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Passive scalar structures in peripheral regions of random flows

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We investigate statistical properties of the passive scalar near walls in random flows assuming weakness of its diffusion. Then at advanced stages of the passive scalar mixing its unmixed residue is concentrated in a narrow diffusive layer near the wall. We conducted numerical simulations and revealed structures responsible for the passive scalar transport to bulk, they are passive scalar tongues pulled from the diffusive boundary layer. The passive scalar integrated along the wall possesses well pronounced scaling behavior. We propose an analytical scheme giving exponents of the integral passive scalar moments, the exponents agree reasonably with numerics in $3d$.

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Stochastic dynamics of such fields, as temperature or concentration of pollutants, in random flows is of great importance in different physical contexts, from cosmology to microfluidics. Speaking about random flows we have in mind turbulent [1, 2] or chaotic flows [3]. Some intermediate position between the turbulent and chaotic flows has the so-called elastic turbulence [4]. The main effect of the random flows is mixing leading to essential acceleration of the temperature or concentration homogenization. Our consideration can be extended to chemical reactions that are accelerated in the random flows due to the same mixing effect [5].

If the feedback of the field to the flow is negligible then the field is called usually passive scalar. Dynamical and statistical properties of the passive scalar in random flows are extensively examined during the last two decades experimentally, analytically and numerically, see, e.g., Refs. [6 – 12]. It was established that the passive scalar has complicated statistics exhibiting such features as multifractality and intermittency. It is a consequence of a complicated interplay of diffusion and random advection determining the passive scalar evolution.

The passive scalar field θ in an external flow is described by the advection-diffusion equation

$$\partial_t\theta + \mathbf{v}\nabla\theta = \kappa\nabla^2\theta, \tag{1}$$

where $\mathbf{v}$ is the flow velocity and κ is the diffusion (thermodiffusion) coefficient. The equation (1) should be supplemented by boundary conditions at the walls that are zero normal gradient for density of pollutants and a fixed value of θ for temperature. We assume that the Peclet or the Schmidt number is large (that is κ is small in comparison with the fluid kinematic viscosity ν). Below, the fluid is assumed to be incompressible (that is $\nabla\mathbf{v} = 0$).

In this paper, we investigate the passive scalar statistics in peripheral regions of a vessel where developed turbulence or a chaotic flow is excited. Speaking about the peripheral regions, we imply a laminar (viscous) sublayer formed near walls where the velocity field can be treated as smooth, it varies on distances of the order of the thickness of the peripheral region, being a random function of time. We are interested in advanced stages of the passive scalar decay. Then the unmixed fraction of the passive scalar (or of unreacted chemicals) is located mainly in peripheral regions near walls. The same is true for a statistically stationary situation related, say, to a permanent heat flow through the walls. A theoretical approach to the problem was developed in Refs. [13 – 16], general predictions of the approach were confirmed by mixing experiment with elastic turbulence, see Ref. [17].

Since the Peclet or the Schmidt number is assumed to be large, the passive scalar dynamics in the peripheral region is slow in comparison with bulk. Therefore at the first stage the passive scalar is well mixed in bulk, that leads to a homogeneous distribution with $\theta = \text{const}$. We assume that bulk can be treated as a big reservoir, then the constant is practically time independent. Below, we imply that the passive scalar field is shifted by the constant, that gives $\theta = 0$ in bulk. And after the first stage the passive scalar evolution remains appreciable in the peripheral regions. In the case the unmixed passive scalar is concentrated mainly in a narrow diffusive layer near the wall, thinner than the thickness of the peripheral region [13, 14]. Then the passive scalar transport to bulk goes through the peripheral region outside the diffusive layer. Just this peripheral region plays a crucial role in formation of statistical characteristics of the transport. Note that our assumptions lead to an exponential decay of the passive scalar [14]. A power decay observed in Ref. [18] is probably related to the experimental setup, where bulk cannot be treated as a big reservoir.

