Shedding and interaction of solitons in weakly disordered optical fibers

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The propagation of the soliton pattern through optical fiber with weakly disordered dispersion coefficient is considered. Solitons perturbed by this disorder radiate and, as a consequence, decay. The average radiation profile is found. Emergence of a long-range intrachannel interaction between the solitons (mediated by this radiation) is reported. We show that soliton in a multisoliton pattern experiences a random jitter: intersoliton separation is zero mean Gaussian random field. Fluctuations of this separation are estimated by $\delta y \sim Dz^2\sqrt{\mu}$, where D measures the disorder strength, z is the propagation distance, and μ stands for the transmission rate (number of solitons per unit length of the fiber). Direct numerical simulations are used to validate theoretical predictions for single soliton decay and two-soliton interaction. Relevance of these results to fiber optics communication technology is discussed.

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I. INTRODUCTION

Fibers are not ideal, i.e., inability of production to achieve 100% guaranteed control of fiber parameters in the process of fiber pulling and preform manufacturing results in irregularities of the fiber structure. Structural disorder is built in the fiber. The effect of this disorder on the propagation and interaction of pulses accumulates with propagation, i.e., the longer some pulse (pattern of pulses) travels along the fiber, the more strongly disorder affects it. Even weak disorder may cause essential damage to pulse integrity. A strong effect of weak disorder in the fiber dispersion coefficient on the shedding and interaction of pulses, a problem which is crucial for progress in modern nonlinear fiber optics and related communication technology, is described in this paper.

In fiber optics communication a pulse is used as a bit of information. For an ideal fiber, working in the regime of nonlinear transmission, a pulse of the electric field is described by a stationary solution (soliton) of the self-focusing nonlinear Shrödinger equation (NLSE) with constant coefficients [1–3]. Stationarity, in particular, means that soliton propagates through the fiber with a constant speed. (For a detailed derivation of the NLSE from Maxwell's equations in a very general fiber optics setup see, e.g., Ref. [4].) The stationary solution is a result of a fine balance between the fiber dispersion and nonlinearity [2–4].

A sequence of pulses launched in the fiber forms a pattern, which codes the transmitted message. Ideally, this pattern is a sequence of solitons, each positioned in the center of a slot allocated for the respective information bit, where the soliton is a stationary single-pulse solution of NLSE. State ''1'' is assigned to a slot if the soliton is present there, and the state of the slot is ''0'' if the slot contains no soliton. The disorder, built in this fiber, breaks this ideal picture. (Some other potentially important corrections to NLSE are discussed in Refs. [5,6].) In the present manuscript we describe dynamics of single- and multi-soliton patterns in the presence of weak disorder in the dispersion coefficient. Some preliminary results of this study, detailed and corrected here, were briefly described in Ref. [7].

A soliton, propagating through a fiber, emits radiation due to disorder and, consequently, loses its energy. However, in the case of weak disorder (weakness of disorder is actually required for successful fiber performance) the destruction of the soliton is slow, thus making an adiabatic description of this problem possible. The adiabaticity implies separation of dynamical degrees of freedom into slow and fast modes. [See Refs. [8–10] for the general description of the adiabatic perturbation approach to partial differential equations and Refs. [11,12] for applications of the general method to various regular perturbation expansions about the soliton solution of the one-dimensional (1D) NLSE.] Slow modes describe evolution of the soliton itself while the fast modes correspond to the radiation. The soliton keeps its shape (so that, at each instant, the soliton is close to a stationary solution of the noiseless NLSE) with the soliton parameters (position, width, phase, and phase velocity) evolving slowly. Waves shed by a soliton are moving away from it. The average intensity of the radiation (at $1 \ll t < z$) is estimated as $D\eta^4 \ln(z/t)$. Here, η is the soliton amplitude, D measures the intensity of the disorder (which is assumed to be weak, $D \ll 1$), z stands for position along the fiber, and t is the retarded time, i.e., time counted from the moment when the soliton passes through a given position z. (All the quantities are measured in the respective soliton units: the time unit is the soliton width, and the length unit corresponds to the distance passed by soliton during one turn on 2π of the soliton phase.) In the domain where $|t| \gg z$ the radiation decays exponentially with t/z. Thus, one can say, that the radiation propagates away from the soliton (in t) with velocity, which is $O(1)$. Amplitude of the front forerunner (i.e., the domain of t where $|t| \gg z$) decays exponentially with t/z. One finds that at any z, however large, the radiation in an immediate vicinity of the soliton is much less intense than the soliton itself, i.e., the soliton is always distinguishable from the radiation. Since the soliton losses its energy into radiation, its amplitude η decays with z. The degradation law is deterministic in spite of the original setting stochasticity. This is due to the fact that the variation of η is determined by an integral over z, which is a self-averaged quantity at large z. The soliton degradation law (valid at any $z \gg 1$) is

$$\eta = (1 + 32Dz/15)^{-1/4}. \tag{1.1}$$

(Quantitative definition of the noise intensity D is given in the next section.) Notice, that the degradation of the soliton amplitude in the presence of disorder in the dispersion coefficient was previously considered in Ref. [13], where estimations consistent with the analytic expression (1.1) were derived. Equation (1.1) shows that the soliton starts to degrade essentially at $z \sim 1/D$.

Next, we examine interaction of solitons at $1 \ll z \ll 1/D$ (when the soliton amplitude decrease is still negligible) emerging under the influence of the radiation. We show that the interaction is extremely long range, due to the one-dimensional (1D) nature of the system and also because of the reflectionless feature of the radiation. At any z all solitons separated from a given one by $|t| \lesssim z$ act on this soliton with a force, which is zero on average. Fluctuations of the force result in a Gaussian jitter of the soliton position. We find that in the two soliton case (i.e., for the pattern consisting of two solitons only, so that no other solitons are present anywhere in the $|t| \lesssim z$ vicinity of the pair) fluctuations in their relative position δy are determined by

$$\langle (\delta y)^2 \rangle = 0.37[1 + \cos(2\alpha)]D^2 z^3, \tag{1.2}$$

where α is the intersoliton phase mismatch. Angular brackets in Eq. (1.2) stand for averaging over many realizations of disorder (i.e., over different fibers). In the general multisoliton case fluctuations in the ith soliton position are estimated as

$$\langle (\delta y_i)^2 \rangle \sim N D^2 z^3, \tag{1.3}$$

where N is the number of solitons in the same channel (propagating on a given frequency, i.e., with a given group velocity) in the $|t| \lesssim z$ vicinity of the pair. (To avoid confusion, note, that effects of multichannel interaction are not discussed here.) At $z \sim N^{-1/3} D^{-2/3}$ the effect of interaction on the soliton displacement becomes dangerous, i.e., $O(1)$. This interaction length $N^{-1/3} D^{-2/3}$ is shorter than the degradation length D^{-1}. Thus our approximation is justified: solitons acquire significant shifts in their positions well before any essential decrease of the soliton amplitude (or, generally, essential distortion of its shape) is observed. Notice, that Eq. (1.3) also applies to the case of an infinite pattern, correspondent to the continuous flow of information. In this case, $N = \mu z$, where μ is the information rate, i.e., number of solitons per unit length of the fiber.

The material in the paper is organized as follows. General fiber optics relations relevant to our analysis are presented in Sec. II. The single soliton results are detailed in Sec. III. Section IV is devoted to the two-soliton interaction analysis. Generalization of the two-soliton picture for the multisoliton case is discussed in Sec. V. Effect of the recently proposed pinning of disorder in dispersion [14–17] on single pulse degradation and intersoliton interaction is addressed in Sec. VI. Direct numerical simulations for single-soliton and two-soliton cases, confirming the theoretical analysis, are discussed in Sec. V. Section VIII is reserved for conclusions. Some calculation details are described in appendixes.

II. BASIC RELATIONS

This section is devoted to general introduction into the problem of optical signal nonlinear propagation through imperfect fiber. Basic equations governing propagation of a pulse through such a fiber are introduced in Sec. I A. Section I B is devoted to discussion of real fibers parameters used in communication technology. Section I C introduces the formalism of a signal separation into localized modes (solitons) and delocalized modes (radiation). General consequences of the weakness of disorder for the separation formalism are discussed in Sec. I D.

A. NLS with frozen disorder

Optical fibers are waveguides relying on the effect of complete internal reflection. A typical fiber consists of core with higher refractive index and of a cladding with lower refractive index. The diameter of the fiber core corresponds to the first transverse mode at the carrier frequency of a signal. Therefore, light pulses can be described in terms of a single mode electromagnetic field, propagating along the fiber. Then, the field can be treated as one dimensional. Imperfections of the fiber (disorder, built in the fiber) is mainly coming from variations in its diameter and chemical composite. Since the signal propagating through the fiber decays, amplifiers should be inserted in the fiber line to maintain the signal's amplitude. Below, we discuss equations averaged over the inter-amplifier distance, thus assuming that attenuation is compensated by amplification.

The universal description of the signal envelope dynamics in the reference frame moving with the wave packet group velocity is given by the NLSE (see, e.g., Ref. [4])

$$-i\partial_z \Psi = d(z)\partial_t^2 \Psi + 2|\Psi|^2 \Psi, \tag{2.1}$$

explaining dynamics of electromagnetic wave packet with envelope $\Psi(z,t)$. This signal propagates in z (which is position along the fiber) being a subject to dispersion in retarded time t (i.e., time counted from the moment when soliton passes through a given position, z) and to the Kerr nonlinearity. Equation (2.1) assumes that fluctuations in the chromatic dispersion coefficient $d(z)$ characterizing irregularity of the fiber, have a greater effect on propagation of pulses than fluctuations of any other coefficients in the equation, say of the Kerr nonlinearity (which is, therefore, constant, rescaled to 2 in this equation). Equation (2.1) is a result of averaging of Maxwell's equations. This averaging accounts for geometrical features of the fiber core and cladding. Additional averaging, also accounted for in Eq. (2.1), is performed over the amplifier spacing. Real-world problems in fiber-optics communication may require an account for corrections to Eq. (2.1), e.g., for subleading corrections coming from averaging over amplifier spacing [18]. We argue in Sec. I B that such extra terms produce only small, irrelevant corrections to the soliton interaction discussed in this paper.

Only recently has the chromatic dispersion profile $d(z)$ became experimentally accessible. High-precision measurements demonstrated significance of the dispersion randomness [19,20]. Chromatic dispersion in optical fibers comes from two sources. The first source is the medium itself. Material dispersion in modern fibers is a relatively stable parameter, uniformly distributed along the fiber. That is why we assume here, that the dispersion does not fluctuate in time. The second source is due to specific geometry of the waveguide profile. Existing technology does not provide accurate control of the wave-guide geometry in fibers, so that the actual dependence of the dispersion coefficient on the wavelength is complicated. As a result, the typical magnitude d_{var} of random variations of fiber chromatic dispersion $d(z)$, can achieve, or in some cases even become greater than, that of the mean dispersion. The typical scale of the disorder variations z_{var} is much less than all relevant scales in the problem. (See Sec. I B for discussion of real-world numbers and estimations.)

It is convenient to separate the constant part of d (which we rescale to unity) and its fluctuating part ξ: $d=1+\xi$, where ξ is a random function of z, correlated on the scale z_{var}. We examine statistical properties of the fibers, which represent averaging over many realizations of the disorder $\xi(z)$ (over many fibers). Those objects allow establishment of both typical fluctuations and probability of large deviations from the typical value for different quantities. Being interested in phenomena occurring on scales z, larger than z_{var}, one can treat the disorder ξ as a short- (δ-) correlated one. Then the first two cumulants of ξ are

$$\langle \xi \rangle = 0, \quad \langle \xi(z_1)\xi(z_2)\rangle = D\,\delta(z_1-z_2), \tag{2.2}$$

where $\langle \cdots \rangle$ marks averaging over many realizations of disorder (i.e., over many different fibers). The coefficient D (to be called noise intensity) is estimated as $D \sim z_{\text{var}} d_{\text{var}}^2$. High-order cumulants of ξ are negligible as containing higher powers of z_{var}. In other words, statistics of ξ is Gaussian. The smallness of z_{var} (in comparison with other relevant z scales) is due to the fact that the disorder is weak, $D \ll 1$. This weakness of disorder is, actually, a necessary condition for successful fiber performance.

Note that in describing propagation of a signal, we adopt mixed optical-quantum mechanical notations and terminology. Indeed, the traditional optical notation t is reserved for retarded time, since, experimentally, the envelope of the electromagnetic field is measured as a function of time, and also because t in Eq. (2.1) is a "descendant" of the real time in the original Maxwell's equations, Eq. (2.1) was derived from. From the other side, the retarded time is proportional to real time minus position along the fiber z (divided to the velocity of light) and, therefore, t is also carrying a certain spatial sense. In addition to Eq. (2.1) which is called the nonlinear Schrödinger equation in direct analogy with the famous linear Shrödinger equation, is a parabolic equation with second order derivative over time t, and not over the coordinate along the fiber z. The analogy with quantum mechanics is extremely helpful and is used in later discussions and derivations. It explains why we treat t as more of a spatial variable rather than a temporal one, marking oscillations in t by "wave vectors," which would be natural to call "frequencies" in a pure optical context. (To avoid misunderstanding, let us stress, that the frequencies have no relation to the carrier frequency of the original electromagnetic wave.)

Another remark is about relevance of the physics described by Eq. (2.1) for the phenomenon of localization of light in disordered medium [21]. As was mentioned above, the disorder term ξ originates from fluctuations of the waveguide dispersion, and it is not related to the material component of the dispersion. Therefore, fluctuations of the material disorder were not accounted for in Eq. (2.1). Nevertheless, we find it useful to briefly discuss here its effect on propagation of light. Material disorder is associated with irregularities of the fiber core and cladding (impurities) on very short, atomic scales. Light scattering on the impurities leads to the well known phenomenon of localization of light, taking place at some larger scale, usually called the localization length [21]. The localization length is inversely proportional to the strength of the material disorder. Material used for manufacturing modern fibers are usually very clean, so that the localization length essentially exceeds the distance between filters, which, in the typical fiber lines, are placed at the amplifier stations. Filters cut the back scattering of light thus destroying coherence, required for emergence of the localization phenomenon. As a result, presence of (very low intensity) material disorder does not play any significant role in fiber optics communications. Notice, also, that the scale of the waveguide disorder variations z_{var} essentially exceeds the wavelength of light, thus allowing us not to take into account the back-scattering of light due to waveguide disorder. [Notice, that it is this separation of scales which allows us to reduce the hyperbolic Maxwell equations to the parabolic equation (2.1) in the envelope approximation.] Therefore, no localization phenomena due to material disorder is possible. For the sake of generality, let us also note, that the role of disorder in the context of the localization-delocalization transition was investigated for the nonlinear Schrödinger equation (see, e.g., Ref. [22]). However, in solid state physics frozen disorder means that noise is z-independent (in our notations). The t-dependent noise is very different from the z-dependent one, studied here, and to the best of our knowledge, the former case does not correspond to any situation of interest in fiber optics communications.

