3.12) can be rewritten:

$$\begin{aligned} &\frac{(d-1)}{2} D (d\partial_\xi + \partial_\xi^2) F_{2n}(e^\xi \mathbf{r}_i) \\ &= -\sum_{ij} \chi(e^\xi |\mathbf{r}_i - \mathbf{r}_j|) F_{2n-2}(e^\xi \mathbf{r}_{k'}). \end{aligned} \tag{3.14}$$

This is an ordinary differential equation which has to be solved with the following boundary conditions: $F_{2n}(e^\xi \mathbf{r}_i)$ tends to zero if $\xi \to +\infty$ and remains finite if $\xi \to -\infty$. The solution is

$$\begin{aligned} F_{2n}(\mathbf{r}_i) = &\frac{2}{d(d-1)D} \int_{-\infty}^{+\infty} d\xi \exp\left[-\frac{d}{2}(|\xi| - \xi) \right] \\ &\times \sum_{ij} \chi(e^\xi |\mathbf{r}_i - \mathbf{r}_j|) F_{2n-2}(e^\xi \mathbf{r}_{k'}). \end{aligned} \tag{3.15}$$

If the separations $e^\xi |\mathbf{r}_i - \mathbf{r}_j|$ are much larger than L, then the right-hand side of (3.14) can be neglected and we conclude that $F_{2n} \propto \exp(-d\xi)$. Thus we deal with an extremely strong intermittency when the scaling exponents are independent of n. If all separations are of the same order r, then we get from (3.14) an estimate

$$F_{2n} \sim \left(\frac{P_2}{D} \right)^n (L/r)^d. \tag{3.16}$$

Note that if the distances strongly differ then it follows from (3.15) that it is the largest distance that gives the main contribution into (3.16).

The analogous procedure can be applied to the odd correlation functions of the passive scalar θ. The only difference is that now we should also take into account the third-order correlation function of the pumping. Then we get

$$\begin{aligned} &\frac{(d-1)}{2} D \left[d \sum r_{ij} \frac{\partial}{\partial r_{ij}} + \left(\sum r_{ij} \frac{\partial}{\partial r_{ij}} \right)^2 \right] F_{2n+1} \\ &= -\sum_{ij} \chi(|\mathbf{r}_i - \mathbf{r}_j|) F_{2n-1}(t, \mathbf{r}_{k'}) \\ &\quad - \sum_{ijm} \chi_3(|\mathbf{r}_i - \mathbf{r}_j|, |\mathbf{r}_i - \mathbf{r}_m|, |\mathbf{r}_j - \mathbf{r}_m|) F_{2n-2}(t, \mathbf{r}_{k''}), \end{aligned} \tag{3.17}$$

where $\mathbf{r}_{k'}$ is the set $\mathbf{r}_k$ with $\mathbf{r}_i$ and $\mathbf{r}_j$ excluded and $\mathbf{r}_{k''}$ is the set $\mathbf{r}_k$ with $\mathbf{r}_i$, $\mathbf{r}_j$ and $\mathbf{r}_m$, excluded. Considering all the separations of the order of r we get from (3.17)

$$F_{2n+1} \sim \frac{P_3}{D} \left(\frac{P_2}{D} \right)^{n-1} \left(\frac{L}{r} \right)^d, \tag{3.18}$$

where $P_3 = \chi_3(0,0,0)$. The same r dependence of the odd correlation functions as in (3.16) is accounted for by the same structure of the differential operator in the left-hand sides of (3.12) and (3.17).

IV. DIMENSIONALITY TWO

As we mentioned above, the $2d$ case needs to be separately considered because of an additional degeneracy of equations for the correlation functions of the passive scalar. The degeneracy is associated with the area conservation law of any triangle, vertices of which move along Lagrangian trajectories.

A. Triple correlation function

As explained in Sec. II, the three-point correlation function has a sharp peak for the collinear geometry, whereas for the general position of points the answer is determined by the tails of the pumping function and is nonuniversal. Therefore, only the collinear answer is of interest, which has already been obtained in Sec. III B. Here, we just rederive the result in a systematic way, starting directly from the equation $D\hat{\mathcal{L}}F_3=-\chi_3$. Introducing the variables

$$x_1=\frac{r_{13}}{r_{12}}\cos\vartheta,\quad x_2=\frac{r_{13}}{r_{12}}\sin\vartheta,\quad s=r_{12}r_{13}\sin\vartheta,\tag{4.1}$$

the operator $\hat{\mathcal{L}}$ (3.4) can be recast to the following simple form[11,12]

$$\hat{\mathcal{L}}=2x_2^2(\partial_1^2+\partial_2^2)\,.\tag{4.2}$$

Here ϑ is the angle between $\mathbf{r}_{12}$ and $\mathbf{r}_{13}$, and s is the doubled area of the triangle, with vertices in $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$. Thus, the solution can be easily found[11,12] [see also (5.5)]

$$F_3=\int_{-\infty}^{+\infty}dx_1'\int_0^{+\infty}\frac{dx_2'}{x_2'^2}\ln\left[\frac{(x_1-x_1')^2+(x_2+x_2')^2}{(x_1-x_1')^2+(x_2-x_2')^2}\right]\times\frac{\chi_3(r_{12}',r_{23}',r_{31}')}{8\pi D}.\tag{4.3}$$

One should substitute into (4.3) the transformations inverse to (4.1),

$$r_{12}=\sqrt{\frac{s}{x_2}},\quad r_{13}=\sqrt{\frac{s(x_1^2+x_2^2)}{x_2}},\quad r_{23}=\sqrt{\frac{s([x_1-1]^2+x_2^2)}{x_2}},\tag{4.4}$$

and the analogous relations between r_{12}',r_{23}',r_{31}' and s,x_1',x_2'.

One can easily check, that if $s\gtrsim L^2$, there are no such values of x_1' and x_2', that r_{12}', r_{13}', and r_{23}' are smaller than L. Therefore, in this case the value of F_3 will be determined by the nonuniversal behavior of the function χ_3 for the values of its arguments larger than L. Consequently, properties of the correlation function F_3 are nonuniversal, in agreement with the qualitative discussion of Sec. II.