We assume that the walls of the vessel are smooth and that the boundary layer width is much less than the curvature radii of the wall. Then it can be treated as flat in the main approximation. Let us introduce the reference system with the Z-axis perpendicular to the wall and assume that the fluid occupies the region $z > 0$. Smoothness of the velocity leads to the proportionality laws $v_x, v_y \propto z$ and $v_z \propto z^2$ for the velocity components along and perpendicular to the wall, respectively. The laws are consequences of the non-slipping boundary condition $\mathbf{v} = 0$ at the wall and of the incompressibility condition $\nabla\mathbf{v} = 0$. Below we assume that the velocity statistics is homogenous in time, and also assume its homogeneity along the wall. However, the flow statistics is non-homogeneous in z-direction and highly anisotropic.

Since the velocity tends to zero at approaching the wall and the diffusion is assumed to be weak, the passive scalar dynamics in the peripheral region, determined by an interplay of advection and diffusion, is slower than the velocity dynamics. Thus at investigating the passive scalar evolution the velocity can be treated as short correlated in time, and therefore closed equations can be derived for the passive scalar correlation functions (see, e.g., [8, 14]). Say, the equation for the first moment of θ, $\langle\theta\rangle$ (where angular brackets mean time averaging), is written as [14]

$$\partial_t\langle\theta\rangle = \partial_z\left(\mu z^4 \partial_z\langle\theta\rangle\right) + \kappa\partial_z^2\langle\theta\rangle, \qquad (2)$$

where the second term is caused by random advection, its z-dependence is related to the law $v_z \propto z^2$ and the factor μ characterizes strength of the velocity fluctuations.

Comparing the advection and the diffusion terms in the equation (2) one finds a characteristic diffusion length $r_{bl} = (\kappa/\mu)^{1/4}$. Due to smallness of κ the diffusion length is much less than the thickness of the peripheral region. At advanced stages of the passive scalar decay or in the statistically stationary situation (caused by the permanent passive scalar flux through the wall) the passive scalar is concentrated mainly in the diffusive boundary layer, at $z \lesssim r_{bl}$. However, we are interested in the passive scalar transport through the region $z > r_{bl}$, where the passive scalar is carrying from the diffusive boundary layer to bulk. There it is possible to neglect the diffusion term in Eq. (2) and we arrive at the law $\langle\theta\rangle \propto z^{-3}$.

To check theoretical predictions we conducted extensive numerical simulations of the problem. Details of the simulations will be published elsewhere, here we give a short overview of our results. We have chosen the scheme where dynamics of a large number of particles subjected to flow advection and Langevin forces is examined. The set of the particles is used instead of the passive scalar field θ, that can be treated as density of the particles. A big advantage of the approach is ability to produce simulations for arbitrary dimension of space d.

To establish principal qualitative features of the passive scalar transport, we perform mainly $2d$ simulations. The setup is periodic in x (coordinate along the wall) and in majority of simulations the velocity was

$$\begin{aligned} v_x &= z\left[\xi_1\cos(2\pi x/L) + \xi_2\sin(2\pi x/L)\right]L/\pi, \\ v_z &= z^2\left[\xi_1\sin(2\pi x/L) - \xi_2\cos(2\pi x/L)\right], \end{aligned} \qquad (3)$$

where L is period (we have chosen $L = 10$) and ξ_1 and ξ_2 are independent random functions of time. The velocity field satisfies the incompressibility condition $\partial_x v_x + \partial_z v_z = 0$ for any functions $\xi_1(t)$ and $\xi_2(t)$. They possess identical Gaussian probability distributions, then the velocity (3) has statistics homogeneous in x (along the wall). Since the velocity correlation time is much less than the passive scalar mixing time, one should assume that $\xi_1(t)$ and $\xi_2(t)$ are short correlated in time. We model the functions by telegraph processes, where both functions, ξ_1 and ξ_2, remain constants inside time slots of a small duration τ, and the values of ξ_1 and ξ_2 inside the slots are chosen to be random variables with identical normal distributions. In our simulations the variants were $\langle\xi_1^2\rangle = \langle\xi_2^2\rangle = 1$, that gives $\mu = \tau/2$.