B. Real-world transmission parameters

Equation (2.1) is written in dimensionless units, which are related to the real-world fiber units through the following rules. The envelope of the electric field is in the form $E = \text{Re}[\sqrt{P_0}\Psi e^{i\omega_0 t}]$, where P_0 is the peak pulse power and ω_0 is the carrier frequency of the signal. The propagation variable is $z=Z(\alpha_K P_0/2)$, where Z is the distance along the fiber and α_K is the Kerr nonlinearity coefficient. The Kerr coefficient can be expressed in terms of other fiber parameters $\alpha_K = 2\pi n_2/(\lambda S_{\text{eff}})$, where n_2 is the nonlinear component of the fiber refractive index, λ is the operating wavelength, and S_{eff} is an effective area of the fiber core. The other coordinate is $t=(T-Z/c)/\tau_0$, where T is time, c is the velocity of light in the medium (i.e., $T-Z/c$ is just the retarded time), and τ_0 is the pulse width. The dispersion coefficient is $d=2\beta_2/(\alpha P_0\tau_0^2)$, where β_2 is the second order dispersion parameter. To give an example, a typical set of parameters for dispersion shifted fiber is as follows: $\beta_2=0.1\ \mathrm{ps}^2/\mathrm{km}$, $\alpha=2\ \mathrm{W}^{-1}\ \mathrm{km}^{-1}$, $\lambda=1550$ nm, $\tau_0=7$ ps, $P_0=2$ mW.

The typical scale of the disorder variations, z_{var}, can be extracted from experimental measurements [19,20] showing that z_{var} is shorter than $\sim 1-2$ km. Notice, that this number actually comes from experimental resolution, while one expects that the typical scale of the variations is at least one to two orders of magnitude shorter, $\sim 10-100$ m, i.e., the scale is fixed by the size of the production facility (fiber pulling device). In any case, z_{var} appears to be essentially shorter than any other, relevant for long-haul transmission, scales. It was also reported in Ref. [19] that fluctuations of the dispersion coefficient in a sample of ''dispersion shifted'' fiber are of the order of its average value, i.e., $\delta\beta_2\sim 0.5\ \mathrm{ps}^2/\mathrm{km}$. Therefore, for a pulse width of ~ 7 ps (which would correspond to a single-channel transmission rate of 28 Gb/s) and for a nonlinear length of, $z_{nl}=(\alpha P_0)^{-1}\sim 250$ km, the noise intensity $D=z_{\mathrm{var}}d_{\mathrm{var}}^2$ is estimated by $10^{-3}-10^{-2}$. Then, the soliton interaction is seen at $z_{\mathrm{int}}=1/\sqrt{D}\sim 2500$–$7500$ km. Notice, also that the decrease of the pulse width by a factor q (correspondent to a factor of q increase of the transmission rate) leads to q^2 decrease in z_{int}.

Let us now discuss applicability criteria of the approximations leading to Eq. (2.1) for the real-world situation in fiber-optics communication technology. An important additional scale in optical communication systems is imposed by fiber losses γ. Compensation of energy losses require use of in-line optical amplifiers separated by $z_{\mathrm{amp}}\sim\gamma^{-1}$. The value of z_{amp} is usually 40–70 km. Soliton based optical communications is possible if the dispersion length $z_{\mathrm{disp}}=\tau_0^2/\beta_2$, the length of nonlinearity and amplification spacing are related to each other as $z_{\mathrm{disp}}\sim z_{nl}\gg z_{\mathrm{amp}}$. Averaging over the shortest scale z_{amp} we arrive at Eq. (2.1). Subleading corrections, not accounted for in Eq. (2.1), are $O((z_{\mathrm{amp}}/z_{\mathrm{disp}})^2)$ [18]. This small parameter $(z_{\mathrm{amp}}/z_{\mathrm{disp}})^2$, is $\approx 10^{-2}$ in the aforementioned example of the dispersion-shifted fiber. Therefore, exclusion of the correction term from Eq. (2.1), as well as the validity of the averaging procedure over ξ, both require some additional justification. The correction term provides deterministic and stochastic contributions to optical pulse. Deterministic contribution does not produce any additional continuous radiation and it results only in a weak deformation of the optical soliton shape. The second, stochastic, contribution is $(z_{\mathrm{disp}}/z_{\mathrm{amp}})^2$ times smaller than the main stochastic contribution considered in the paper. Therefore, averaging over the amplifier spacing does not change the value of D, only affecting the value of the effective noise correlation length $(z_{\mathrm{var}}\to z_{\mathrm{amp}})$. This latter scale is still much smaller than all other relevant scales. One concludes that Eq. (2.1) does explain situation of practical interest for fiber optics communications.

C. Separation into localized-delocalized modes

One assumes that at the fiber entrance $z=0$, the signal Ψ is close to an N-soliton solution of the no-disorder NLSE, i.e., of Eq. (2.1) with $d=1$. The disorder term ξ disturbs the ideal N-soliton pattern. Our task here is to describe evolution of Ψ under action of this disorder. The weakness of disorder ξ and localized nature of the initial soliton profile $\Psi(0,t)$ suggest the following decomposition

$$\Psi=\Psi_{\mathrm{sol}}+\Psi_{\mathrm{con}}, \tag{2.3}$$

where Ψ_{sol} is the localized (soliton) part of the envelope and Ψ_{con} stands for the radiation (delocalized part). If there is no disorder ($\xi=0$) Ψ_{sol} is a solution of the $\xi=0$ version of Eq. (2.1) and $\Psi_{\mathrm{con}}=0$. Therefore, Ψ_{con} is $O(\xi)$.

In the single-soliton case one has

$$\Psi_{\mathrm{sol}}=\frac{\eta}{\cosh[\eta(t-y)]}\exp[i\varphi+i\beta(t-y)], \tag{2.4}$$

where η, y, φ, and β are amplitude, position, phase, and phase velocity of the soliton. The disorder ξ is a reason for complicated z dependence of the soliton parameters η, φ, β, and y, whereas in the absence of disorder (i.e., when $\xi=0$) η and β are z independent, and y and φ are linear functions of z. It is convenient to change from the soliton phase $\varphi(z)$ to the auxiliary (z independent in the absence of disorder) object α,

$$\varphi=\alpha+\int_0^z dz'\,\eta^2(z'). \tag{2.5}$$

It is also convenient to change from the radiation field Ψ_{con} to a new field v, which differs from Ψ_{con} by the single-soliton phase factor

$$v=\exp[-i\varphi-i\beta(t-y)]\Psi_{\mathrm{con}}. \tag{2.6}$$

By analogy with expansion over plane waves in the homogeneous case, one can present v here in the form of the following decomposition:

$$\begin{pmatrix} v \\ v^* \end{pmatrix}=\int_{-\infty}^{+\infty}\frac{dk}{2\pi}[a_k\varphi_k(x)+a_k^*\overline{\varphi}_k(x)], \tag{2.7}$$

where φ, $\overline{\varphi}$ are eigenfunctions

$$\hat{L}_\eta\varphi_k=(k^2+\eta^2)\varphi_k,\quad \hat{L}_\eta\overline{\varphi_k}=-(k^2+\eta^2)\overline{\varphi_k}, \tag{2.8}$$

of the operator

$$\hat{L}_\eta\equiv(\partial_t^2-\eta^2)\hat{\sigma}_3+\frac{2\eta^2}{\cosh^2[\eta(t-y)]}(2\hat{\sigma}_3+i\hat{\sigma}_2), \tag{2.9}$$

describing evolution of a linear perturbation about the single soliton profile (2.4) of the no-disorder NLSE. The eigenfunctions can be presented as $\varphi_k=f_{k/\eta}(x)$ and $\overline{\varphi_k}=\overline{f_{k/\eta}}(x)$, where $x=\eta(t-y)$, and f_k, $\overline{f_k}$ are the eigenfunctions of $\hat{L}_\eta$ at $\eta=1$, defined in Appendix A. This complete system of the eigenfunctions was found by Kaup in Ref. [11]. The coefficients a_k and a_k^* in Eq. (2.8) are functions of z. The eigenfunctions φ_k and $\overline{\varphi_k}$ depend on z via $\eta(z)$ and $y(z)$. The functions φ_k, $\overline{\varphi_k}$ are orthogonal to the four localized modes, corresponding to variations of the four soliton parameters in Eq. (2.4) (see Appendix A), and the orthogonality conditions can be written as

$$\int_{-\infty}^{+\infty} dt \cosh^{-1}(x)(v+v^*)=0,$$

$$\int_{-\infty}^{+\infty} dt \tanh(x)\cosh^{-1}(x)(v-v^*)=0,$$

$$\int_{-\infty}^{+\infty} dtx \cosh^{-1}(x)(v+v^*)=0,$$

$$\int_{-\infty}^{+\infty} dt[x\tanh(x)-1]\cosh^{-1}(x)(v-v^*)=0. \quad (2.10)$$

The relations (2.10) fix uniquely (even though inexplicitly) the soliton parameters, introduced by Eq. (2.4), for a given function $\Psi(z,t)$ in the decomposition (2.3) where $\Psi_{\rm con}$ is related to v through Eq. (2.6).

Let us rewrite Eq. (2.1) in terms of the new variables. Substitution of expressions (2.3)–(2.6) into Eq. (2.1) (where d has to be replaced by $1+\xi$), and subsequent expansion over ξ and v results in

$$\begin{aligned} &i\eta\partial_z\alpha f_0(x)-\partial_z\eta f_3(x)+\eta^2(\partial_z y-2\beta)f_1(x)+i\eta\partial_z\beta f_2(x)\\ &\quad+\partial_z\begin{pmatrix} v\\ v^*\end{pmatrix}-i\hat{L}_\eta\begin{pmatrix} v\\ v^*\end{pmatrix}+\cdots\\ &\quad=i\xi\eta^3\left[\frac{1}{\cosh x}-\frac{2}{\cosh^3 x}\right]\begin{pmatrix}1\\ -1\end{pmatrix}, \end{aligned} \quad (2.11)$$

where $x=\eta(t-y)$. The ellipses in Eq. (2.11) stand for high-order terms in v and β. Then, the equations for the soliton parameters and the continuous spectrum amplitudes a_k can be found by projecting Eq. (2.11) onto respective eigenfunctions of $\hat{L}_\eta$ (2.9). Let us present an expansion of the right-hand side of Eq. (2.11) into a series over the eigenfunctions:

$$\begin{aligned} &i\left[\frac{1}{\cosh x}-\frac{2}{\cosh^3 x}\right]\begin{pmatrix}1\\ -1\end{pmatrix}\\ &\quad=\eta^{-1}\int\frac{dk}{2\pi}(b_{k/\eta}\varphi_k+b^*_{k/\eta}\overline{\varphi_k})-if_0(x), \end{aligned} \quad (2.12)$$

$$b_q=\frac{\pi i}{2}\frac{(q+i)^2}{\cosh(\pi q/2)}, \quad (2.13)$$

where Eqs. (A11), (A12) from Appendix A were used.

In the multisoliton case a localized part of Ψ, $\Psi_{\rm sol}$, can be approximated as an N-soliton solution of the no-disorder NLSE with $4N$ parameters, varying in z. If individual solitons in the N-soliton pattern are well-separated from each other (i.e., if the intersoliton separations are all much larger than a single soliton width), $\Psi_{\rm sol}$ can be approximated (with an exponential accuracy over the intersoliton separations) by a sum of the single-soliton contributions

$$\begin{aligned} \Psi_{\rm sol}=&\sum_{i=1}^{N}\frac{\eta_i}{\cosh[\eta_i(t-y_i)]}\\ &\times\exp\left[i\alpha_i+i\int_0^z dz'\,\eta_i^2+i\beta_i(t-y_i)\right], \end{aligned} \quad (2.14)$$

labeled by index i, $i=1,\ldots,N$. Here η_i, y_i, α_i, and β_i are real parameters, standing for amplitudes, positions, phases and phase velocities of the solitons. In the absence of disorder, the soliton parameters are z independent with the same exponential accuracy. The disorder drives a z dependence of the parameters. The $4N$ parameters of $\Psi_{\rm sol}$ are determined for a given Ψ through $4N$ conditions generalizing the relations (2.10). The conditions manifest orthogonality between the continuous spectrum and localized modes of differential operator defined for linear perturbation of the no-disorder version of Eq. (2.1) about its N-soliton solution.

We assume that a sequence of identical (of the same unit amplitude and zero initial phase velocity) ideal solitons are launched into the fiber at $z=0$. Thus the initial conditions for Ψ are

$$\eta_i(0)=1, \quad \beta_i(0)=0, \quad \Psi_{\rm con}(0,t)=0. \quad (2.15)$$

The initial positions of the solitons $y_i(0)$ are parameters coding the transmitted information. Solitons phases $\alpha_i(0)$ are particular initial data.

D. Weakness of disorder

The separation (2.3) of the entire solution of Eq. (2.1) into the localized and delocalized parts is natural in the case of weak disorder. The weakness of disorder ($D\ll 1$) has two important consequences: first, the radiation emitted by soliton is also weak, i.e., $\Psi_{\rm con}=O(\xi)$, and second, parameters of the soliton vary slowly in z, while dynamics of the radiation field $\Psi_{\rm con}$ is relatively fast. The weakness of the radiation intensity, $|\Psi_{\rm con}|\ll 1$, suggests a linear description for $\Psi_{\rm con}$. Let us, however, stress, that, generally, the decomposition (2.3), determined by Eqs. (2.4)–(2.9) for a single soliton (and by analogous relations for the multisoliton case), does not require any smallness of $\Psi_{\rm con}$. The generality of the approach will help us to construct a consistent perturbation theory (which, as we demonstrate below, requires an account for some higher order terms).

An important part of our further analysis will be focused on derivation and solution of a linear (as the radiation is weak) equation for $\Psi_{\rm con}$. The equation gets a rather complex structure, which, generally, requires an accurate, case specific, analysis. However, the asymptotic behavior of $\Psi_{\rm con}$, away from all the solitons, is simple and general, and it is certainly worth discussing it here. Far from solitons the radiation field $\Psi_{\rm con}$ is described by the linear wave equation

$$-i\partial_z\Psi_{\rm con}=\partial_t^2\Psi_{\rm con}. \quad (2.16)$$

Thus, in the asymptotic domain the field $\Psi_{\rm con}$ can be expanded over the set of plain waves $\propto\exp(-ik^2z+ikt)$. In the reference frame, moving with the speed of light through the fiber, a wave packet with the wave vector k propagates along the z axis with the group velocity $2k$. Therefore, the group velocity decays as the wavelength k^{-1} increases. This means, in particular, that short waves arrive first at some remote point.