Let us consider F_3 at $s\lesssim L^2$. Since we assumed that both r_{12} and r_{13} are much larger than L, the condition $s\lesssim L^2$ gives the inequality $\vartheta<L^2/(r_{12}r_{13})\ll 1$, that is we consider geometry close to collinear. In this case, the main contribution to the integral (4.3) is made by the region of integration, where all r' are smaller than L. In particular, $r_{12}'\lesssim L$, which allows one to estimate $x_2'\gtrsim s/L^2$, which is the same as $x_2'/x_2\gtrsim r_{12}^2/L^2\gg 1$. Therefore, we can expand the resolvent in (4.3) and write

$$F_3=\frac{x_2}{2\pi D}\int\frac{dx_2'}{x_2'}\int\frac{\chi_3(r_{12}',r_{13}',r_{23}')dx_1'}{(x_1-x_1')^2+x_2'^2}.\tag{4.5}$$

In the limit $s\ll L^2$, it can be further simplified since the main contribution to the integral (4.5) is associated with the region $x_2'\ll\min(1,x_1)$:

$$F_3=\frac{1}{D}\int_0^\infty\chi_3(r_{12}\xi,r_{13}\xi,r_{23}\xi)\xi d\xi\approx\frac{P_3L^2}{2D\max(r_{12}^2,r_{13}^2,r_{23}^2)}.\tag{4.6}$$

Expression (4.6) is in accordance with estimate (3.18). Note that (4.6) has no singularity when any two points coincide as long as at least one distance is finite.

B. Four-point correlation function

In this section we derive the result for the four-point correlation function starting directly from (3.2). Again, there are two regimes for which one can find the answer. For the collinear geometry, the consideration is very similar to the one done in Sec. IV A and reproduces the result (3.15). Here we will find the answer for the noncollinear geometry. Note that its estimate is already known from Sec. II. Equation (3.2) for the four-point correlation function F_4 is

$$-D\hat{\mathcal{L}}F_4=\chi(r_{12})F_2(r_{34})+\text{permutations.}$$

The property of the operator (3.4) (characteristic of the large-scale advecting velocity) is that the solution of this equation is reducible into pieces, corresponding to each term on its right-hand side:

$$F_4=\tilde{F}_4(\mathbf{r}_{12},\mathbf{r}_{34})+\tilde{F}_4(\mathbf{r}_{34},\mathbf{r}_{12})+\tilde{F}_4(\mathbf{r}_{13},\mathbf{r}_{24})+\tilde{F}_4(\mathbf{r}_{24},\mathbf{r}_{13})+\tilde{F}_4(\mathbf{r}_{14},\mathbf{r}_{23})+\tilde{F}_4(\mathbf{r}_{23},\mathbf{r}_{14}).\tag{4.7}$$

To find $\tilde{F}_4$ we should solve the equation

$$-D\hat{\mathcal{L}}\tilde{F}_4(\mathbf{r}_{12},\mathbf{r}_{34})=\chi(r_{12})F_2(r_{34}).\tag{4.8}$$

In terms of the variables (ϑ is the angle between $\mathbf{r}_{12}$ and $\mathbf{r}_{34}$)

$$x_1=\frac{r_{34}}{r_{12}}\cos\vartheta,\quad x_2=\frac{r_{34}}{r_{12}}\sin\vartheta,\quad s=r_{12}r_{34}\sin\vartheta,\tag{4.9}$$

the operator $\hat{\mathcal{L}}$ for $d=2$ is[11,12]

$$\hat{\mathcal{L}}=2x_2^2(\partial_1^2+\partial_2^2).\tag{4.10}$$

The solution of (4.8) can be written as a double integral,

$$\tilde{F}_4=\frac{1}{8\pi D}\int_{-\infty}^{+\infty}dx_1'\int_0^{+\infty}\frac{dx_2'}{x_2'^2}\chi(r_{12}')F_2(r_{34}')\times\ln\left[\frac{(x_1-x_1')^2+(x_2+x_2')^2}{(x_1-x_1')^2+(x_2-x_2')^2}\right],\quad r_{12}'=\sqrt{\frac{s}{x_2'}},\quad r_{34}'=\sqrt{\frac{s(x_1'^2+x_2'^2)}{x_2'}}.\tag{4.11}$$

We shall calculate the integral in the limit $s\gg L^2$ when there are several simplifications. First, since $r_{12}'\lesssim L$, we can write $x_2'\gtrsim s/L^2$. Hence, $r_{34}'>\sqrt{sx_2'}\gtrsim s/L\gg L$, and we can use the asymptotic form (3.7) of F_2. Second, like for the three-point correlation function one can show that $x_2'\gg x_2$, and one can expand the logarithm. Finally, we can write

$$\tilde{F}_4=\frac{\chi_2 x_2}{4\pi^2 D^2 s}\int_0^{+\infty} dx_2'\chi\left(\sqrt{\frac{s}{x_2'}}\right)\times\int_{-\infty}^{+\infty}\frac{dx_1'}{[x_1'^2+x_2'^2][(x_1'-x_1)^2+x_2'^2]}. \tag{4.12}$$

The integral over x_1' can be easily calculated and we get

$$\tilde{F}_4=\frac{\chi_2 x_2}{2\pi D^2 s}\int_0^{+\infty}\frac{dx_2'}{x_2'}\frac{1}{x_1^2+4x_2'^2}\chi\left(\sqrt{\frac{s}{x_2'}}\right). \tag{4.13}$$

If x_1 is not anomalously large, we can disregard it in the integrand, and find

$$\tilde{F}_4=\frac{\chi_2 C x_2}{4\pi D^2 s^3},\quad C=\int_0^{+\infty}\chi(\xi)\,\xi^3 d\xi. \tag{4.14}$$

V. DIMENSIONALITIES LARGER THAN TWO

Here we treat correlation functions of the passive scalar for $d>2$. In this case, the degeneracy inherent to $d=2$ is absent, and the consideration is the same for all d, which is thus considered as a parameter. Of course, direct physical meaning can be attributed only to $d=3$.