In our scheme a particle trajectory $\boldsymbol{\varrho}(t)$ obeys the equation $\partial_t\boldsymbol{\varrho} = \mathbf{v}(t, \boldsymbol{\varrho}) + \boldsymbol{\zeta}(t)$, where the first term represents the particle advection and the second term represents the Langevin forces. Let us stress that the variables $\boldsymbol{\zeta}$ are independent for different particles whereas the variables ξ_1 and ξ_2 are identical for all particles, according to physical meaning of the variables. Again, $\boldsymbol{\zeta}$ is modeled by a telegraph process with the same time slot duration τ and with normal distributions of the values in the slots. To ensure a given value of the diffusion coefficient κ, one should accept $\langle\zeta_x^2\rangle = \langle\zeta_z^2\rangle = 2\kappa/\tau$. In majority of simulations we have chosen $\kappa = \tau/2$, and therefore the diffusive length was $r_{bl} = 1$.

Inside a time slot all, ξ_1, ξ_2 and $\boldsymbol{\zeta}$, are time-independent constants and the equation $\partial_t\boldsymbol{\varrho} = \mathbf{v}(t, \boldsymbol{\varrho}) +$

$+ \boldsymbol{\zeta}(t)$ becomes an autonomous ordinary differential equation. It was solved as follows. A time slot was divided into a number of time intervals and the equation was solved (without the Langevin force) using the second order Runge-Kutta method. The number of intervals is z-dependent, it is inversely proportional to z at $z > 2.5$. For $z > 12$ we solved equations for ϱ_x and $1/\varrho_z$. Both features are motivated by the strong dependence of the velocity on z, $v_z \propto z^2$. After solving the equation inside a slot a term produced by the Langevin force was added. To examine role of diffusion outside the diffusive boundary layer, in some simulations we switched off diffusion (Langevin forces) at distances $z > z_d$ (where a choice of z_d is different for different cases).

The particles are permanently injected near the wall in random positions at the beginning of each time slot. The simulations were performed in the interval $0 < z < 100$, the particles crossing the lines $z = 0$ and $z = 100$ were excluded from the consideration. A balance between the particle injection and losses leads to a statistical equilibrium achieving gradually in our numerics. Thus, our simulations cover the statistically stationary passive scalar transport. It corresponds both to the steady temperature distribution supported by a heat flux from the wall and to the decay of the concentration of pollutants that can be treated adiabatically.

The simulations reveal specific structures responsible for the passive scalar transport to bulk. It is related to jets carrying the passive scalar from the wall towards bulk, the jet produces a passive scalar tongue with width (cross-section) diminishing as z grows. The property is a consequence of the law $v_z \propto z^2$ reading that the z-component of the tongue velocity increases as z grows. Sometimes tongues are pulled upto z-infinity, and then a portion of the passive scalar is pushed to bulk. After some time the tongue is tilted and then pressed to the diffusive layer. Then next tongue is pulled, usually from the bump remaining at bottom of the previous tongue, and is, in turn, pressed to the diffusive layer. As a result, a complicated structure is formed, an example of such structure is drawn in Fig.1, that represents a snapshot generated in our simulations.

In terms of the particles, the passive scalar field θ is defined as a number of particles per unit area, that is as a number of particles inside a box divided by the box area. Of course, the definition works well provided the box is small enough and the number of particles inside the box is large. To satisfy these contradictory conditions one should deal with a large enough total number of particles. That is why the injection rate in our numerics is chosen to produce a large number of particles, $10^5 \div 10^6$, in the statistical equilibrium. Statistical characteristics

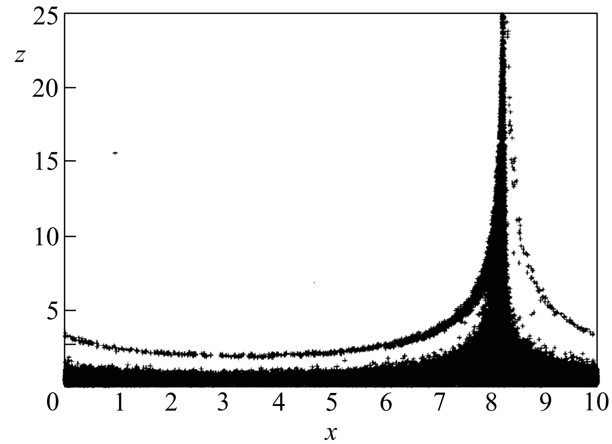

Fig.1. An example of a passive scalar structure formed near the wall in the random flow. Different particles are designated by small crosses

of the passive scalar can be characterized by correlation functions (moments) that are obtained in our numerics by averaging over a time interval $t \sim 10^6 \div 10^7 \tau$.