III. SHEDDING OF RADIATION BY A SINGLE SOLITON

The symmetry of the single-soliton allows reduction in the number of essential degrees of freedom. Since both the Eq. (2.1) and the single-soliton version of the initial condition (2.15) are invariant under time inversion $t\to -t$ neither soliton position y nor its phase velocity β are varying with z. The integral quantity $E=\int dt|\Psi|^2$ (which is also natural to call energy, since it corresponds to the energy of the original electromagnetic field) is conserved. This conservation law is due to the gauge symmetry of Eq. (2.1). The single-soliton version of the conservation law is

$$2\eta+\int dt|v|^2=2. \qquad (3.1)$$

Equation (3.1) gives an instantaneous relation between soliton amplitude and the integral over t of the radiation intensity. The soliton phase, α, although evolving under the action of disorder, does not enter Eq. (3.1). Notice that the relation (3.1) is valid generally, regardless of the relative strength of the two terms on the left-hand side of Eq. (3.1).

The weakness of disorder ($D\ll 1$) is essential for the next two steps.

(1) Linear approximation, reducing calculations directly to account for the leading order in the radiation ξ terms in the basic dynamical equation. We will show below that the direct perturbation expansion is valid at $z\ll 1/D$, where deviations of η from unity are small.

(2) Quasilinear approximation, explaining generalization of the pure linear approximation to the case of moderate- ($z\sim 1/D$) and long- ($z\gg 1/D$) haul transmissions. For such z a cumulative change of the soliton amplitude, η, becomes essential, while the radiation shed is still (as in the linear case) weak at any given position.

Equations for z dependence of the parameters η, β, α, y, a_k, and a_k^* are presented and discussed below separately for the linear ($z\ll 1/D$) and quasilinear ($z\gtrsim 1/D$) cases. An essential part of this analysis (especially complex in the quasinonlinear case) is the proof of the following asymptotic statement: the higher-order terms [ellipses in (2.11)] do not contribute to the leading asymptotic description of the radiation profile v at any $t,z\gg 1$. Notice, however, that some of the higher-order terms have to be taken into account in the asymptotic equations for the soliton parameters.

A. Linear approximation

The linear (first order in ξ) approximation is examined here. Recalling that the parameters α, β, and η (and, also, y, if the soliton is not moving) are z independent in the nodisorder ($\xi=0$) case. One finds that z derivatives of the slow variables are $O(\xi)$ or smaller. The radiation v is also $O(\xi)$, i.e., it is small due to the smallness of ξ. According to the conservation law (3.1), $\eta=1+O(v^2)$, i.e., it can be simply replaced by unity in the approximation. The observations make it simple to linearize Eq. (2.11) with respect to ξ.

Once the linearized equation is found, one can derive equations for the soliton parameters and the expansion coefficients a_k, introduced by Eq. (2.7) by projecting this equation on the respective eigenfunctions of the operator $\hat{L}$ (see Appendix A). Projection on the eigenfunctions of the discrete spectrum gives the following equations for the soliton parameters:

$$\partial_z\alpha=-\xi,\quad \partial_z\eta=0,\quad \partial_z\beta=0,\quad \partial_z y=2\beta, \qquad (3.2)$$

where we used the expansion (2.12). In agreement with what was already discussed, Eq. (3.2) shows that neither y nor β depend on z. Below we put $\beta=0$ in accordance with the initial conditions, and assume $y=0$ (without any loss of generality). Equation (3.2) confirms an already mentioned observation that η does not have z dependence in the first order in ξ. The equation for the continuous spectrum coefficients of the radiation expansion, a_k, derived from Eqs. (2.11),(2.12),(2.13), is

$$\partial_z a_k-i(k^2+1)a_k=b_k\xi, \qquad (3.3)$$

where b_k is defined by Eq. (2.13). The solution of Eq. (3.3) is written as

$$a_k(z)=\int_0^z dz'\,\xi(z')b_k\exp[i(k^2+1)(z-z')]. \qquad (3.4)$$

Substituting Eq. (3.4) into Eq. (2.7) and considering the radiation far away from the soliton (that implies $t\gg 1$) one gets

$$v\approx-\frac{i}{4}\int_0^z dz'\,\xi(z')\exp[-i(z-z')]\mathcal{J}(t,z-z'), \qquad (3.5)$$

$$\mathcal{J}(t,s)=\int dq\frac{(q-i)^2}{\cosh[\pi q/2]}\exp(-iqt-iq^2s). \qquad (3.6)$$

A stationary phase calculation of the integral on the right-hand side of Eq. (3.6) gives

$$\mathcal{J}(t,s)\approx\sqrt{\frac{\pi}{is}}\left(\frac{t}{2s}+i\right)^2\exp\left(i\frac{t^2}{4s}\right)\cosh^{-1}\left(\frac{\pi t}{4s}\right). \qquad (3.7)$$

The asymptotic expression given by Eq. (3.7) is valid at $s\gg 1$.

To describe the space-time dependence of the radiation, we examine the radiation intensity $|v|^2$, averaged over realizations of the disorder ξ, in the asymptotic domain of large z and t, $z,t\gg 1$. Multiplying together two replicas of Eq. (3.5) and averaging the result over disorder, in accordance with Eq. (2.2), one finds

$$\langle |v|^2\rangle = \frac{D}{16}\int_0^z dz' |\mathcal{J}(t,z-z')|^2. \tag{3.8}$$

At $t \gg 1$ one can replace $\mathcal{J}$ in Eq. (3.8) by its asymptotics (3.7).

First, let us consider relatively short time, $t \ll z$. For z' in Eq. (3.8), restricted by $z-z' \gg t$, one gets $|\mathcal{J}|^2 \approx \pi/(z-z')$, resulting in the logarithmic divergence of the integral in Eq. (3.8) at small values of $z-z'$. The divergence is cut at $z-z' \sim t$, leading to the following radiation intensity profile:

$$t \ll z \ll 1/D, \quad \langle |v|^2\rangle \approx \frac{\pi}{16} D \ln\frac{z}{t}. \tag{3.9}$$

Within the domain of the radiation forerunner, defined by $t \gg z$, cosh in Eq. (3.7) can be replaced by its exponential asymptotics. Then, the integral in Eq. (3.8) is formed in the region of the shortest z' allowed in the domain. Calculating the integral explicitly, one derives the following asymptotics for the radiation forerunner profile:

$$z \ll 1/D, \quad z \ll t, \quad \langle |v|^2\rangle \approx \frac{Dt^3}{32z^3}\exp\left[-\frac{\pi t}{2z}\right]. \tag{3.10}$$

The two asymptotic expressions (3.9) and (3.10) match at $z \sim t$.

It is instructive to present a qualitative explanation for the logarithmic profile (3.9). At small k the source of the radiation (localized at the soliton) can be treated as a pointlike one. Therefore waves with the wave vectors $k<1$ are excited by the disorder with approximately equal probability. Nevertheless, they have different group velocities. Among all the waves shed by the soliton (at $t \sim 1$ and $z'<z$) only those special with the wave vector (group velocity) $k \geq t/z$ contribute to $\langle |v(t)|^2\rangle$ at the given z and t. On the other hand, emission of waves with $k>1$ is suppressed. Thus, the main contribution to $\langle |v(t)|^2\rangle$ is proportional to $\int_{t/z}^1 dk/k = \ln(z/t)$, where the $1/k$ factor originates from the group velocity.

We conclude this subsection by establishing the region of validity for the linear approximation explained above. The first, and immediate, consequence of the linear approximation is the smallness of the soliton amplitude degradation. This means that the amount of energy shed by the soliton into radiation is negligible in comparison with the energy still left in the soliton $E_{\rm sol} \approx 2$. According to Eqs. (3.9),(3.10), the average energy shed into the radiation is $E_{\rm rad} = \langle \int dt |v|^2\rangle$. One finds, that the radiation energy is mainly stored in the region separating the logarithmic and the exponential profiles, i.e., $E_{\rm rad} \sim Dz$. Since, according to Eq. (3.1), the overall energy is conserved, one finds that the linear approximation is justified, i.e., $E_{\rm sol} \gg E_{\rm rad}$, if z is essentially shorter than the degradation scale $z_{\rm degr} = 1/D$.

B. Quasilinear approximation

Let us first draw a qualitative picture of what is happening at scales larger than the degradation scale. Once z exceeds $z_{\rm degr} = 1/D$, the balance of energy between the soliton and the radiation shifts towards the radiation. However, the differential (per unit z) release of energy into the radiation remains small and, actually, continues to decrease with z. The radiation emitted by the soliton moves out of the soliton with a speed, fixed by the instantaneous value of the soliton amplitude η at the moment of emission z. Once emitted the radiation never returns back to the soliton, i.e. it does not affect η later (at larger z). Therefore, since the density of radiation was small at the relatively short z, $z \ll 1/D$ (a fact proved in the previous subsection) it cannot increase at larger z, quite the opposite, it may only decrease, i.e., $|v| \ll 1$ at any t and z. This feature of the linear approximation will be, therefore, carried over larger z. The only new ingredient (not considered at shorter z) is accounting for slow degradation of the soliton amplitude with z. Physically, the quasilinear approximation works because the waves shed by soliton leave it, while the soliton travels a distance $\delta z \sim 1/\eta^2$, and the soliton amplitude η does not get any essential change during δz (since $D \ll 1$).

Our first task here is, assuming some given dependence of η on z, to study the radiation profile v. One derives from Eqs. (2.11)–(2.13)

$$\partial_z a_k - i(k^2+\eta^2)a_k = \eta^2 b_{k/\eta}\xi. \tag{3.11}$$

Some terms, originating from the z dependence of η were omitted in Eq. (3.11). This step will be justified below. The solution of Eq. (3.11) is

$$a_k(z) = \int_0^z dz' \xi(z') \eta^2(z') b_{k/\eta(z')} \times \exp\left[ik^2(z-z') + i\int_{z'}^z dz'' \eta^2(z'')\right]. \tag{3.12}$$

Substituting Eq. (3.12) into Eq. (2.7) and considering the radiation away from the soliton ($\eta t \gg 1$) one gets

$$v \approx -\frac{i}{4}\int_0^z dz' \xi(z')\eta^3(z') \times \exp\left[-i\int_{z'}^z d\zeta \eta^2(\zeta)\right]\mathcal{J}(\eta(z')t, \eta^2(z')(z-z')), \tag{3.13}$$

where the function $\mathcal{J}$ is defined by Eq. (3.6).

Equation (3.13) is fundamental for further calculation of both η's dependence on z, and the average radiation intensity profile dependence on t and z. (The following two subsections are devoted specifically to the two aforementioned subjects.) However, it is very important to justify beforehand the validity of those few but crucial assumptions made in the course of derivation of Eq. (3.13) from Eq. (2.11). The rest part of the present subsection is devoted to this task.

The key question here is could some small terms in Eq. (2.12), neglected in the course of derivation of Eq. (3.13), be accumulated at large z? The major result here will be a negative answer to the question. To prove this general validity of Eq. (3.13) one divides the entire t domain into two distinct regions, of a τ-wide soliton vicinity, $\tau\gg 1/\eta$, and the rest (remote region of t). The two regions will be considered separately. First, the validity of Eq. (3.13) should be proved for t from the box $[-\tau,\tau]$. Then, in the second step, one should take into account a term, omitted in the derivation of Eq. (3.13), originating from a z dependence (via η) of the eigenfunctions φ_k and $\overline{\varphi}_k$, in Eq. (2.7).

The generalized version of Eq. (3.11) accounting for the dangerous term, is

$$\partial_z a_k - i(k^2+\eta^2)a_k + \hat{A}a_k = \eta^2 b_{k/\eta}\xi,$$

where $\hat{A}$ is a linear nonlocal over k nonsingular operator, estimated by, $\hat{A}\sim\partial_z\eta/\eta$. Assuming that the $\hat{A}$ correction is small, one arrives at the following modification of Eq. (3.13)

$$v\approx -\frac{i}{4}\int_0^z dz'\,\xi(z')\,\eta^3(z')\exp\left[-i\int_{z'}^z dz''\,\eta^2(z'')\right]\times\left[1+\int_{z'}^z dz''\hat{A}\right]\mathcal{J}[\eta(z')t,\eta^2(z')(z-z')]. \quad (3.14)$$

For $|t|\leq\tau$ integration over z' from the right-hand side of Eq. (3.13) is formed at $z-z'\sim\tau/\eta$. Therefore, correction to the integrand of Eq. (3.13) due to the $\hat{A}$ term in Eq. (3.14) is estimated by

$$(z-z')\hat{A}\sim\frac{\tau}{\eta^2}\partial_z\eta\sim D\tau\eta^3, \quad (3.15)$$

where one substitutes the law (1.1), announced in the Introduction and derived in the next subsection. The correction (3.15) is small provided $\tau\ll D^{-1}\eta^{-3}$. The later inequality is obviously compatible with the only restriction we have imposed so far on the size of the box $\tau\gg\eta^{-1}$.

Next, we discuss the region of remote t, $|t|>\tau$, where the soliton part of the solution Ψ is negligible, while $\Psi_{\rm con}$ satisfies the linear wave equation (2.16). One can find $\Psi_{\rm con}$ outside the box by solving Eq. (2.16) with proper boundary conditions, where $\Psi_{\rm con}(\pm\tau)$ was determined in the previous step, and it is also assumed that the radiation only escapes the τ box but never reenters. Fortunately, the result of this procedure coincides with Eq. (3.13). Indeed, it is straightforward to check that $\Psi_{\rm con}$ related to v via the phase factor change (2.6), satisfies the linear equation (2.16), if v is given by Eq. (3.13). It is also seen from Eq. (3.7), that v contains only waves leaving the τ box. All this proves that there are no essential corrections to Eq. (3.13) originating from the domain of the remote t.