We will calculate the three- and four-point correlation functions. Exactly as it was for $d=2$, the operator $\hat{\mathcal{L}}$ has the same form for both correlation functions. Namely, we have to solve the following equations:

$$-\hat{\mathcal{L}}F_3=\chi_3,\quad -\hat{\mathcal{L}}\tilde{F}_4=\chi(r_{12})F_2(r_{34}), \tag{5.1}$$

where $\tilde{F}_4$ is defined by (4.7). Then, we can introduce the variables (4.1) for F_3 and (4.9) for $\tilde{F}_4$. In these variables the operator $\hat{\mathcal{L}}$ has the following rather simple form:

$$\hat{\mathcal{L}}=dx_2^2(\partial_1^2+\partial_2^2)+(d-2)(\partial_\tau^2+d\,\partial_\tau),\qquad \tau=\ln(s/L^2).$$

Therefore, in order to solve Eq. (5.1) we have to find the resolvent $\mathcal{R}$ of the operator $\hat{\mathcal{L}}$ which satisfies the equation

$$-\hat{\mathcal{L}}\mathcal{R}=\delta(x_1-x_1')\,\delta(x_2-x_2')\,\delta(\tau-\tau')$$

and the following boundary conditions: First, $\mathcal{R}$ should go to zero when $x_1\to\pm\infty$, $x_2\to+\infty$, $x_2\to 0$, and $\tau\to+\infty$. Then $\mathcal{R}$ should tend to a constant at $\tau\to-\infty$. It is more convenient to work with Hermitian operators, therefore it is useful to make a substitution,

$$\mathcal{R}=\frac{x_2}{dx_2'}\exp\left[-\frac{d}{2}(\tau-\tau')\right]R(x_1,x_1',x_2,x_2',\tau,\tau')$$

Then we obtain

$$x_2^2\frac{\partial^2 R}{\partial x_1^2}+x_2\frac{\partial^2(x_2 R)}{\partial x_2^2}+\frac{d-2}{d}\left(\frac{\partial^2 R}{\partial\tau^2}-\frac{d^2}{4}R\right)=-\delta(x_1-x_1')\,\delta(x_2-x_2')\,\delta(\tau-\tau').$$

It is natural to seek the solution in the following form:

$$R=\frac{1}{\sqrt{x_2}}\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\int_{-\infty}^{+\infty}\frac{d\alpha}{2\pi}u(k,\alpha,x_2,x_2')e^{ik(x_1-x_1')+i\alpha(\tau-\tau')}. \tag{5.2}$$

The function u satisfies

$$x_2^2\frac{\partial^2 u}{\partial x_2^2}+x_2\frac{\partial u}{\partial x_2}-(k^2x_2^2+\nu^2)u=-\sqrt{x_2'}\,\delta(x_2-x_2'),$$

where

$$\nu=\sqrt{\frac{(d-1)^2}{4}+\frac{d-2}{d}\alpha^2}. \tag{5.3}$$

The solution of Eq. (5.3) can be readily expressed in terms of the Bessel functions of imaginary argument (see, e.g., Ref. 13),

$$u=\frac{1}{\sqrt{x_2'}}\{\theta(x_2-x_2')K_\nu(|k|x_2)I_\nu(|k|x_2')+\theta(x_2'-x_2)K_\nu(|k|x_2')I_\nu(|k|x_2)\}.$$

Here $\theta(x)$ is the step function, equal to one if $x>0$ and zero otherwise.

Now we should substitute u back into (5.2). The integral over k can be calculated analytically with the help of Eqs. (6.672) and (8.820) from Ref. 13. Then we get

$$R=\frac{1}{2\sqrt{\pi}x_2x_2'}\int_{-\infty}^{+\infty}\frac{d\alpha}{2\pi}\frac{\Gamma(\nu+1/2)}{\Gamma(\nu+1)}\times\left[\frac{x_2x_2'}{x_2^2+x_2'^2+(x_1-x_1')^2}\right]^{\nu+1/2}\times F\left(\frac{\nu}{2}+\frac{3}{4},\frac{\nu}{2}+\frac{1}{4};\nu+1;\left[\frac{2x_2x_2'}{x_2^2+x_2'^2+(x_1-x_1')^2}\right]^2\right)\times\exp[i\alpha(\tau-\tau')]. \tag{5.4}$$

Here $F(\alpha,\beta;\gamma;x)$ is the hypergeometric function and $\Gamma(x)$ is the Euler gamma function.

In $d=2$, the integral over α is trivial, and the resolvent can be easily reproduced,[11,12]

$$\mathcal{R}=\frac{1}{8\pi x_2'^2}\ln\left[\frac{(x_1-x_1')^2+(x_2-x_2')^2}{(x_1-x_1')^2+(x_2+x_2')^2}\right]\delta(\tau-\tau'). \tag{5.5}$$

Convolution of the resolvent with the right-hand sides of Eq. (5.1) depends on the properties of the sources. Therefore, below we separately consider both correlation functions.

A. Three-point correlation function

The solution of (5.1) for the three-point correlation function can be written in the following form:

$$F_3=\int_0^\infty dx_2'\int_{-\infty}^{+\infty}dx_1'\int_{-\infty}^{+\infty}d\tau'\,\mathcal{R}(x_1,x_1',x_2,x_2',\tau,\tau')\times\frac{\chi_3(r_{12}',r_{13}',r_{23}')}{D}. \tag{5.6}$$

The relations between the variables $r'_{12}, r'_{13}, r'_{23}$ and τ', η', ϑ' are as in (4.4). Recall that $\tau = \ln s/L^2$.