We computed moments of θ and realized that the law $\langle\theta\rangle \propto z^{-3}$ is perfectly satisfied for $z \gg r_{bl}$ whereas higher moments decay faster. Thus, we obtain results inconsistent with the behavior $\propto z^{-3}$ that can be derived for all moments of θ provided diffusion is neglected [14]. Based on the notion of tongues, one can explain relevance of the diffusion term at $z \gg r_{bl}$. Indeed, the term $\kappa\nabla^2$ in Eq. (1) can be estimated as κ/l^2 where l is a characteristic tongue width that diminishes as z grows. If l diminishes faster than z^{-1} then the diffusion term appears to be relevant, as can be understood by comparing it with the advection term in Eq. (2). A dependence of l on z is a subject of special investigation. Any case, neglect of the diffusion term should be specially grounded. The only exclusion is the first moment since the x,y-diffusion drops from the equation for the moment due to homogeneity of the system in the directions.

To exclude the effect of diffusion, we introduce another object, Θ that is an integral of the passive scalar field along a surface parallel to the wall. Due to narrowness of the tongues, one expects that the passive scalar is short correlated along the wall at $z \gg r_{bl}$. Then we obtain closed equations for the Θ moments

$$\partial_t\langle\Theta^n\rangle = \mu\left[z^4\partial_z^2 + 4nz^3\partial_z + 4n(n-1)z^2\right]\langle\Theta^n\rangle. \quad (4)$$

In the stationary (or quasi-stationary) case (where $\partial_t\langle\Theta^n\rangle$ is negligible) we obtain a homogeneous differential equation for the n-th moment that admits a power solution $\langle\Theta^n\rangle \propto z^{-\zeta_n}$, where

$$\zeta_n = 2n - 1/2 + \sqrt{2n + 1/4}. \quad (5)$$

The positive sign of the square root is chosen to reproduce the reference value $\zeta_1 = 3$.

In our numerics, Θ is represented by a number of particles in a stripe parallel to the wall, divided by its width. The moments of Θ, $\langle\Theta^n\rangle$, were computed in $2d$ by time averaging over a long time $\sim 10^7\tau$. To check robustness of the results we performed computations for different time slots, $\tau = 0.001, 0.002, 0.004$, and for four different values of the diffusion coefficient κ. The Fig.2 demonstrates that the values of each moment collapse to a single curve in the coordinates $\ln(z/r_{bl})$ and $\ln(\langle\Theta^n\rangle/C_n)$ where the factors C_n are of the order of the corresponding moments near the wall. We also checked that the moments of Θ are insensitive to diffusion, switching diffusion off at $z > 12$ in some simulations. One can observe no difference between the data. One can also try to use a more complicated than (3) velocity field. We calculated the moments for a velocity field with four random factors instead of two, as in Eqs. (3). Again, there are no visible differences in comparison with Fig.2.