C. Degradation law for soliton amplitude

The energy balance between the soliton and the radiation controls the law of the soliton amplitude decay with z. From the basic equation (2.1) one gets

$$\partial_z|\Psi|^2 = id(z)\partial_t(\Psi^*\partial_t\Psi-\Psi\partial_t\Psi^*). \quad (3.16)$$

This equation describes dynamics of the energy density $|\Psi|^2$, and leads to the conservation law (3.1). Integrating Eq. (3.16) over the τ-wide box, introduced in the previous subsection, one obtains a relation between the amount of energy shed by soliton and the flux of energy coming through the boundaries of the box. We choose τ to be large enough so that the t-integral of $|\Psi|^2$ gets the major contribution from the soliton itself, and is equal to 2η. The integral of the right-hand side of Eq. (3.16) is reduced to two boundary terms at $t=\pm\tau$. At the boundaries one can replace Ψ by $\Psi_{\rm con}$ and also replace ξ by zero. The result is

$$\partial_z\eta(z) = i(v^*(z,\tau)\partial_\tau v(z,\tau)-v(z,\tau)\partial_\tau v^*(z,\tau)). \quad (3.17)$$

We show below that the dependence of η on z can be established from Eq. (3.17) with its right-hand side replaced by its average value. Performing this averaging (over the statistics of disorder) one arrives at

$$\partial_z\eta = \frac{iD}{8}\int_0^z dz'\,\eta^6(z')\mathcal{I}^*\partial_\tau\mathcal{I}, \quad (3.18)$$

where $\mathcal{I}(z,t)=\mathcal{J}[\eta(z')\tau,\eta^2(z')(z-z')]$, and the function $\mathcal{J}$ is defined by Eq. (3.6). In Eq. (3.18) the function can be approximated by its asymptotic expression (3.7), resulting in

$$\partial_z\eta = -\frac{\pi D}{8}\int_0^z dz'\frac{\tau\eta^4(z')}{(z-z')^2}\frac{(\zeta^2+1)^2}{\cosh^2(\pi\zeta/2)},$$

where $\zeta=\tau/[\eta(z')(z-z')]$. The integral over z' in the expression is formed at $z-z'\sim\tau/\eta$. The size of the box τ can be chosen to be much smaller than ηz (if $z\gg 1$). Then, for relevant z', $z-z'\ll z$, and $\eta(z)$ can be substituted for $\eta(z')$. Passing from z' to the integration variable ζ and extending the integration region over ζ down to 0 (this is possible since $\tau/z\ll 1$) one gets

$$\partial_z\eta = -\frac{\pi D}{8}\eta^5\int_0^\infty d\zeta\frac{(\zeta^2+1)^2}{\cosh^2(\pi\zeta/2)} = -\frac{8D}{15}\eta^5. \quad (3.19)$$

Integration of the differential equation (3.19) gives the final result for the degradation law (1.1) announced in the Introduction.

The law of the soliton decay given by Eq. (1.1) is deterministic in spite of the randomness of the initial setup described by Eq. (2.1). This remarkable fact is due to the self-averaged feature of η. The rest part of this section is devoted to the proof of this statement: we demonstrate below that deviation of η (for a given realization of the disorder ξ) from its average value is small.

To establish statistical properties of η we turn to the auxiliary quantity,

$$\mathcal{V}(z)\equiv i(v^*(z,\tau)\partial_\tau v(z,\tau)-v(z,\tau)\partial_\tau v^*(z,\tau)), \quad (3.20)$$

extracted from the right-hand side of Eq. (3.17). The irreducible pair correlation function (cumulant) of $\mathcal{V}$,

$$\mathcal{K}(z_1,z_2)=\langle\mathcal{V}(z_1)\mathcal{V}(z_2)\rangle-\langle\mathcal{V}(z_1)\rangle\langle\mathcal{V}(z_2)\rangle, \qquad (3.21)$$

is presented, according to Eqs. (2.2),(3.13), as a double integral over $z'_{1,2}$. One examines Eq. (3.21) at large values of $z_1>z_2, \eta\tau, z_{1,2}, \eta\tau\gg 1$, and also assumes that the two inequalities $z_1-z_2\gg\eta^{-2}\tau$ and $(z_1-z_2)\partial_z\eta\ll 1$ are valid. Then, using Eq. (3.7), one finds

$$|\mathcal{K}|<D^2\eta^5\frac{\tau^5}{(z_1-z_2)^2}, \qquad (3.22)$$

where the phase factor $t^2/4s$ in Eq. (3.7) was dropped. [An account for the phase would decrease the value of the right-hand side in Eq. (3.22), thus turning the inequality (3.22) into equality.] Integrating Eq. (3.22) over some z_0-wide vicinity of $z=z_1$, one derives

$$\int_{z-z_0}^{z}dz'|\mathcal{K}(z,z')|<D^2\eta^9\tau, \qquad (3.23)$$

where $z_0\gg\tau/\eta$. Evaluating the inequality (3.23) further, one gets

$$\left\langle\left[\int_{z-z_0}^{z}dz'\mathcal{V}(z')\right]^2\right\rangle\left\langle\int_{z-z_0}^{z}dz'\mathcal{V}(z')\right\rangle^{-2}-1<\frac{\tau}{\eta z_0}\ll 1.$$

The integral $\Delta\eta\equiv\int_{z-z_0}^{z}dz'\mathcal{V}(z')$ determines variations of $\eta(z')$ for z' from the interval bounded by $z-z_0$ and z. We established that fluctuations of $\Delta\eta$ are weak. On the other hand, we are free to choose such z_0 that $\Delta\eta\ll\eta$. To conclude, evolution of η can be described in terms of the deterministic equation (3.19).

D. Average radiation

This subsection is devoted to derivation of the average radiation intensity profile from Eqs. (1.1),(2.2),(3.13). We examine it in the asymptotic domain of large z and t, $z,t\gg 1$. Averaging the radiation intensity $|v|^2$ in accordance with Eq. (2.2) one finds

$$\langle|v|^2\rangle=\frac{D}{16}\int_0^z dz'\,\eta^6|\mathcal{J}[\eta t,\eta^2(z-z')]|^2, \qquad (3.24)$$

where $\eta=\eta(z')$, and $\mathcal{J}$ is defined by Eq. (3.6).

The radiation profile at $z\gg 1/D$ gets a more complicated structure than in the domain of short z, $z\ll 1/D$, studied above in Sec. II A. Using the asymptotic expression (3.7) for the auxiliary function $\mathcal{J}$ and substituting Eq. (1.1) into Eq. (3.24) one derives

$$\langle|v|^2\rangle=\frac{15\pi}{512}\int_0^z\frac{dz'}{(z-z')[z'+(15/32)D^{-1}]}\times\left[\frac{t^2}{4\eta^2(z-z')^2}+1\right]^2\cosh^{-2}\left[\frac{\pi t}{4\eta(z-z')}\right]. \qquad (3.25)$$

Analysis of this expression shows that there are three different asymptotic domains of t for any given z:

(a) $t\ll[z^3/D]^{1/4}$ and $zD\gg 1$,

(b) $[z^3/D]^{1/4}\ll t\ll z$ and $zD\gg 1$,

(c) $t\gg z$ and $zD\gg 1$.

In domain (a) two different asymptotic regions of z', $1/D\ll z'\ll z$, and $t/\eta\ll z-z'\ll z$, give the major contribution to the integral on the right-hand side of Eq. (3.25). Collecting the major logarithmic terms, one obtains

$$\text{(a)}\quad\langle|v|^2\rangle=\frac{15\pi}{512z}\ln\frac{D^{3/4}z^{7/4}}{t}. \qquad (3.26)$$

In domain (b) the major contribution is coming from the $1\ll Dz'\ll(z/t)^4$ region of the z' integration in Eq. (3.25), leading to

$$\text{(b)}\quad\langle|v|^2\rangle=\frac{15\pi}{128z}\ln(z/t). \qquad (3.27)$$

And finally, at $t\gg z$ the integral in Eq. (3.25) is formed at $Dz'\lesssim z/t$, where cosh can be replaced by its exponential asymptotics. This leads to

$$\text{(c)}\ \langle|v|^2\rangle=\frac{15t^3}{256z^4}\exp\left(-\frac{\pi t}{2z}\right). \qquad (3.28)$$

Now that all the asymptotics (for relatively short z in the previous subsection, and for long z here) have been presented, let us describe a general picture of the radiation distribution. The radiation front runs out of the soliton with constant speed, $t/z\sim 1$. A logarithmic profile is formed behind the front, while the radiation forerunner decays exponentially with $t/z\gg 1$. The energy of the radiation is contained mainly in the boundary region between the logarithmic and the exponential profiles. At $z\ll 1/D$, the logarithmic profile (3.9) is simple, and the preexponential factor depends on D, as it seen from Eq. (3.10). At larger z, $z\gg 1/D$, when the soliton has already shed almost all its energy into the radiation, the logarithmic profile splits into two parts described by Eqs. (3.26) and (3.27), respectively, and the exponential asymptotics is modified to Eq. (3.28). Regime (a) is formed by the waves with $k<\eta$ emitted continuously at different z', whereas regimes (b) and (c) are formed by the "fast" waves, emitted at z' far from the observation point z. The boundary between regimes (a) and (b) is determined by the condition $t\sim\eta z$ (that is the "distance" passed by waves with $k\sim\eta$). The profile in regime (a) knows about the current amplitude η of the soliton, whereas in regimes (b) and (c) the radiation is insensitive to the current value of η. Note that the universal profile, formed in the regions (b) and (c), does not depend on the intensity of the disorder D and the only information stored in the asymptotics is about the initial soliton profile. The universal profile (b), (c) is self-similar: $\langle|v^2|\rangle=z^{-1}\Phi(t/z)$. At first sight, this type of self-similarity $t\sim z$ seems to contradict the asymptotic equation (2.16). This confusion has a simple resolution. The main dependence of v on t is associated with its phase, which, as it is seen from Eq. (3.7), has a normal kind of self-similarity $z\sim t^2$. However, the phase drops from the absolute value $|v|^2$, so that the self-similarity of the latter object is determined by the subleading $\sim 1/z$ terms in the eikonal approximation. Notice, that the phase (normal) self-similarity will be seen in the two soliton interaction discussed in the next section.

IV. INTERACTION OF TWO SOLITONS

Propagation of a two-soliton pattern at moderate z, $1\ll z\ll D^{-1}$, is discussed in this section. As was shown in Sec. III, dynamics of a single pulse within the range of scales bounded from above by the degradation scale D^{-1} is trivial: y and β do not evolve, while the change of the soliton amplitude η is $O(zD)$, i.e., negligible. The major observation following from our analysis here is that the intersoliton separation y_2-y_1 coupled to the phase velocities $\beta_{1,2}$ of the two solitons, gets a nontrivial dynamics at scales much shorter than the single soliton degradation scale D^{-1}. We show that the intersoliton interaction mediated by disorder is essential at shorter scales. The soliton parameters $\beta_{1,2}$ are $O(\Psi_{\rm con}^2)$, while $\Psi_{\rm con}$ itself is $O(\xi)$. Therefore, we divide our analysis into the following sub-steps. First, the radiation $\Psi_{\rm con}$ will be related to ξ in the linear approximation. Second, $\beta_{1,2}$, and then $y_{1,2}$, will be presented as a second order form in $\Psi_{\rm con}$. Finally, we calculate statistics of the forces acting on the solitons and, therefore, explain soliton jitter.

A. Radiation generated by two solitons

We consider the $N=2$ case of the general setting (2.3),(2.14) when the solitons are well separated, that is $y=y_2-y_1\gg 1$ ($y_2>y_1$ is assumed). At $z\ll D^{-1}$ one can substitute $\eta_1=\eta_2=1$, and the localized part of Ψ (2.14) is reduced to

$$\Psi_{\rm sol}=\frac{e^{i\alpha_1+iz+i\beta_1(t-y_1)}}{\cosh(t-y_1)}+\frac{e^{i\alpha_2+iz+i\beta_2(t-y_2)}}{\cosh(t-y_2)}. \tag{4.1}$$

The delocalized part $\Psi_{\rm con}$ of the complete solution (2.3) of Eq. (2.1) is built according to the general scheme outlined in Sec. II.

As in Eq. (2.6), one introduces an auxiliary radiation field v, $v=\Psi_{\rm con}\exp(-i\alpha_1-iz)$, accounting for the phase shift of the soliton, positioned at y_1. The field v can be written in the form of the expansion (2.7) over the continuous spectrum eigenfunctions of an auxiliary perturbation problem. The auxiliary problem is fixed by the operator $\hat{\mathcal{L}}$, which is a two-soliton generalization of the single-soliton operator (2.9). With exponential (with respect to the separation $y=y_2-y_1$) accuracy, the differential operator $\hat{\mathcal{L}}$ is $\hat{\mathcal{L}}=\hat{L}(t-y_1)$ at $t<(y_1+y_2)/2$ and $\hat{\mathcal{L}}=\hat{L}_\alpha(t-y_2)$ at $t>(y_1+y_2)/2$. Here $\alpha=\alpha_2-\alpha_1$ is the phase mismatch, $\hat{L}$ and $\hat{L}_\alpha$ are defined in Appendix A by Eqs. (A3),(A13). We adopt the same general notations, φ_k, $\overline{\varphi_k}$ for the continuous spectrum eigenfunctions of $\hat{\mathcal{L}}$, i.e., $\hat{\mathcal{L}}\varphi_k=(k^2+1)\varphi_k$, $\hat{\mathcal{L}}\overline{\varphi_k}=-(k^2+1)\overline{\varphi_k}$. The eigenfunctions are fixed by their asymptotic behavior at large negative t:

$$t\to-\infty\,\varphi_k\to\left(\frac{k-i}{k+i}\right)^2\exp(ikt-iky_1)\begin{pmatrix}0\\1\end{pmatrix}. \tag{4.2}$$

Then, with exponential accuracy, $\varphi_k=f_k(t-y_1)$ if $t<y_2$ and

$$\varphi_k(t)=\frac{(k+i)^2}{(k-i)^2}\exp(iky+i\alpha)f_{\alpha,k}(t-y_2)$$

if $t>y_1$. Here $y=y_2-y_1$ and the functions f_k, $f_{\alpha,k}$ are defined by the expressions (A6),(A7),(A14). In the transient region $1\ll t-y_1$, $y_2-t\gg 1$, the two asymptotics of φ_k, presented above, coincide. One should also add $\overline{\varphi_k}=\hat{\sigma}_1\varphi_k^*$ to the set of eigenfunctions to make it complete. The orthogonality properties of φ_k, $\overline{\varphi_k}$ are identical to the ones given by Eqs. (A11).

The linear equation for v follows from direct expansion of the basic equation (2.1),

$$\partial_z\begin{pmatrix}v\\v^*\end{pmatrix}-i\hat{\mathcal{L}}\begin{pmatrix}v\\v^*\end{pmatrix}+\ldots=g\xi, \tag{4.3}$$

$$g=i\left[\frac{2}{\cosh^3(t-y_1)}-\frac{1}{\cosh(t-y_1)}\right]\begin{pmatrix}1\\-1\end{pmatrix}+i\left[\frac{2}{\cosh^3(t-y_2)}-\frac{1}{\cosh(t-y_2)}\right]\begin{pmatrix}e^{i\alpha}\\-e^{-i\alpha}\end{pmatrix}. \tag{4.4}$$

Here ellipses stand for terms corresponding to the localized modes, and $\hat{\mathcal{L}}$ was already introduced above. Substituting the decomposition (2.7) into Eq. (4.3) and expanding its right-hand side over the eigen-functions of the operator $\hat{\mathcal{L}}$, one gets

$$\partial_z a_k-i(k^2+1)a_k=B_k\xi, \tag{4.5}$$

$$B_k=b_k\left[1+\frac{(k-i)^2}{(k+i)^2}e^{-iky-i\alpha}\right], \tag{4.6}$$

where b_k are defined by Eq. (2.13). [In the derivation we did not account for a z dependence of φ_k, since $\partial_z\varphi_k=O(\xi)$.] The solution of Eq. (4.5) is

$$a_k(z)=\int_0^z dz'\,\xi(z')\exp[i(k^2+1)(z-z')]B_k, \tag{4.7}$$

analogously to Eq. (3.4).