Like it was in $d=2$, the behavior of F_3 is very different for the cases of $\vartheta \ll 1$ and $\vartheta \sim 1$. Let us first consider the case of not very small angles, namely $\vartheta \gg L^2/(r_{12}r_{13})$. Since both r_{12} and r_{13} are much larger than L, the area s of the triangle is much larger than L^2, which means that $\tau \gg 1$. On the other hand, since χ_3 decreases very rapidly when any of r_{ij} is larger than L, the area s' cannot be much larger than L^2. Therefore, τ' is of the order unity in the integral (5.6). Thus, we see that $\tau - \tau'$ is always positive and much larger than unity. On the other hand, from the condition $\vartheta \gg L^2/r^2$, it is easy to check that for a typical configuration contributing in (5.4) the condition $\ln A \ll \tau$ holds. Therefore, shifting the contour of integration in (5.4) into the upper half-plane, we will meet a branch point of the integrand, which originates from ν and is situated at $\alpha = i(d-1)\sqrt{d/(d-2)}$. Because of the large value of τ, the integral will be determined by a near vicinity of the branch point. Therefore, we can write

$$\mathcal{R} \propto \exp\left[-\frac{\Delta_3}{2}\tau\right], \tag{5.7}$$

$$\Delta_3 = d + (d-1)\sqrt{d/(d-2)}. \tag{5.8}$$

Substituting the expression into (5.6) we get

$$F_3 \sim (P_3/D)(L/r)^{\Delta_3}, \tag{5.9}$$

Note that $\Delta_3 > 2d$.

Let us now consider the limit $\vartheta \ll L^2/r^2$ and reproduce the collinear result. In this case, as we shall see, the main contribution to (5.6) is made by $\vartheta' \ll 1$. The resolvent $\mathcal{R}$ in the limit $\vartheta, \vartheta' \ll 1$ can be found directly from the representation (5.2). The smallness of the angles implies that $x_2, x'_2 \ll 1$. Using the asymptotic expansion of the Bessel functions one gets

$$u = \frac{1}{2\nu\sqrt{x'_2}}\left\{\theta(x_2 - x'_2)\left(\frac{x'_2}{x_2}\right)^{\nu} + \theta(x'_2 - x_2)\left(\frac{x_2}{x'_2}\right)^{\nu}\right\}. \tag{5.10}$$

We see that in the main approximation the k dependence disappears from u and we can integrate over k in (5.2). Then we get

$$R = \frac{\delta(x_1 - x'_1)}{2\sqrt{x_2 x'_2}}\int_{-\infty}^{\infty}\frac{d\alpha}{2\pi\nu(\alpha)} \times \exp\left[i\alpha(\tau - \tau') - \nu(\alpha)\left|\ln\frac{x_2}{x'_2}\right|\right]. \tag{5.11}$$

The integral (5.11) can be calculated analytically,

$$R = \frac{1}{\pi}\sqrt{\frac{x_2}{x_2'^3 d(d-2)}}K_0(\mathcal{X}) \times \exp\left[-\frac{d}{2}(\tau - \tau')\right]\delta(x_1 - x'_1), \tag{5.12}$$

$$\mathcal{X} = \frac{d-1}{2}\sqrt{\frac{d}{d-2}(\tau - \tau')^2 + \ln^2\frac{x_2}{x'_2}}. \tag{5.13}$$

The δ function forces the ratio r'_{12}/r'_{13} to be equal to r_{12}/r_{13}. Integration over one of the distances, say r'_{12}, makes both of r' to be of the order L (we believe that $r_{12} \sim r_{13}$). Let us consider the integral over the angle ϑ'. It is easy to see that the argument of K_0 in (5.12) is always large. Therefore we can use the asymptotic form of this function and write

$$F_3 \sim \int d\vartheta' \frac{\vartheta^{1/2}}{\vartheta'^{3/2}}\exp\left[-\frac{d}{2}\ln\left(\frac{r^2}{L^2}\frac{\vartheta}{\vartheta'}\right) - \frac{d-1}{2}\sqrt{\ln^2\frac{\vartheta}{\vartheta'} + \frac{d}{d-2}\ln^2\left(\frac{r^2}{L^2}\frac{\vartheta}{\vartheta'}\right)}\right].$$

The main contribution to the integral is made by the vicinity of $\vartheta' = \vartheta r^2/L^2$. Hence $F_3 \sim (L/r)^d$. From the assumption $\vartheta' \ll 1$, we see that there should be $\vartheta \ll L^2/r^2$, otherwise the main contribution comes from $\vartheta' \sim 1$, and the expression for the resolvent (5.12) is inapplicable.

B. Four-point correlation function

From (5.1) it follows that the answer for the four-point correlation function can be written in the form (4.7) where

$$\tilde{F}_4(\mathbf{r}_{12}, \mathbf{r}_{34}) = \frac{1}{D}\int_0^{\infty} dx'_2 \int_{-\infty}^{+\infty} dx'_1 \int_{-\infty}^{+\infty} d\tau' \mathcal{R}(x_1, x'_1, x_2, x'_2, \tau, \tau') \times \chi(r'_{12}) F_2(r'_{34}), \tag{5.14}$$

The variables r'_{12} and r'_{34} are expressed via x'_1, x'_2, and τ' by (4.11). The analogous relations hold for the variables r_{12} and r_{34}. It can be more convenient for the present purposes to pass from the integration over x'_1, x'_2, and τ' to the integration over ϑ', r'_{12}, and r'_{34}. Then (5.14) can be replaced by

$$\tilde{F}_4 = \frac{1}{2d\pi^{3/2}}\int_{-\infty}^{\infty}\frac{dr'_{12}}{r'_{12}}\chi(r'_{12})\int_{-\infty}^{\infty}\frac{dr'_{34}}{r'_{34}}F_2(r'_{34}) \times \int_0^{\pi}\frac{d\vartheta'}{\sin^2\vartheta'}\int_{-\infty}^{\infty}d\alpha\frac{\Gamma(\nu+1/2)}{\Gamma(\nu+1)}\left(\frac{A}{2}\right)^{\nu+1/2} \times F\left(\frac{\nu}{2}+\frac{3}{4}, \frac{\nu}{2}+\frac{1}{4}; \nu+1; A^2\right)\left(\frac{r_{12}r_{34}\sin\vartheta}{r'_{12}r'_{34}\sin\vartheta'}\right)^{i\alpha - d/2}.$$