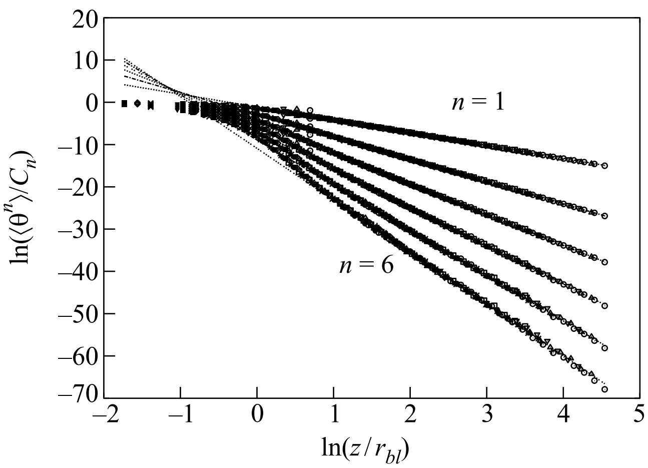

Fig.2. Moments of Θ, $\langle\Theta^n\rangle$, in log-log coordinates, $n = 1 \div 6$. In the region $z > r_{bl}$ the results collapse onto single curves for three times $\tau = 0.001, 0.002, 0.004$ and four different values of the diffusion coefficient κ

We observe that the moments of Θ are decreasing functions of z that are power-like, $\langle\Theta^n\rangle \propto z^{-\zeta_n}$, in the region $z > r_{bl}$. Extracting the scaling exponents ζ_n for $n = 1 \div 6$ we obtain values that are presented in Fig.3 as the upper set of points (some smooth curve is drawn through the points for better visualization). We conducted analogous simulations for higher dimensions, upto $d = 5$. The results are depicted in the same Fig.3. We see that the exponents ζ_n depend on d, however, they are close for $d \geq 3$, and are close to the theoretical values (5). Deviations of the numerical values of ζ_n from the theoretical predictions (5) can be explained by long correlation of the passive scalar along the wall that can be produced by the multi-fold structures of the type drawn in Fig.1. It is naturally to expect that the fold effect becomes less pronounced in high dimensions. Indeed, Fig.3 shows that the deviations from the values (5) diminish as the space dimensionality d grows. That confirms our explanation. Really, the exponents (5) are close to ones, found in numerics, already in the dimensionality $d = 3$.

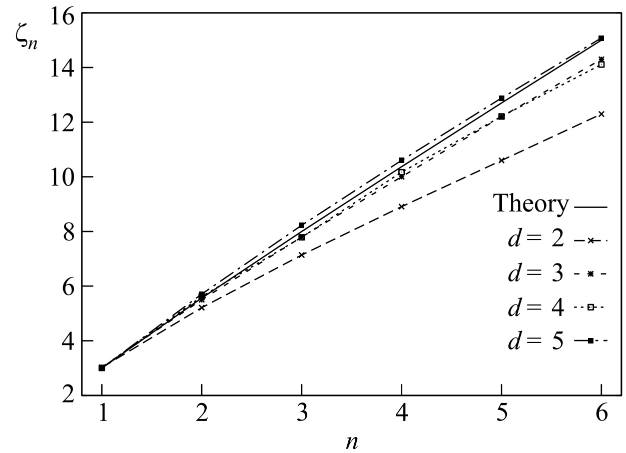

Fig.3. Exponents of the moments $\langle\Theta^n\rangle$, for $n = 1 \div 6$ and space dimensions $d = 2 \div 5$. For comparison the theoretical curve $\zeta_n = 2n - 1/2 + \sqrt{2n + 1/4}$ is plotted

Any case, we observe anomalous scaling, that is a non-linear dependence of ζ_n on n. The scaling behavior leads to estimates $\langle\Theta^n\rangle \sim \Theta_0^n (r_{bl}/z)^{\zeta_n}$, where Θ_0 is a characteristic value of Θ inside the diffusive layer. Since the curve ζ_n is convex (see Fig.3), we conclude that higher moments are much larger than their naive estimates in terms of the first ones. Say, $\langle\Theta^n\rangle \gg \langle\Theta\rangle^n$. By other words, the probability distribution function of the moments has "thick" tails, reflecting strong intermittency, naturally explained by the tongues.

There remains a set of questions. We were not examined tongue shapes in $3d$, particularly, aspect ratio of their cross-section. There was no average flow in our numerics whereas an average shear-like flow occurs usually near the wall. Another natural extension of our approach is related to chemical reactions in random flows. One can also note polymer solutions, where the polymer elongation is very sensitive to the character of the flow. The problem is significant, e.g., for the elastic turbulence. We considered smooth walls. A roughness of the wall can modify essentially our conclusions. We consider all the problems as subjects for future investigations.

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