In the linear approximation over ξ, the soliton's parameters can be examined in the framework of the same Eqs. (4.3),(4.4). The resulting equations for the soliton parameters are

$$\partial_z\alpha_{1,2}=-\xi,\quad \partial_z\beta_{1,2}=0,\quad \partial_z y_{1,2}=2\beta_{1,2}, \tag{4.8}$$

similarly to Eq. (3.2). Note, that according to Eqs. (4.8), $\partial_z(\alpha_2-\alpha_1)=0$, i.e., the phase mismatch $\alpha=\alpha_2-\alpha_1$ is independent of z.

B. Evolution of soliton parameters

As follows from Eqs. (4.8), the soliton parameters $y_{1,2}$ and $\beta_{1,2}$ do not get any z dependence in the first order in v. One expects that in the second order, the β equations acquire some nonzero contribution, so that $\partial_z\beta_{1,2}\sim|v|^2$. Then, according to the y equations [the last ones in Eq. (4.8)] fluctuations of the separation $y=y_2-y_1$ are estimated by $z^2|v|^2$, and can be significant in the interesting range of scales $z \ll D^{-1}$. The estimations also show that higher order $O(v^3)$ corrections to the equations for $\beta_{1,2}$ are not essential. Further, it is easy to check that the equations for $\beta_{1,2}$ contain phases $\alpha_{1,2}$ only in the combination, $\alpha=\alpha_2-\alpha_1$. According to the first equation in Eq. (4.8), α does not evolve in the first order in ξ, while second order correction to the equation for α is inessential in the range of z, $z\ll 1/D$. To conclude, the only thing left to be studied is the second order in v contributions to the equations for $\beta_{1,2}$.

To find the contribution, one expands the basic equation (2.1) up to the second order in v,

$$\partial_z\begin{pmatrix}\Psi\\ \Psi^*\end{pmatrix}=\ \ldots+i\xi\partial_t^2\begin{pmatrix}\Psi_{\rm con}\\ -\Psi^*_{\rm con}\end{pmatrix}+2i\begin{pmatrix}2|\Psi_{\rm con}|^2 & \Psi^2_{\rm con}\\ -(\Psi^*_{\rm con})^2 & -2|\Psi_{\rm con}|^2\end{pmatrix}\begin{pmatrix}\Psi_{\rm sol}\\ \Psi^*_{\rm sol}\end{pmatrix}, \tag{4.9}$$

where ellipses stand for the first-order terms. Extracting terms, proportional to $\partial_z\beta_1$, $\partial_z y_1$ from the left-hand side of Eq. (4.9) and making the respective projections one arrives at

$$\partial_z\beta_1=\mathcal{F}(z)=\mathcal{F}_{vv}(z)+\mathcal{F}_{\xi v}(z)+\mathcal{F}_{\xi\alpha}(z), \tag{4.10}$$

$$\mathcal{F}_{vv}=\int dx\frac{\tanh x}{\cosh^2 x}[4|v|^2+v^2+(v^*)^2], \tag{4.11}$$

$$\mathcal{F}_{\xi v}=\xi\,\mathrm{Re}\int dx\frac{\tanh x}{\cosh x}\partial_x^2 v, \tag{4.12}$$

$$\mathcal{F}_{\xi\alpha}=-\partial_z\alpha_1\,\mathrm{Re}\int dx\frac{\tanh x}{\cosh x}v, \tag{4.13}$$

$$\partial_z y_1=2\beta_1+\mathcal{P}_1, \tag{4.14}$$

$$\mathcal{P}_1=i\int\frac{dx\,x}{\cosh^2 x}[v^2-(v^*)^2], \tag{4.15}$$

where $x=t-y_1$. For completeness, we calculated the second-order term in the equation for y_1, which in Eq. (4.14) is added to the first-order one. Expressions for the soliton positioned at $t=y_2$, can be obtained in a similar way. Using a mechanical analogy, one can call β momentum of the soliton. Then $\mathcal{F}$ is the force acting on the soliton, and $\mathcal{P}_1$ is an additional impulse.

One is interested describing fluctuations (statistics) of y_1 as a function of z, assuming that the intersoliton separation $y=y_2-y_1$ is much larger than unity, but much less than z. Integrating Eqs. (4.10),(4.14), we obtain

$$\delta y_1=\int_0^z dz'(2\beta_1+\mathcal{P}_1),\quad \beta_1=\int_0^z dz'\mathcal{F}(z_2). \tag{4.16}$$

According to the central limit theorem [23] at large z, $\beta_{1,2}$, and $y_{1,2}$, as z integrals of random functions, are Gaussian random processes. This Gaussianity allows us to estimate fluctuations of various quantities (about respective average values) for particular realization of the disorder: $|\delta y_1|$ fluctuates about $\langle(\delta y_1)^2\rangle^{1/2}$ with the same amplitude $\langle(\delta y_1)^2\rangle^{1/2}$.

The main contribution to δy_1 is related to the force $\mathcal{F}$. As it is shown in Appendix B, the average value of $\mathcal{F}$ is negligible [more accurately it is exponentially small in y, $\sim\exp(-y)$ and vanishes algebraically with $z\to\infty$]. This fact (lack of a $\sim D$ contribution into the average value of the force $\mathcal{F}$) is a consequence of the reflectionless feature of the soliton radiation. Thus, fluctuations of β_1 are controlled by the pair correlation function of $\mathcal{F}$ (calculated in detail in Appendix B). The main contribution to the correlation function is

$$\langle\mathcal{F}(z)\mathcal{F}(z')\rangle=D^2G\,\delta(z-z'), \tag{4.17}$$

where G is given by the integral (B27), $G\approx0.14$. One therefore obtains from Eqs. (4.16),(4.17)

$$\langle\beta_1^2(z)\rangle=D^2Gz,\quad \langle(\delta y_1)^2\rangle=\frac{4}{3}GD^2z^3. \tag{4.18}$$

One concludes, that the typical shift of the soliton position (counted from its initial value at $z=0$) is estimated as $\delta y_1\sim Dz^{3/2}$. The soliton leaves its slot (in the soliton pattern), i.e., δy becomes $O(1)$, at $z\sim D^{-2/3}$. Since $D\ll1$, this happens well before the soliton amplitude acquires any significant reduction, therefore justifying our approximation.

Note that the average of the impulse $2\int dz\mathcal{F}+\mathcal{P}_1$ is equal to $2D/3$ (see Appendix B). This implies a systematic drift $2Dz/3$ in y_1. This drift is negligible in comparison with the fluctuating part of y_1, $\delta y_1\sim Dz^{3/2}$, at $z\gg1$.

It is also of interest to examine the relative motion of the solitons. One finds that the cross correlation term of the forces is dependent on the solitons phase mismatch α. It results in the following expression for the fluctuations of the relative position $y=y_2-y_1$ (see Appendix B for details of the derivation):

$$\langle(\delta y)^2\rangle=\frac{8[1+\cos(2\alpha)]}{3}D^2Gz^3. \tag{4.19}$$

Substituting the approximate value of G found in Appendix A, $G\approx0.14$, one arrives at Eq. (1.2). Equation (4.19) shows that fluctuations of the intersoliton separation are sensitive to the phase mismatch (e.g., the fluctuations are strongly suppressed at $\alpha=\pi/2$).

V. MULTISOLITON CASE

Let us discuss the effect of soliton interaction in a multisoliton pattern. The reflectionless feature of the radiation guarantees lack of radiation screening. In other words, all solitons positioned on distances $\lesssim z$ from a given soliton are affected by the radiation shed by the soliton. Therefore, the radiation v in a vicinity of a soliton is determined by a superposition of single-soliton radiative contributions, which differ only by shifted phases from the two-soliton case. Each of the contributions is weakly dependent on the intersoliton separation, provided the separation between the solitons is less than z (then the analysis, similar to one explained in Appendix B, is applicable). To conclude, the force acting on a single soliton should grow with the number of solitons N affecting the given soliton through emitted radiation.

To obtain quantitative conclusions, one has to extend the analysis of Appendix B to the multisoliton case. Average force, applied to a soliton, vanishes. [This is valid at large z and if the exponential, in y, corrections, $\sim\exp(-y)$, are not taken into account.] Fluctuations of y_i,β_i are Gaussian again (due to the central limit theorem). One finds that the pair correlation function of the force acting on a soliton (and also the pair correlation function of the given soliton position shift) is $\propto N$. Notice also, that as in the two-soliton case, forces acting on the solitons, and, consequently, their mutual shifts, are sensitive to the relative phases of all the N solitons. However, unlike in the two-soliton case, it is impossible simply adjusting phases to suppress fluctuations of all the intersoliton separations.

One concludes that in the multisoliton case Eqs. (4.18) for the velocity and the soliton position get an extra factor N on their right hand sides. If the information rate in a fiber is fixed, N grows linearly with z, i.e., δy is estimated by $\sim\sqrt{\mu}Dz^2$, where μ is the number of solitons per unit length of the fiber.

VI. PERIODICALLY PINNED NOISE

A new method of periodical "pinning" of disorder was suggested recently [16,17]. This method comes in two modifications of "distributed" and "point" pinning. "Distributed pinning" applies to new fiber lines (not yet installed in the ground). The method requires controlling the integral dispersion (its fluctuating part) of a fiber piece prior to its connection to the line. A profile of the integral of the fluctuating part of the dispersion coefficient should be found, first, and then the suggestion is to cut this fiber at a zero point for the fluctuating part of the integral dispersion (closest to the end of the fiber piece). The other type of pinning, "point" pinning, was suggested for implementation in already installed fiber optics lines. At the points of access to the fiber optics line (at amplifier stations, placed periodically, or quasiperiodically along the fiber) it is suggested to measure the integral of the fluctuating part of dispersion, and then to compensate it to zero by inserting a small peace of a fiber with a very well controlled integral dispersion. If the pinning period, l is short (i.e., if it is the shortest scale in the problem $l\ll 1$), the coarse-grained dynamics of Ψ at the larger scales $z\gg 1$ is described by Eq. (2.1) with the noise term ξ replaced by $\tilde{\xi}$ described by

$$\langle\tilde{\xi}(z_1)\tilde{\xi}(z_2)\rangle=-\frac{Dl^2}{12}\delta''(z_1-z_2). \tag{6.1}$$

($\tilde{\xi}$ actually corresponds to the "distributed pinning" case, while in the case of the "point pinning" the replacement should be $\xi\to 2\tilde{\xi}$.)

Recalculation of all the major results of the paper for the pinned noise (6.1) is straightforward. First of all, one gets

$$\partial_z\eta=\frac{Dl^2}{96}\int_0^z dz'\,\eta^4(z')\times\left|\int dq\frac{(q/\eta+i)^2(q^2+\eta^2)}{\cosh[\pi q/(2\eta)]}e^{iq\tau+iq^2(z-z')}\right|^2 \tag{6.2}$$

instead of Eq. (3.18). Calculating the integrals over k and q in Eq. (3.18) and integrating the resulting equation one arrives at

$$\eta=\left(1+\frac{2^{10}Dl^2z}{315}\right)^{-1/8}. \tag{6.3}$$

This expression, contrasted against Eq. (6.1), shows an essential suppression in the soliton decay law at large z, $z\gg 1/D$. At moderate z, $z\ll 1/D$, Eq. (6.1) and Eq. (6.3) show the same (linear with z) law of the soliton amplitude decay, so that the difference is only in the decay rate. One finds, however, that at this small z the major effect of pinning is in the soliton jitter due to intersoliton interaction. We derive that the pinned case version of Eq. (4.17) is

$$\langle\mathcal{F}(z)\mathcal{F}(z')\rangle=D^2\tilde{G}\delta''(z-z'), \tag{6.4}$$

where $\tilde{G}$ is defined by the right-hand side of Eq. (B26) with an additional factor $(1+k^2)^2(1+q^2)^2$ introduced in the integral, so that the pinned version of Eq. (B27) [with $(1+k^2)^4$ in the integrand replaced by $(1+k^2)^8$] gives $\tilde{G}\approx 21.03$. Thus, an analog of Eq. (4.18) becomes $\langle\beta_1(z)\beta_1(z')\rangle=D^2\tilde{G}\delta(z-z')$ and $\langle(\delta y_1)^2\rangle=D^2\tilde{G}z$. This shows that the jitter of the soliton position is essentially suppressed if pinning is applied.

VII. DIRECT NUMERICAL SIMULATION

We discuss here direct numerical simulations of the one- and two-soliton patterns. The major numerical problem here is due to the long haul (large z) nature of the transmission. The radiation moves away from the soliton pattern and eventually hits the boundaries of the computational domain, which, in reality, cannot be infinite. Therefore, it is important to design a numerical method which allows the radiation not to retract from the boundaries, but instead to evolve like it would not feel the artificial boundaries. The problem of numerically absorbing boundary conditions design is one of the typical computational problems in wave-type equations, and numerous efforts have been made to overcome these numerical artifacts [24–27]. A common approach, widely used to overcome the numerical problem, is to apply an artificial damping at the vicinity of edges to suppress the radiation in the far region. However, during evolution of the soliton, the transmission and reflection of waves takes place simultaneously. In other words, damping, inevitably creates a parasite back refraction of waves.

We solve this problem in another way. Namely, we introduce boundary conditions that the reflectionless feature of the artificial boundaries is controlled analytically. The only, but crucial, assumptions of the approach is that the intensity of the signal at the boundaries of the computational domain is low enough, so that one can linearize the basic Eq. (2.1) there. Let us consider regions $|t|\gg 1$ where one should observe the radiation going away from the solitons. In this region one can use the equation

$$(i\partial_z+\partial_t^2)\Psi=0, \tag{7.1}$$

which is just the linear Schrödinger equation (without potential). The radiative boundary conditions, imposed on a solution of Eq. (7.1) at the boundaries of the computational domain $t=\pm T$ can be written as

$$-i\partial_t\Psi(z,T)=\sqrt{i\partial_z}\Psi(z,T),$$

$$-i\partial_t\Psi(z,-T)=-\sqrt{i\partial_z}\Psi(z,-T), \tag{7.2}$$

where $\sqrt{i\partial_z}$ is a nonlocal (integral) operator

$$\sqrt{i\partial_z}\Psi\equiv\sqrt{\frac{i}{\pi}}\int^z\frac{dz_1}{\sqrt{z-z_1}}\partial_z\Psi(z_1).$$

(The condition $T\gg 1$ should also be satisfied.) Notice, that a similar scheme for the transient linear Schrödinger equation with a potential bounded in a finite domain was suggested in Ref. [27]. Furthermore, for the one-dimensional NLSE, the transparent boundary conditions have been discussed and introduced in several articles from various application fields (see Ref. [28,29]).