The expression for A can be written as follows:

$$A = \frac{2\sin\vartheta\sin\vartheta'}{r_{12}r'_{34}/(r'_{12}r_{34}) + r'_{12}r_{34}/(r_{12}r'_{34}) - 2\cos\vartheta\cos\vartheta'}. \tag{5.15}$$

The case $\vartheta \ll L^2/r^2$ can be analyzed in a way presented in Sec. V A for the three-point correlation function, leading to the expression (3.15) and to the law (3.16). Below we analyze the case $\vartheta \gg L^2/r^2$. In this case, there are two different regions in the integral over ξ'_2 in the integral making contributions to $\tilde{F}_4$. The first region is determined by the inequality $\xi_2' \lesssim 1$, which corresponds to $r_{34}' \simeq L$. By the order of magnitude, this contribution is equal to the three-point correlation function, because in this case the value of τ' in (5.14) is of the order of unity, while $\tau \gg 1$, and therefore all the arguments presented in Sec. V A are valid. The second region is defined by the condition $\xi_2' \gtrsim 1$. It corresponds to large values of r_{34}', for which we can use the asymptotic behavior of F_2, given by (3.7). Another simplification is that in this region $r_{34}' \gg r_{12}'$ and therefore we can believe

$$A \approx 2 \sin\vartheta \sin\vartheta' \frac{r_{34} r_{12}'}{r_{12} r_{34}'} \ll 1. \tag{5.16}$$

Then, we can put the hypergeometric function to be equal to 1, and write

$$\begin{aligned}\tilde{F}_4 \approx & \frac{2\chi_d}{\pi^{3/2} d^2 (d-1) D^2} \int d\alpha \frac{\Gamma(\nu+1/2)}{\Gamma(\nu+1)} \left(\frac{r_{34}}{r_{12}} \sin\vartheta \right)^{\nu+1/2} \\ & \times (r_{12} r_{34} \sin\vartheta)^{i\alpha - d/2} \int \frac{du}{u} \chi(u)\, u^{\nu+1/2-i\alpha+d/2} \\ & \times \int \frac{dv}{v} v^{-d/2-\nu-1/2-i\alpha} \int d\vartheta' (\sin\vartheta')^{\nu-3/2-i\alpha+d/2}.\end{aligned}$$

The integral over angle ϑ' can be easily expressed via the Euler Γ functions. The integral over v should be calculated with the cutoff on L. Therefore, we get the following integral over α:

$$\begin{aligned}\tilde{F}_4 \propto & \int d\alpha \frac{\Gamma(\nu+1/2)}{\Gamma(\nu+1)} \frac{\Gamma(\nu/2 - 1/4 - i\alpha/2 + d/4)}{\Gamma(\nu/2 + 1/4 - i\alpha/2 + d/4)} \\ & \times \left(\frac{r_{12} r_{34}}{L^2} \sin\vartheta \right)^{i\alpha - d/2} \frac{(r_{34} \sin\vartheta / r_{12})^{\nu+1/2}}{d/2 + \nu + 1/2 + i\alpha} \\ & \times \int \frac{du}{u} \chi(u) \left(\frac{u}{L^2} \right)^{\nu+1/2-i\alpha+d/2}.\end{aligned} \tag{5.17}$$

Again, we can shift the contour of integration up to the branch point determined by the same condition $\nu = 0$, and we get the same answer as for the three-point correlation function. Thus, both contributions possess an identical r dependence giving

$$F_4 \sim \frac{P_2^2}{D^2} \left(\frac{L}{r} \right)^{\Delta_3}, \tag{5.18}$$

with the same exponent (5.8).

The contribution (5.18) to F_4 is the leading one only if $d > \sqrt{2}+1$. If $d < \sqrt{2}+1$, then along with (5.18) there appears an additional contribution into F_4 due to a pole of the integrand in (5.17). It originates from the zero of the denominator $d/2 + \nu + 1/2 + i\alpha$, existing only at $d < \sqrt{2}+1$. The contribution behaves like

$$\propto \left(\frac{L}{r} \right)^{d(d+1)/(d-1)}. \tag{5.19}$$

Comparing the law (5.18) with (5.19), we conclude that in the region of its existence, that is at $d < 1 + \sqrt{2}$, the term (5.19) is the leading contribution to $\tilde{F}_4(\mathbf{r}_{12}, \mathbf{r}_{34})$. Particularly, this is the case in $d = 2$, where the contribution (5.19) behaves as $(L/r)^6$, in accordance with (4.14).

C. Instanton

To understand better the underlying Lagrangian dynamics let us outline briefly another method of calculation, based on the fact that they are rare events that contribute to the correlation functions at large scales. Therefore, some kind of instanton formalism can be applied; the main task here is to recognize the relevant degrees of freedom. In this way, we shall establish a relation between the scaling exponent Δ_3 and the Lyapunov exponents of the smooth flow. The Lagrangian distances $\mathbf{R}_{ij}$ are all determined by a single matrix $W = T \exp(\int \sigma\, dt)$ via $\mathbf{R}_{ij} = W \mathbf{r}_{ij}$. To find the correlation functions, we should be able to average over the statistics of the matrix W. The way to do it was proposed in Ref. 14, we shall follow Ref. 15 and write

$$W = RT, \tag{5.20}$$

where R is an orthogonal matrix and T is an upper-triangular matrix: $T_{ij} = 0$ if $i > j$. The matrix R can be excluded from the consideration due to isotropy. Then, representing T as

$$T_{ii} = \exp(\rho_i), \quad T_{ij} = \exp(\rho_i)\, \eta_{ij} \quad \text{if } i < j, \tag{5.21}$$

we can write the action describing the stochastic dynamics of ρ and η,

$$\begin{aligned}\mathcal{L} = & \sum_{i=1}^{d} m_i \left[\partial_t \rho_i + D \frac{d(d-2i+1)}{2} \right] + \frac{\imath D}{2} \left[d \sum_i m_i^2 \right. \\ & \left. - \left(\sum_a m_a \right)^2 \right] + \imath D d \sum_{i<j} \exp(2\rho_i - 2\rho_j) \mu_{ij}^2 \\ & + 2\imath D d \sum_{i<k<j} \mu_{ij} \mu_{ik} \exp(2\rho_k - 2\rho_i)\, \eta_{kj} \\ & + \sum_{i<j} \mu_{ij} \partial_t \eta_{ij} + \imath D d \sum_{i<k<m,n} \mu_{im} \mu_{in} \eta_{km} \eta_{kn} \\ & \times \exp(2\rho_k - 2\rho_i).\end{aligned} \tag{5.22}$$

Here m_i and μ_{ij} are auxiliary fields, conjugated to ρ_i and η_{ij}, respectively.