Implementing this transparent boundary condition with a symplectic scheme for NLSE, we examined, first, degradation of single soliton, and then, interaction of two solitons caused by fluctuations of the dispersion coefficients. We use a standard random number generator to produce a Gaussian zero-mean random process correlated at $z_{\rm var}$ with amplitude $d_{\rm var}$. Choosing small $z_{\rm var}$ ($z_{\rm var}$ is 0.05 in our numerical experiments) we guarantee that the numerical random process approximates the zero mean δ-correlated uniform noise for ξ described by $\langle\xi(z_1)\xi(z_2)\rangle=D\delta(z_1-z_2)$, with $D=d_{\rm var}^2 z_{\rm var}$. The results of this numerical simulation are shown in the figures.

Figure 1 shows dependence of the soliton amplitude on z, with the strength of disorder D equal to 0.0225. (D is chosen to be a small number to allow a quantitative comparison with

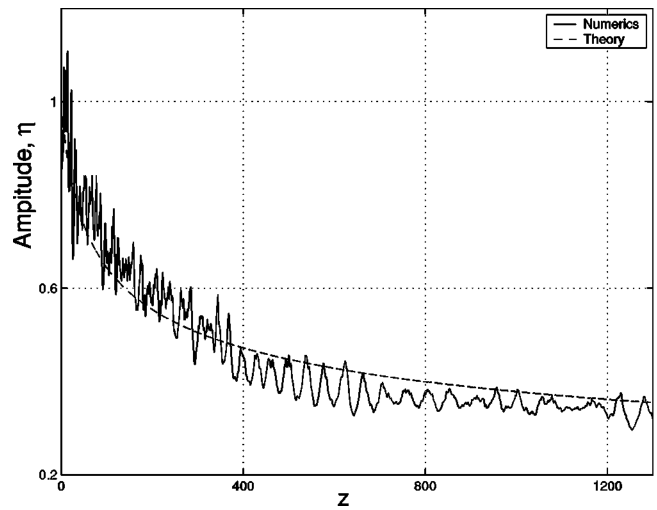

FIG. 1. Dependence of the soliton amplitude, measured in units of its initial value, on the dimensionless coordinate along the fiber z, is shown for disorder strength $D=0.0225$. Solid and dashed curves represent theory, resulted in Eq. (1.1), and numerics for a representative realization of the disorder, respectively.

the asymptotic theory, valid at $D\ll 1$.) Solid and dashed curves represent theory, resulted in Eq. (1.1), and numerics for a representative realization of the disorder respectively. Good quantitative agreement between the theory and numerics is reported over an extremely long range of z.

We also perform a numerical study of two-soliton interaction. Notice that the two-soliton case requires an accurate numerical definition of the soliton position at any given z. Since the soliton amplitude only weakly deviates from unity, the position of a soliton was found by minimizing $\Sigma_i[|\Psi(t_i,z)|-1/\cosh(t_i-y)]^2$, where i numbers the temporal grid points in a vicinity of a special point, where $|\Psi(t_i,z)|$ reaches its maximum. Figure 2 shows dependence of the dispersion in the intersoliton separation fluctuations $\langle(\delta y)^2\rangle$ on z at the phase mismatches $\alpha=0,\pi/4,\pi/2$. Our averaging is done over 15 realizations (for each α). Numerical curves are solid, dashed curves correspond to theoretical predictions of Eq. (1.2). The strength of the disorder is chosen to be much smaller here than in the single soliton numerics $D=0.0125^2$ on purpose. We aimed to separate $r_{\rm degr}=1/D$ and the interaction scale $r_{\rm int}\sim D^{-2/3}$, as much as we can to be able to study the intersoliton dynamics of the solitons with bare (nonperturbed) shape ($\eta=1$) at $z\sim z_{\rm int}$. The initial separation $y(0)$ was chosen to be large enough [$y(0)=20$ for the data shown in Fig. 2] to avoid interference of the effects driven by disorder in the dispersion coefficient with the direct interaction of solitons. (The direct effect gives a subleading, exponentially small in y, correction [30,31].) The figure shows good agreement between our theory and the numerics. To illustrate the δy statistics we show 15 ($\alpha=0$) realizations of δy in Fig. 3. For comparison, the root-mean-square displacement is also shown in the figure.

VIII. CONCLUSION AND DISCUSSION

Let us recall the different stages and scales characterizing evolution of soliton patterns in the weak disorder regime D

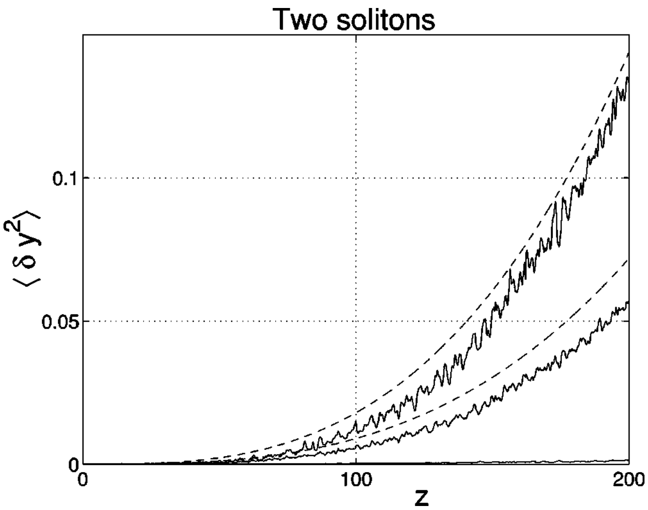

FIG. 2. Dependence of the mean square value of the intersoliton separation $\langle(\delta y)^2\rangle$ measured in units of the soliton width square, on the dimensionless position along the fiber z is shown. The disorder strength is $D=0.0125^2$. Three different sets of curves for the three different values of the intersoliton phase mismatch $\alpha=0,\pi/4,\pi/2$ are presented. Dashed curves represent the analytical result given by Eq. (1.2). Solid curves represent results of numerics. (Each curve is the result of averaging over 15 different realizations of disorder.)

$\ll 1$. The distance passed by a soliton during one full turnover of its phase is unity in our notations. Soliton starts to degrade, i.e., its amplitude change becomes of the order of its initial value, at $z_{\rm degr}=1/D$. An important observation of this paper is that an interesting physics is also taking place at much shorter z when the intersoliton interaction caused by radiation leads to an essential shift of the solitons at $z\sim z_{\rm int}$, $z_{\rm int}=N^{-1/3}D^{-2/3}$, where N is the number of solitons in the channel.

The major effect reported in the paper is the emergence of the separation independent, fluctuating in z interaction between solitons, mediated by their mutual radiation. A frozen (t-independent), disorder (which produces a multiplicative noise in the NLSE) stimulates the shedding of radiation by solitons, which, in turn, mediates the intersoliton interaction. The interaction causes the soliton to jitter randomly. The soliton displacement δy is a zero mean Gaussian random variable, with the typical value estimated by $\delta y\sim Dz^{3/2}N^{1/2}$. If N does not grow with z (e.g., there are only finite number of solitons propagating in the channel) the z dependence of the jitter is the same as the one given by the Elgin-Gordon-Haus jitter [32–35] developed under the action of random additive noise (short-correlated both in t and z noise of amplifiers in the fiber system). However, if the flow of information is continuous, i.e., if the front of radiation shed by the given soliton sweeps more and more solitons with increasing z, $N\propto z$, the efficiency of the interaction grows with z in a faster, $\delta y\propto z^2$, pace, thus overwhelming the Elgin-Gordon-Haus jitter in long-haul transmission. Notice, however, that as was shown above in Sec. V, the destructive effect of the disorder term in the dispersion coefficient can be essentially suppressed by pinning [16,17], so that the radiation mediated jitter estimated by $\delta y\sim\sqrt{Nz}$, becomes less important asymptotically than the Elgin-Gordon-Hauss jitter.

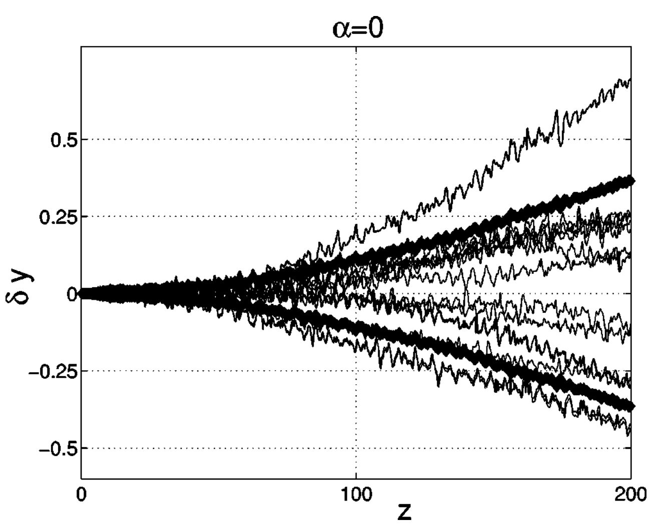

FIG. 3. Dependence of δy (measured in the units of pulse width) on the dimensionless separation z is shown for 15 different realizations of disorder, all for $\alpha=0$. The bold curves correspond to the root-mean-square expectation $\pm\sqrt{\langle(\delta y)^2\rangle}$.

The intersoliton interaction discussed in this manuscript is zero on average. This cancellation (in the mean value of the force) is due to reflectionless feature of radiation scattering on solitons. However, the scattering becomes reflective in some cases, described by nonintegrable generalizations of the NLS equation that are of physical importance, e.g., pattern dynamics in some fibers with essential birefringence [36] can be of this kind. The reflectivity leads to essential changes in the properties of the radiation and the intersoliton interaction, e.g., the force exerted on a soliton acquires a nonzero mean.

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APPENDIX A: KAUP PERTURBATION TECHNIQUE

Recall some properties of the perturbations near an ideal soliton described by the nonlinear Schrödinger equation [11,12]

$$-i\partial_z\Psi=\partial_t^2\Psi+2|\Psi|^2\Psi. \qquad (A1)$$

Substituting the expression

$$\Psi=[\cosh^{-1}(t)+v\,]\exp(iz+i\alpha)$$

into Eq. (A1) and expanding the result over v one finds

$$i\partial_z\begin{pmatrix} v \\ v^* \end{pmatrix}+\hat{L}\begin{pmatrix} v \\ v^* \end{pmatrix}=0, \tag{A2}$$

where the operator $\hat{L}$ is

$$\hat{L}=(\partial_t^2-1)\hat{\sigma}_3+\frac{2}{\cosh^2[t]}(2\hat{\sigma}_3+i\hat{\sigma}_2), \tag{A3}$$

and the standard notations for the Pauli matrixes $\hat{\sigma}_{1,2,3}$ are used. $\hat{L}$ satisfies the following set of relations:

$$\hat{\sigma}_1\hat{L}\hat{\sigma}_1=-\hat{L}^*, \quad \hat{L}^+=\hat{\sigma}_3\hat{L}\hat{\sigma}_3. \tag{A4}$$

The eigenset of the operator $\hat{L}$ is defined by

$$\hat{L}f=\lambda f, \tag{A5}$$

where f is the eigenfunction correspondent to the eigenvalue λ. The general solution of Eq. (A5) is

$$f_k=\exp[ikt]\left\{1-\frac{2ik\exp(-t)}{(k+i)^2\cosh(t)}\right\}\begin{pmatrix} 0 \\ 1 \end{pmatrix}+\frac{\exp(ikt)}{(k+i)^2\cosh^2(t)}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda_k=k^2+1, \tag{A6}$$

where k runs from $-\infty$ to $+\infty$. According to Eq. (A4), $\overline{f_k}\equiv\hat{\sigma}_1 f_k^*$ are the other eigenfunctions of $\hat{L}$,

$$\overline{f_k}=\exp(-ikt)\left\{1+\frac{2ik\exp(-t)}{(k-i)^2\cosh[t]}\right\}\begin{pmatrix} 1 \\ 0 \end{pmatrix}+\frac{\exp(-ikt)}{(k-i)^2\cosh^2(t)}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda_k=-(k^2+1). \tag{A7}$$

The eigenset of $\hat{L}$ also contains the following marginally stable modes:

$$f_0=\frac{1}{\cosh(t)}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad f_1=\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{\tanh(t)}{\cosh(t)}, \tag{A8}$$

where $\lambda_0=\lambda_1=0$. The existence of double poles at $k=\pm i$ means that two more functions must be added to the eigenset for completeness

$$f_2=\frac{t}{\cosh(t)}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \hat{L}f_2=-2f_1, \tag{A9}$$

$$f_3=\frac{t\tanh(t)-1}{\cosh(t)}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \hat{L}f_3=-2f_0. \tag{A10}$$

Next, $f_k^+\hat{\sigma}_3$ and $\overline{f_k^+}\hat{\sigma}_3$ (where the upper index $+$ stands for transposition and complex conjugation) are the left eigenfunctions of $\hat{L}$, which satisfy

$$\int_{-\infty}^{+\infty}dt\overline{f_k^+}\hat{\sigma}_3\overline{f_q}=-\int_{-\infty}^{+\infty}dtf_k^+\hat{\sigma}_3 f_q=2\pi\delta(k-q), \tag{A11}$$

$$\int_{-\infty}^{+\infty}dtf_2^+\hat{\sigma}_3 f_1=2, \quad \int_{-\infty}^{+\infty}dtf_0^+\hat{\sigma}_3 f_3=-2. \tag{A12}$$

Let us now modify the definition of v:

$$\Psi=[e^{i\alpha}\cosh^{-1}(t)+v]\exp(iz).$$

Then, the operator describing the linearized dynamics of v is

$$\hat{L}_\alpha=(\partial_t^2-1)\hat{\sigma}_3+\frac{2}{\cosh^2 t}\left[2\hat{\sigma}_3+\begin{pmatrix} 0 & e^{2i\alpha} \\ -e^{-2i\alpha} & 0 \end{pmatrix}\right]. \tag{A13}$$

The operator $\hat{L}_\alpha$ satisfies the same identities (A4) as $\hat{L}$ does. The eigenfunctions of the operator (A13) can be obtained from Eqs. (A6),(A7) by an obvious phase shift. One gets

$$f_{\alpha,k}(t)=\exp(ikt)\left\{1-\frac{2ik\exp(-t)}{(k+i)^2\cosh(t)}\right\}\begin{pmatrix} 0 \\ e^{-i\alpha} \end{pmatrix}+\frac{\exp(ikt)}{(k+i)^2\cosh^2(t)}\begin{pmatrix} e^{i\alpha} \\ e^{-i\alpha} \end{pmatrix},$$

$$\overline{f}_{\alpha,k}(t)=\exp(-ikt)\left\{1+\frac{2ik\exp(-t)}{(k-i)^2\cosh(t)}\right\}\begin{pmatrix} e^{i\alpha} \\ 0 \end{pmatrix}+\frac{\exp(-ikt)}{(k-i)^2\cosh^2(t)}\begin{pmatrix} e^{i\alpha} \\ e^{-i\alpha} \end{pmatrix}. \tag{A14}$$

The eigenfunctions (A14) possess the same orthogonality properties (A11) as f_k, $\overline{f_k}$ do.

APPENDIX B: INTERACTION OF TWO SOLITONS

Here we examine statistics of the force from the right-hand side of Eq. (4.10). One starts analyzing $\mathcal{F}_{vv}$ given by Eq. (4.11). Substituting φ_k and $\overline{\varphi_k}$ into Eqs. (2.7) one derives

$$v^2+(v^*)^2+4|v|^2=\int \frac{dqdk}{(2\pi)^2}\frac{e^{ikx-iqx}a_k a_q^*}{(k+i)^2(q-i)^2}\left\{\frac{2}{\cosh^2 x}[(q-i\tanh x)^2+(k+i\tanh x)^2]+4\cosh^{-4}x+4(q-i\tanh x)^2\right.$$
$$\left.\times(k+i\tanh x)^2\right\}+\int \frac{dqdk}{(2\pi)^2}\frac{e^{ikx+iqx}a_k a_q}{(k+i)^2(q+i)^2}\left\{\frac{1}{\cosh^4 x}+\left[(k+i)^2-2ik\frac{e^{-x}}{\cosh x}+\frac{1}{\cosh^2 x}\right]\right.$$
$$\left.\times\left[(q+i)^2-2iq\frac{e^{-x}}{\cosh x}+\frac{1}{\cosh^2 x}\right]+\frac{4}{\cosh^2 x}\left[(q+i)^2-2iq\frac{e^{-x}}{\cosh x}+\frac{1}{\cosh^2 x}\right]\right\}$$
$$+\int \frac{dqdk}{(2\pi)^2}\frac{e^{-ikx-iqx}a_k^* a_q^*}{(k-i)^2(q-i)^2}\left\{\frac{1}{\cosh^4 x}+\left[(k-i)^2+2ik\frac{e^{-x}}{\cosh x}+\frac{1}{\cosh^2 x}\right]\right.$$
$$\left.\times\left[(q-i)^2+2iq\frac{e^{-x}}{\cosh x}+\frac{1}{\cosh^2 x}\right]+\frac{4}{\cosh^2 x}\left[(q-i)^2+2iq\frac{e^{-x}}{\cosh x}+\frac{1}{\cosh^2 x}\right]\right\}. \tag{B1}$$

Substituting Eq. (B1) into Eq. (4.11) and taking integrals over x one finds

$$\mathcal{F}_{vv}=\int \frac{dkdq}{24\pi}\frac{ia_k a_q^*(k^2-q^2)^2(1+k^2+kq+q^2)}{(k+i)^2(q-i)^2\sinh[\pi(k-q)/2]}+\int\frac{dkdq}{2\pi}\frac{ia_k a_q(k+q)^2(1+k^2+q^2-kq)(2+k^2+q^2)}{24(k+i)^2(q+i)^2\sinh[\pi(k+q)/2]}$$
$$-\int\frac{dkdq}{2\pi}\frac{ia_k^* a_q^*(k+q)^2(1+k^2+q^2-kq)(2+k^2+q^2)}{24(k-i)^2(q-i)^2\sinh[\pi(k+q)/2]}. \tag{B2}$$

From Eq. (2.13),(4.6),(4.7) and Eq. (B2) one derives $\mathcal{F}_{vv}=F+\Phi+\Phi^*$ where the new quantities F and Φ are defined as

$$F=\frac{\pi i}{3\cdot 2^5}\int \frac{dkdq(k^2-q^2)^2(1+k^2+q^2+kq)}{\cosh[\pi k/2]\cosh[\pi q/2]\sinh[\pi(k-q)/2]}$$
$$\times\int_0^z dz_1 dz_2\,\xi(z_1)\xi(z_2)e^{i(k^2+1)(z-z_1)-i(q^2+1)(z-z_2)}$$
$$\times\left[\left(\frac{k-i}{k+i}\right)^2\left(\frac{q+i}{q-i}\right)^2 e^{i(q-k)y}+\left(\frac{k-i}{k+i}\right)^2 e^{-iky-i\alpha}\right.$$
$$\left.+\left(\frac{q+i}{q-i}\right)^2 e^{iqy+i\alpha}\right], \tag{B3}$$

$$\Phi=-\frac{\pi i}{3\cdot 2^6}\int \frac{dkdq(k+q)^2(1+k^2+q^2-kq)(2+k^2+q^2)}{\cosh(\pi k/2)\cosh(\pi q/2)\sinh[\pi(k+q)/2]}$$
$$\times\int_0^z dz_1 dz_2\,\xi(z_1)\xi(z_2)e^{i(k^2+1)(z-z_1)+i(q^2+1)(z-z_2)}$$
$$\times\left[\left(\frac{k-i}{k+i}\right)^2\left(\frac{q-i}{q+i}\right)^2 e^{-i(q+k)y-2i\alpha}+\left(\frac{k-i}{k+i}\right)^2 e^{-iky-i\alpha}\right.$$
$$\left.+\left(\frac{q-i}{q+i}\right)^2 e^{-iqy-i\alpha}\right]. \tag{B4}$$

The second term in the force (4.12) can be analogously presented as

$$\mathcal{F}_{\xi v}=\frac{\pi}{60}\xi(z)\int_0^z dz'\,\xi(z')\int \frac{dkk(k^2+1)(16+9k^2)}{\cosh^2(\pi k/2)}$$
$$\times\mathrm{Re}\left[e^{i(k^2+1)(z-z')}\left(\frac{k-i}{k+i}\right)^2 e^{-i\alpha-iky}\right]. \tag{B5}$$

The third term, originating from the phase α_1 dependence on z [in the leading first order over ξ, see Eq. (4.8)] in the force (4.13) is given by

$$\mathcal{F}_{\xi\alpha}=\frac{\xi(z)}{2\pi}\int dk\,\mathrm{Re}\left[\int \frac{dx\tanh x}{\cosh x}a_k(z)(f_k^{(1)}+f_k^{(2)})\right]$$
$$=\frac{\xi(z)}{2\pi}\int dk\,\mathrm{Re}\left[a_k(z)\int\frac{dx e^{ikx}\tanh x}{\cosh x}\right.$$
$$\left.\times\left(1-\frac{2ike^{-x}}{(k+i)^2\cosh x}+\frac{2}{(k+i)^2\cosh^2 x}\right)\right]$$
$$=\frac{\xi(z)}{6}\int\frac{dkk}{\cosh(\pi k/2)}\mathrm{Re}\frac{ia_k(z)(k-i)}{(k+i)}$$
$$=-\frac{\pi}{12}\xi(z)\int_0^z dz'\,\xi(z')\int\frac{dk,k(k^2+1)}{\cosh^2(\pi k/2)}$$
$$\times\mathrm{Re}\left[e^{i(k^2+1)(z-z')}\left(\frac{k-i}{k+i}\right)^2 e^{-i\alpha-iky}\right]. \tag{B6}$$

The expressions (B3),(B4),(B5),(B6) will be used below to examine statistics of the overall force $\mathcal{F}_{vv}+\mathcal{F}_{\xi v}+\mathcal{F}_{\xi\alpha}$ acting on the soliton.

The overall force can also be presented as

$$\mathcal{F}_{vv}+\mathcal{F}_{\xi v}+\mathcal{F}_{\xi\alpha}=\partial_z(\tilde{P}+P+P^*)+\Lambda, \tag{B7}$$

$$\tilde{P}=\frac{\pi}{3\cdot 2^5}\int \frac{dkdq(k^2-q^2)(1+k^2+q^2+kq)}{\cosh[\pi k/2]\cosh[\pi q/2]\sinh[\pi(k-q)/2]}\\ \times\int_0^z dz_1dz_2\xi(z_1)\xi(z_2)e^{i(k^2+1)(z-z_1)-i(q^2+1)(z-z_2)}\\ \times\left[\left(\frac{k-i}{k+i}\right)^2\left(\frac{q+i}{q-i}\right)^2 e^{i(q-k)y}+\left(\frac{k-i}{k+i}\right)^2 e^{-iky-i\alpha}\right.\\ \left.+\left(\frac{q+i}{q-i}\right)^2 e^{iqy+i\alpha}\right], \tag{B8}$$

$$P=-\frac{\pi}{3\cdot 2^6}\int \frac{dkdq(k+q)^2(1+k^2+q^2-kq)}{\cosh(\pi k/2)\cosh(\pi q/2)\sinh[\pi(k+q)/2]}\\ \times\int_0^z dz_1dz_2\xi(z_1)\xi(z_2)e^{i(k^2+1)(z-z_1)+i(q^2+1)(z-z_2)}\\ \times\left[\left(\frac{k-i}{k+i}\right)^2\left(\frac{q-i}{q+i}\right)^2 e^{-i(q+k)y-2i\alpha}+\left(\frac{k-i}{k+i}\right)^2 e^{-iky-i\alpha}\right.\\ \left.+\left(\frac{q-i}{q+i}\right)^2 e^{-iqy-i\alpha}\right], \tag{B9}$$

$$\Lambda=\frac{\pi\xi(z)}{8}\mathrm{Re}\int \frac{dkk(1+k^2)^2}{\cosh^2(\pi k/2)}\int_0^z dz'\,\xi(z')e^{i(k^2+1)(z-z')}\\ \times\left[\left(\frac{k-i}{k+i}\right)^2 e^{-i\alpha-iky}\right], \tag{B10}$$

where exponentially small in y terms are omitted. [The terms are produced by integrals, say, over k, with oscillating, $\sim\exp(-iky)$ and z-independent integrands. Then, the integration contour can be shifted to surround a pole, nearest to the real axis, and a residue at the pole gives the main contribution, exponentially small over y.]

Straightforward calculations show that for the force acting on the second soliton one can, actually, use Eqs. (B3),(B4),(B5),(B6),(B10) with the expressions under square brackets on the right-hand side of each of those formulas replaced by their complex conjugates.

1. Average impulse

Here we calculate the average over statistics of ξ of the overall force $\mathcal{F}_{vv}+\mathcal{F}_{\xi v}$ given by (B7). Notice that the average of Λ, calculated in accordance with Eqs. (2.2),(B10), is exponentially small ($\sim\exp[-\mathrm{const}\cdot y]$, where $y=y_2-y_1$) and thus it will be neglected below.

It follows from Eq. (B8) that

$$\langle\tilde{P}\rangle=\frac{\pi iD}{96}\int dkdq\{1-\exp[i(k^2-q^2)z]\}\\ \times\frac{1+k^2+q^2+kq}{\cosh(\pi k/2)\cosh(\pi q/2)\sinh[\pi(k-q)/2]}\\ \times\left[\left(\frac{k-i}{k+i}\right)^2\left(\frac{q+i}{q-i}\right)^2 e^{i(q-k)y}+\left(\frac{k-i}{k+i}\right)^2 e^{-iky-i\alpha}\right.\\ \left.+\left(\frac{q+i}{q-i}\right)^2 e^{iqy+i\alpha}\right]. \tag{B11}$$

Let us change the integration variables from k,q to $k_\pm=k\pm q$. The first contribution to the average impulse originates from the first term inside the brackets in Eq. (B11)

$$\langle\tilde{P}\rangle_1=\frac{\pi iD}{192}\int \frac{dk_+dk_-}{\sinh(\pi k_-/2)}\{1-e^{ik_+k_-z}\}e^{-ik_-y}\\ \times\frac{1+k^2+q^2+kq}{\cosh(\pi k/2)\cosh(\pi q/2)}\frac{(k-i)^2}{(k+i)^2}\frac{(q+i)^2}{(q-i)^2}.$$

The integral is formed at the smallest k_-. One gets

$$\langle\tilde{P}\rangle_1=\frac{\pi D}{48}\int_0^\infty dk_+\frac{1+3k_+^2/4}{\cosh^2(\pi k_+/4)}=\frac{D}{6}, \tag{B12}$$

where terms exponentially small in y are omitted. The second contribution to the average impulse coming from the last two terms inside the square brackets in Eq. (B11) is formed at small $k_\pm$ and can be written as

$$\langle\tilde{P}\rangle_2=\frac{iD}{96}\int \frac{dk_-dk_+}{k_-}\{1-\exp[ik_+k_-z]\}\\ \times[e^{-i(k_+y/2+k_-y/2+\alpha)}+e^{i(k_+y/2-k_-y/2+\alpha)}]\\ =\frac{\pi D}{12y}\sin(\alpha+y^2/4z).$$

One finds that at large y the contribution given by Eq. (B12) is dominant.

Let us now consider the average

$$\langle P\rangle=-\frac{\pi iD}{3\cdot 2^6}\int \frac{dkdq(k+q)^2(1+k^2+q^2-kq)}{\cosh(\pi k/2)\cosh(\pi q/2)}\\ \times\frac{[1-e^{i(k^2+1)z+i(q^2+1)z}]}{(2+k^2+q^2)\sinh[\pi(k+q)/2]}\left[\left(\frac{k-i}{k+i}\right)^2 e^{-iky-i\alpha}\right.\\ \left.+\left(\frac{q-i}{q+i}\right)^2 e^{-iqy-i\alpha}+\left(\frac{k-i}{k+i}\right)^2\left(\frac{q-i}{q+i}\right)^2 e^{-i(q+k)y-2i\alpha}\right].$$

The term in the expression which does not contain a z dependence produces an exponentially subleading in y contribution to $\langle P\rangle$. The z-dependent contribution is formed at $q,k\sim 1/\sqrt{z}$, and it is, therefore, $\sim y/z^2$. (Notice also, that the term rapidly oscillates with z.) Therefore, the averages $\langle P\rangle$ and $\langle P^*\rangle$ are negligible at large z in comparison with the contribution given by Eq. (B12). To conclude, at large z the average force is zero and the main contribution to the impulse of the force $\mathcal{F}$ is $D/6$.