The variables ρ_i describe stretching of volume elements in the flow, while η_{ij} describe the direction fluctuations of a given vector with respect to the main axis of the strain matrix $\hat{\sigma}$. Note that the constants $\lambda_i = d(d-2i+1)/2$ entering the action (5.22) are the Lyapunov exponents. Using (5.22), we can rewrite F_3 (2.9) as follows:

$$F_3 = \int \mathcal{D}\rho \mathcal{D}\eta \mathcal{D}\mu \mathcal{D}m \exp\left[i \int_{-\infty}^{0} dt\, \mathcal{L} + \ln\left(\int_{-\infty}^{0} dt\, \chi_3 \right) \right]. \tag{5.23}$$

The variables η_{ij} are irrelevant for the evaluation of the scaling and are only responsible for the angular behavior. Since all three R_{ij} always lie on a plane, we can leave only two of ρ_i and write an effective Lagrangian

$$\tilde{\mathcal{L}}=m_a(\partial_t\rho_a+\lambda_a)+m_b(\partial_t\rho_b+\lambda_b)+\frac{\imath D}{2}[d(m_a^2+m_b^2)-(m_a+m_b)^2].$$

Then, the dependence on L/r can be extracted from the following expression:

$$F_3\sim\int \mathcal{D}\rho_{a,b}\mathcal{D}m_{a,b}\exp\left[\imath\int_{-\infty}^0 dt\,\tilde{\mathcal{L}}+\ln\left(\int_{-\infty}^0 dt\,\chi_3\right)\right]. \tag{5.24}$$

Now we can write the instanton equations for the extremum of the exponent in (5.24),

$$\begin{aligned}&\partial_t\rho_a+\lambda_a=-\imath D[(d-1)m_a-m_b],\\&\partial_t\rho_b+\lambda_b=-\imath D[(d-1)m_b-m_a],\\&\imath\partial_t m_a=\frac{1}{F}\frac{\partial\chi_3}{\partial\rho_a},\quad \imath\partial_t m_b=\frac{1}{F}\frac{\partial\chi_3}{\partial\rho_b},\quad F=\int_{-\infty}^0 dt\,\chi_3.\end{aligned} \tag{5.25}$$

The boundary conditions are $m_{a,b}\to 0$ as $t\to-\infty$ and $\rho_{a,b}=0$ at $t=0$. Note that the "energy"

$$E=\imath(m_a\lambda_a+m_b\lambda_b)-\frac{D}{2}[d(m_a^2+m_b^2)-(m_a+m_b)^2]+\frac{1}{F_3}\chi_3 \tag{5.26}$$

is a constant. From the boundary conditions we deduce that it should be zero.

Let us explain the qualitative behavior of the solution. We consider the evolution backwards in time. At small times all R_{ij} are large, and therefore the derivatives of χ_3 on the right-hand side of Eq. (5.25) are zero. Hence $m_{a,b}$ are constants such that $\rho_{a,b}$ diminish and R_{ij} also become smaller. Then, at some moment all R_{ij} become of the order of L. Then, derivatives of χ_3 cannot be disregarded and during some short interval of time when $R_{ij}\sim L$ the values of $m_{a,b}$ will change to vacuum zero values. The derivatives of $\rho_{a,b}$ change sign, and R_{ij} start to grow. Note, that if only one of the R_{ij} is of the order L and the others are still much larger than L, then χ_3 is small, the derivatives in Eqs. (5.25) are ineffective, and the solution will never reach its vacuum stage. Thus, we should tune the conditions so that all R_{ij} will become of the order L simultaneously. Finally, we come to the set of conditions for the initial stage

$$\begin{aligned}&\partial_t\rho_a=\lambda=-\lambda_a-\imath D[(d-1)m_a-m_b],\\&\partial_t\rho_b=\lambda=-\lambda_b-\imath D[(d-1)m_b-m_a],\\&E=\imath(m_a\lambda_a+m_b\lambda_b)-\frac{D}{2}[d(m_a^2+m_b^2)-(m_a+m_b)^2]=0.\end{aligned}$$

From here we find the value of λ,

$$\lambda=-\frac{1}{2}\sqrt{(\lambda_a+\lambda_b)^2+\frac{d-2}{d}(\lambda_a-\lambda_b)^2}. \tag{5.27}$$

Calculating the action, we find that $F_3\sim(L/r)^{\Delta_3^{a,b}}$ with

$$\Delta_3^{a,b}=\frac{\lambda_a+\lambda_b+\sqrt{(\lambda_a+\lambda_b)^2+\frac{d-1}{d}(\lambda_a-\lambda_b)^2}}{d-2}. \tag{5.28}$$

The value of $\Delta_3^{a,b}$ is minimal (that is F_3 is maximal) if we take the two largest Lyapunov exponents: $\lambda_a=\lambda_1=d(d-1)/2$ and $\lambda_b=\lambda_2=d(d-3)/2$. Substituting it into (5.28) we reproduce (5.8). The instanton found has a long lifetime, proportional to $\ln(r/L)$, therefore the above consideration has to be valid also for a velocity finite correlated in time.

D. Discussion

We thus see that many-point correlation functions are not scale invariant because of strong angular dependence. One may consider averaging over different geometries, for instance, integrating over the angle between any two vectors $\mathbf{r}_{ij}$ keeping $R^2=\Sigma r_{ij}^2$ in $2d$. As a result of such averaging, the object appears which depends on R only and is thus scale invariant. Does such averaging also restore the normal scaling? One may notice that the main contribution into the angular integral gives the region of small angles near collinear peak $\vartheta\lesssim(L/r)^2$; since there are $n-1$ angles in the n-point correlation function then one gets $F_n\propto r^{-2n}$, that is normal scaling is restored indeed. That means that the increase at small angles and decrease at large ones (relative to a normal scaling) are of the same order and both caused by the same mechanism. It is unclear what is the way — if any — of natural average over geometries that restores normal scaling in $d>2$.

We thus discovered an intermittency build-up in the direction opposite to the cascade. It is instructive to compare this with an intermittency discovered at small scales when the cascade flows upscale in a compressible flow.[16]

What will be for a finite correlation of $\hat{\sigma}$ in time? It is clear from Secs. II and III that the dependence r^{-d} both for the pair correlation function and for the correlation function of any order for collinear geometry is valid as long as the correlation time of $\hat{\sigma}$ is much less than $D^{-1}\ln(r/L)$. Under the same assumption, all the results obtained in two dimensions will be valid (up to a single numerical factor in front of any correlation function) for a finite-correlated strain as well. As far as higher dimensions are concerned, it is clear that some anticorrelation between contraction of different distances will be present, and it is likely that it will be governed by the same exponent Δ_3. Indeed, (5.28) has to hold in a finite-correlated case as well, Δ_3 is thus determined by the two largest Lyapunov exponents which are likely to be proportional to $(d-1)$ and $(d-3)$, respectively.

VI. NONSMOOTH VELOCITY FIELD

The advection of the passive scalar by the nonsmooth velocity in the framework of the Kraichnan model is described by Gaussian velocity field with the pair correlation function

$$\langle u^{\alpha}(t,\mathbf{r})u^{\beta}(0,0)\rangle$$
$$=\delta(t)\{V_0\delta^{\alpha\beta}-r^{-\gamma}[(d+1-\gamma)r^2\delta^{\alpha\beta}-(2-\gamma)r^{\alpha}r^{\beta}]\}.$$

Here γ is a measure of velocity nonsmoothness, $0\leqslant\gamma\leqslant 2$. The generalization of Eqs. (2.2) and (2.3) for nonzero γ needs replacing $\hat{\mathcal{L}}$ by

$$\sum_{i,j} r_{ij}^{-\gamma}[(d+1-\gamma)r_{ij}^2\delta^{\alpha\beta}-(2-\gamma)r_{ij}^{\alpha}r_{ij}^{\beta}]\boldsymbol{\nabla}_i^{\alpha}\boldsymbol{\nabla}_j^{\beta}.$$

The pair correlation function is now $F_2(r)\propto r^{\gamma-d}$.[5]

An interpretation of the extra factor r^{γ} compared to (3.7) is related to the fact that every Lagrangian distance R generally grow by a power law $t^{1/\gamma}$ as distinct from an exponential law at $\gamma=0$. In other words, stretching is uniform in the logarithm of scale at $\gamma=0$ and decelerating at $\gamma\neq 0$.

We cannot yet find the high-order correlation functions at arbitrary γ. Fortunately, at the limit of extremely irregular velocity $\gamma=2$ the operator turns into

$$\hat{\mathcal{L}}_0=-(d-1)\left[\left(\sum \boldsymbol{\nabla}_i\right)^2-\sum \nabla_i^2\right].$$

For translationally invariant functions the term $(\Sigma\nabla_i)^2$ can be discarded. We are thus left with a diffusion equation; it is straightforward to show that if the pumping is Gaussian then the scalar statistics is Gaussian, too (both at the scales larger and smaller than the pumping scale). If the pumping is non-Gaussian, then the scalar statistics is getting Gaussian at the scales distant form the pumping scale (odd moments and cumulants of even moments decrease with the growth of r/L faster than the respective Gaussian moments). The third moment, for instance, decreases with r/L faster than $F_2^{3/2}$, this can be shown by substituting the resolvent $(d>2)$

$$\mathcal{R}=\hat{\mathcal{L}}_0^{-1}=-\frac{\Gamma(3d/2-1)}{2\pi^{3d/2}D(d-1)}[(\mathbf{x}_1-\mathbf{y}_1)^2+(\mathbf{x}_2-\mathbf{y}_2)^2+(\mathbf{x}_3-\mathbf{y}_3)^2]^{1-3d/2}$$

into

$$F_3^{(0)}=\int d\mathbf{x}_1\,d\mathbf{x}_2\,d\mathbf{x}_3\,\mathcal{R}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3)$$
$$\chi_3(x_{12},x_{13},x_{23})\approx\frac{C_3}{[y_{12}^2+y_{13}^2+y_{23}^2]^{d-1}},$$
$$C_3=\frac{6^{d-1}\Gamma(d-1)}{\pi^d(d-1)}\int d\boldsymbol{\zeta}\,d\boldsymbol{\eta}\,\chi_3(2|\boldsymbol{\eta}|,|\sqrt{3}\boldsymbol{\zeta}+\boldsymbol{\eta}|,|\sqrt{3}\boldsymbol{\zeta}+\boldsymbol{\eta}|). \quad (6.1)$$

One sees that $F_3^{(0)}\propto r^{2-2d}$, which is decaying with r much faster than $F_2^{3/2}\propto r^{3-3d/2}$.

At $\gamma<2$ one can directly check that the scalar statistics is non-Gaussian even for a Gaussian pumping. Employing perturbation theory with respect to $\xi=2-\gamma$, one may try to prove at least that the scaling is normal and the angular anomaly is absent at $0<\gamma\leqslant 2$. Let us do this for the triple correlation function. The operator $\hat{\mathcal{L}}$ to the first order in ξ is $\hat{\mathcal{L}}=\hat{\mathcal{L}}_0+\xi\hat{\mathcal{L}}_1$ with

$$\hat{\mathcal{L}}_1=-\sum_{i<j}\left(\delta^{\alpha\beta}[(d-1)\ln r_{ij}+1]-\frac{r_{ij}^{\alpha}r_{ij}^{\beta}}{r_{ij}^2}\right)\boldsymbol{\nabla}_i^{\alpha}\boldsymbol{\nabla}_j^{\beta}.$$

Then, we should find $\hat{\mathcal{L}}_1F_3^{(0)}$ and integrate it with the resolvent. We have

$$\hat{\mathcal{L}}_1F_3^{(0)}=-C_3\left\{\frac{3(d-1)(3d-2)}{[r_{12}^2+r_{13}^2+r_{23}^2]^d}-\frac{d(d-1)}{[r_{12}^2+r_{13}^2+r_{23}^2]^{d+1}}\left[\frac{(r_{13}^2-r_{23}^2)^2}{r_{12}^2}-\frac{(r_{12}^2-r_{23}^2)^2}{r_{13}^2}-\frac{(r_{12}^2-r_{13}^2)^2}{r_{23}^2}\right]\right.$$
$$\left.+\frac{2d(d-1)^2}{[r_{12}^2+r_{13}^2+r_{23}^2]^{d+1}}\left(r_{12}^2\ln\frac{r_{13}^2r_{23}^2}{r_{12}^4}+r_{13}^2\ln\frac{r_{12}^2r_{23}^2}{r_{13}^4}+r_{23}^2\ln\frac{r_{12}^2r_{13}^2}{r_{23}^4}\right)\right\}.$$

There are three kinds of terms here. From the symmetry reasons, it is enough to know the values of the following integrals:

$$I_1=\int\frac{d\mathbf{x}_1\,d\mathbf{x}_2\,d\mathbf{x}_3}{(x_{12}^2+x_{13}^2+x_{23}^2)^d[(\mathbf{x}_1-\mathbf{y}_1)^2+(\mathbf{x}_2-\mathbf{y}_2)^2+(\mathbf{x}_3-\mathbf{y}_3)^2]^{3d/2-1}},$$

$$I_2=\int\frac{(x_{13}^2-x_{23}^2)^2d\mathbf{x}_1\,d\mathbf{x}_2\,d\mathbf{x}_3}{x_{12}^2(x_{12}^2+x_{13}^2+x_{23}^2)^{d+1}[(\mathbf{x}_1-\mathbf{y}_1)^2+(\mathbf{x}_2-\mathbf{y}_2)^2+(\mathbf{x}_3-\mathbf{y}_3)^2]^{3d/2-1}},$$

$$I_3=\int\frac{x_{12}^2\ln\dfrac{x_{13}^2}{x_{12}^2}d\mathbf{x}_1\,d\mathbf{x}_2\,d\mathbf{x}_3}{(x_{12}^2+x_{13}^2+x_{23}^2)^{d+1}[(\mathbf{x}_1-\mathbf{y}_1)^2+(\mathbf{x}_2-\mathbf{y}_2)^2+(\mathbf{x}_3-\mathbf{y}_3)^2]^{3d/2-1}}.$$

With the logarithmic accuracy one finds

$$I_{1,2,3}=\frac{A_{1,2,3}}{(y_{12}^2+y_{13}^2+y_{23}^2)^{d-1}}\ln\left[\frac{\sqrt{y_{12}^2+y_{13}^2+y_{23}^2}}{L}\right],$$
$$A_1=\frac{2\pi^{3d/2}}{3(d-1)\Gamma(3d/2-1)},$$
$$A_2=\frac{2\pi^{3d/2}}{3d(d-1)\Gamma(3d/2-1)}, \quad (6.2)$$
$$A_3=-\frac{\pi^{3d/2}}{3d(d-1)\Gamma(3d/2-1)}.$$

The result can be expressed in terms of $I_{1,2,3}$,

$$\begin{aligned}F_3^{(1)}&=-\xi\hat{\mathcal{L}}_0^{-1}\hat{\mathcal{L}}_1F_3^{(0)}\\&=-\xi\frac{C_3\Gamma(3d/2-1)}{2\pi^{3d/2}}[(9d-6)I_1-3dI_2+2d(d-1)I_3]\\&=\frac{\xi C_3}{(y_{12}^2+y_{13}^2+y_{23}^2)^{d-1}}\ln\left[\frac{L}{\sqrt{y_{12}^2+y_{13}^2+y_{23}^2}}L\right]\\&=\xi F_3^{(0)}\ln\left[\frac{L}{\sqrt{y_{12}^2+y_{13}^2+y_{23}^2}}\right].\end{aligned}$$

This is the first term of the expansion with respect to ξ of the function $F_3\propto(y_{12}^2+y_{13}^2+y_{23}^2)^{1-d-\xi}\propto r^{\gamma-1-d}$, which is the normal scaling for the triple correlation function.

Analysis of the integrals $I_{1,2,3}$ shows that there is no angular singularity at collinear geometry. By a direct calculation one can check that the scaling of the triple correlation function is the same for three points on a line. This is natural since nonzero γ destroys degeneracy, collinearity is no longer preserved during the Lagrangian evolution so the correlation functions at collinear geometry have no anomaly, similar to what has been established by γ-expansion at small scales.[12,17]

VII. CONCLUSION

We have studied the correlation functions of a passive scalar in the framework of the Kraichnan model on distances larger than the scalar's pumping length. In the Batchelor limit, the collinear anomaly has been found: scaling behavior of many-point correlation functions for the collinear geometry (where some points lie on a line) strongly differs from one for general geometry. The anomalous scaling is observed in the interval of angles which decreases with increasing scale. This violation of a conventional scaling behavior is related to a strong correlation between different Lagrangian trajectories occurring in the Batchelor case that is for distances smaller than the viscous scale of the velocity field. For larger distances (at the inertial interval of turbulence) the scale invariance of scalar statistics (yet not Gaussianity) is likely to be restored (remember that we consider the scales larger than the scale of scalar's pumping) as is confirmed by our calculations in Sec. VI. At even larger scales (beyond velocity correlation scale that is an external scale of turbulence) the scalar statistics has to be Gaussian.

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