2. Fluctuations of the force

One considers here the irreducible part of the pair correlation of $\mathcal{F}_{vv}$, which can be written as

$$\langle\langle \mathcal{F}_{vv}(z_1)\mathcal{F}_{vv}(z_2)\rangle\rangle=\langle \mathcal{F}_1\mathcal{F}_2\rangle-\langle \mathcal{F}_1\rangle\langle \mathcal{F}_2\rangle=\langle\langle F_1F_2\rangle\rangle+\langle\langle \Phi_1\Phi_2^*\rangle\rangle+\langle\langle \Phi_1^*\Phi_2\rangle\rangle, \quad \text{(B13)}$$

where $\langle \mathcal{F}_{vv}\rangle$ is neglected and only nonoscillating terms are kept. The first contribution to Eq. (B13) is

$$\langle F_1F_2\rangle-\langle F_1\rangle\langle F_2\rangle=\frac{\pi^2D^2}{9\times 2^{10}}\int \frac{dk_1dk_2dq_1dq_2(k_1^2-q_1^2)^2(k_2^2-q_2^2)^2}{\cosh[\pi k_1/2]\cosh[\pi q_1/2]\cosh[\pi k_2/2]\cosh[\pi q_2/2]}$$
$$\times\frac{(1+k_1^2+q_1^2+k_1q_1)(1+k_2^2+q_2^2+k_2q_2)}{\sinh[\pi(k_1-q_1)/2]\sinh[\pi(k_2-q_2)/2]}e^{i(k_1^2-q_1^2)\zeta}\left[\left(\frac{k_1-i}{k_1+i}\right)^2e^{-iq_1y-i\alpha}\right.$$
$$+\left(\frac{q_1+i}{q_1-i}\right)^2e^{ik_1y+i\alpha}+\left.\left(\frac{k_1-i}{k_1+i}\right)^2\left(\frac{q_1+i}{q_1-i}\right)^2\right]\left[\left(\frac{k_2-i}{k_2+i}\right)^2\left(\frac{q_2+i}{q_2-i}\right)^2+\left(\frac{k_2-i}{k_2+i}\right)^2e^{-iq_2y-i\alpha}\right.$$
$$\left.+\left(\frac{q_2+i}{q_2-i}\right)^2e^{ik_2y+i\alpha}\right]\frac{1}{k_+k_-q_+q_-}[e^{ik_+k_-z}-1][e^{iq_+q_-z}-1]e^{-ik_-y-iq_-y}, \quad \text{(B14)}$$

where $k_\pm=k_1\pm q_2$, $q_\pm=k_2\pm q_1$, and $z=\min\{z_1,z_2\}$, $\zeta=|z_1-z_2|$. The simultaneous correlation function, corresponding to $\zeta=0$, is the first object to study here. One finds that the dominant contribution, proportional to the logarithms of y and z, originates from the α-independent terms in the integrand of (B14). (The α-dependent contribution is $\sim 1/y$.) There are actually two kinds of such contributions. The first one comes from the product of two different α-independent terms, each from an expression bounded by the square brackets in the integrand of Eq. (B14). Terms of the second kind come from products of two terms cancelling their α dependence in the result. In the contribution of the first kind, the integrals over k_- and q_- are formed at both $k_-,q_-\sim 1/y$. Thus, replacing k_-,q_- in all nonoscillatory terms by zero, one derives

$$\langle\langle F^2\rangle\rangle_1=\frac{\pi^4D^2}{9\times 2^{18}}\int_{y/z}^{\infty}\frac{dk_+dq_+(k_+^2-q_+^2)^4}{\cosh^2[\pi k_+/4]\cosh^2[\pi q_+/4]}\,\frac{(1+k_+^2/4+q_+^2/4+k_+q_+/4)^2}{k_+q_+\sinh^2[\pi(k_+-q_+)/4]}. \quad \text{(B15)}$$

In the remaining two (identical) contributions of the second type, the k_- and q_- integration are not equivalent. One of the wave vectors, say k_- is still $O(1/y)$. Integrating over k_- one gets

$$\langle\langle F^2\rangle\rangle_2=\frac{\pi^4D^2}{9\times 2^{21}}\int_{y/z}^{\infty}\frac{dq_+}{q_+\cosh^2(\pi q_+/4)}\int\frac{dk_+dk_-}{2\pi i k_+k_-}[\exp(ik_+k_-z)-1]\frac{[(k_++k_-)^2-q_+^2]^2[(k_+-k_-)^2-q_+^2]^2}{\cosh[\pi(k_++k_-)/4]\cosh[\pi(k_+-k_-)/4]}$$
$$\times\frac{[4+(k_++k_-)^2+q_+^2+q_+(k_++k_-)][k_-\to -k_-]}{\sinh[\pi(k_++k_--q_+)/4]\sinh[\pi(k_+-k_--q_+)/4]}. \quad \text{(B16)}$$

The major contribution in the integral originates from $q_+\sim 1\gg k_\pm$. Replacing the integrand in Eq. (B16) by its asymptotic value at $k_\pm\to 0$, one finds that integration over $k_\pm$ is $\sim\ln[z]$. Finally, collecting the two major contribution into the simultaneous correlation function one finds

$$\langle\langle F^2\rangle\rangle=\frac{\pi^4D^2}{9\cdot 2^{15}}[\ln(z/y)+\ln z]\int_0^{\infty}\frac{dq\,q^7(1+q^2/4)^2}{\sinh^2(\pi q/2)}$$
$$=0.0068D^2\ln[z^2/y]. \quad \text{(B17)}$$

The result (B17) is asymptotic in the sense that it is valid at $z\gg y$ only. Let us now account for $\zeta\neq 0$ in Eq. (B14), i.e., for $z_1\neq z_2$. It is obvious from the analysis of the simultaneous correlation function that Eq. (B17) is formed at values of the four wave vectors $k_\pm$, $q_\pm$, that only one of the wave vectors is $O(1)$, while the other three are much smaller. Equation (B17) takes the following form:

$$\langle\langle F(z_1)F(z_2)\rangle\rangle=\frac{\pi^4D^2}{9\times 2^{15}}\ln(z^2/y)\times\int_0^{\infty}\frac{dq\,q^7(1+q^2/4)^2}{\sinh^2(\pi q/2)}\cos[q^2\zeta], \quad \text{(B18)}$$

i.e., at $\zeta \gg 1$, the correlations decay algebraically in ζ, $\langle\langle F(z_1)F(z_2)\rangle\rangle \sim D^2 \ln[z^2/y]/\zeta^3$. It also follows from Eq. (B18) that $\int d\zeta \langle\langle F(z+\zeta)F(z)\rangle\rangle = 0$.

Let us calculate the Φ and Φ^* in Eq. (B13). One finds

$$\langle \Phi(z+\zeta/2)\Phi^*(z-\zeta/2)\rangle = -\frac{\pi^2 D^2}{9\times 2^{11}}\int \frac{dk_1 dq_1 (k_1+q_1)^2(1+k_1^2+q_1^2-k_1q_1)(2+k_1^2+q_1^2)}{\cosh(\pi k_1/2)\cosh(\pi q_1/2)\sinh[\pi(k_1+q_1)/2]}$$
$$\times \int \frac{dk_2 dq_2 (k_2+q_2)^2(1+k_2^2+q_2^2-k_2q_2)(2+k_2^2+q_2^2)}{\cosh(\pi k_2/2)\cosh(\pi q_2/2)\sinh[\pi(k_2+q_2)/2]} \frac{e^{i(k_1^2/2+k_2^2/2+1)\zeta}}{k_1^2-k_2^2}[e^{i(k_1^2-k_2^2)|\zeta|/2}$$
$$-e^{i(k_1^2-k_2^2)z}]\frac{e^{i(q_1^2/2+q_2^2/2+1)\zeta}}{q_1^2-q_2^2}[e^{i(q_1^2-q_2^2)|\zeta|/2}-e^{i(q_1^2-q_2^2)z}]\left[\left(\frac{k_1-i}{k_1+i}\right)^2\left(\frac{q_1-i}{q_1+i}\right)^2 e^{-i(q_1+k_1)y-2i\alpha}\right.$$
$$+\left(\frac{k_1-i}{k_1+i}\right)^2 e^{-ik_1y-i\alpha}+\left(\frac{q_1-i}{q_1+i}\right)^2 e^{-iq_1y-i\alpha}\Bigg]\left[\left(\frac{k_2+i}{k_2-i}\right)^2\left(\frac{q_2+i}{q_2-i}\right)^2 e^{i(q_2+k_2)y+2i\alpha}\right.$$
$$\left.+\left(\frac{k_2+i}{k_2-i}\right)^2 e^{ik_2y+i\alpha}+\left(\frac{q_2+i}{q_2-i}\right)^2 e^{iq_2y+i\alpha}\right]. \tag{B19}$$

The first logarithmic contribution to the average (B19) originates from terms, containing 2α. The integrals are formed at small values of k_1-k_2 and q_1-q_2. The result of integration is

$$\langle \Phi(z+\zeta/2)\Phi^*(z-\zeta/2)\rangle_1 = \frac{\pi^4 D^2}{9\times 2^{15}}\int_{y/z}^{\infty}\frac{dk_+ dq_+}{k_+q_+}\frac{(1+k_+^2/4+q_+^2/4-k_+q_+/4)^2(2+k_+^2/4+q_+^2/4)^2}{\cosh^2(\pi k_+/4)\cosh^2(\pi q_+/4)\sinh^2[\pi(k_++q_+)/4]}$$
$$\times (k_++q_+)^4 \exp[i(k_+^2/4+q_+^2/4+2)\zeta], \tag{B20}$$

where $q_+ = q_1+q_2$ and $k_+ = k_1+k_2$. If $\zeta = 0$ then the integral (B20) is reduced to

$$\langle \Phi(z)\Phi^*(z)\rangle_1 = \frac{\pi^4 D^2}{9\cdot 2^{12}}\ln(z/y)\int_0^\infty dk_+ \frac{k_+^3(1+k_+^2/4)^2(2+k_+^2/4)^2}{\sinh^2[\pi k_+/2]}$$
$$\approx 0.017 D^2 \ln(z/y). \tag{B21}$$

The integral of the expression (B20) over ζ is clearly zero.

We now turn to calculation of the second logarithmic correction originating from the terms which contain α. The contribution is

$$\langle \Phi(z+\zeta/2)\Phi^*(z-\zeta/2)\rangle_2 = \frac{\pi^4 D^2}{9\times 2^{12}}\int \frac{dq_+ q_+^3(1+q_+/4)^2(2+q_+^2/4)^2}{\sinh^2[\pi q_+/2]}$$
$$\times e^{i(q_+^2/4+2)\zeta}\int \frac{dk_+ dk_-}{2\pi i k_+ k_-}[e^{ik_+k_-z}-1], \tag{B22}$$

where $k_\pm \ll q_+$, as in Eq. (B16). The integral over k_+ and k_- in Eq. (B22) produces $\ln z$. Therefore, the term $\langle \Phi(z)\Phi^*(z)\rangle_2$ is given by an expression similar to Eq. (B21), with the replacement of $\ln(z/y)$ by $\ln z$. Finally,

$$\langle \Phi(z)\Phi^*(z)\rangle \approx 0.017 D^2 \ln(z^2/y), \tag{B23}$$

$$\int dz \langle \Phi(z+\zeta/2)\Phi^*(z-\zeta/2)\rangle = 0. \tag{B24}$$

The pair correlation function corresponding to the Λ contribution (B10) is

$$\langle \Lambda(z)\Lambda(z')\rangle = D^2 G\,\delta(z-z'), \tag{B25}$$

$$G = \frac{\pi^2}{2^7}\int dk dq \frac{e^{i(k^2-q^2)z}-1}{i(k^2-q^2)}\frac{kq(1+k^2)^2(1+q^2)^2}{\cosh^2(\pi k/2)\cosh^2(\pi q/2)}$$
$$\times\left(\frac{k-i}{k+i}\right)^2\left(\frac{q+i}{q-i}\right)^2 e^{i(q-k)y}. \tag{B26}$$

The major y- and z-independent contribution in G is coming from small values of $k-q$. Taking the integral over this variable, one derives

$$G = \frac{\pi^3}{2^7}\int_0^\infty \frac{dk k(1+k^2)^4}{\cosh^4(\pi k/2)} \approx 0.14. \tag{B27}$$

The correlation function correspondent to the change in the relative position of the two solitons is defined by $\tilde{\Lambda} = \Lambda_1 - \Lambda_2$, where the indexes (1) and (2) stand for the first and second solitons, i.e., Λ_1 is given by Eq. (B10), while Λ_2

is given by Eq. (B10) with the expression on the right-hand side of it replaced by its complex conjugate. One gets

$$\langle\tilde{\Lambda}(z)\tilde{\Lambda}(z')\rangle = 2[1+\cos(2\alpha)]D^2G\,\delta(z-z'). \quad \text{(B28)}$$

3. Fluctuations of the impulse

As in the calculations of the previous subsection one can analyze fluctuations of the impulse $\mathcal{P}=\tilde{P}+P+P^*$. We obtain instead of Eq. (B17)

$$\langle\langle\tilde{P}^2\rangle\rangle = \frac{\pi^4D^2}{9\times 2^{11}}[\ln(z/y)+\ln z]\int_0^\infty \frac{dq q^3(1+q^2/4)^2}{\sinh^2(\pi q/2)} \approx 0.0036D^2\ln(z^2/y). \quad \text{(B29)}$$

An analog of Eq. (B21) is

$$\langle P(z)P^*(z)\rangle = \frac{\pi^4D^2}{9\times 2^{12}}[\ln(z/y) + \ln z]\int_0^\infty dk_+ \frac{k_+^3(1+k_+^2/4)^2}{\sinh^2[\pi k_+/2]} \approx 0.0018D^2\ln(z^2/y). \quad \text{(B30)}$$

Finally, one gets the following answer for the pair simultaneous correlation function of the impulse:

$$\langle\mathcal{P}^2(z)\rangle \approx 0.0073D^2\ln(z^2/y). \quad \text{(B31)}$$

The major contribution to the overall impulse of the force is coming from the Λ term

$$\left\langle\left[\int_0^z dz'\Lambda(z')\right]^2\right\rangle \approx 0.265D^2z. \quad \text{(B32)}$$

The cross correlations are given by

$$\langle\mathcal{P}(z+\zeta)\Lambda(z)\rangle = \frac{\pi^2D^2}{9\times 2^7}\mathrm{Re}\int dk\,dq\,dp\,e^{i(k^2-q^2)\zeta} \times\frac{k(k-q)(1+k^2+q^2+kq)}{\cosh[\pi k/2]\cosh[\pi q/2]\sinh[\pi(k-q)/2]} \times\left(\frac{q+i}{q-i}\right)^2\frac{p(1+p^2)(5+3p^2)}{\cosh^2(\pi p/2)}\left(\frac{p-i}{p+i}\right)^2 \times\frac{e^{i(p^2-q^2)z}-1}{i(p^2-q^2)}e^{i(q-p)y}, \quad \text{(B33)}$$

which is nonzero at $\zeta>0$ only. The integral (B33) is formed at small values of $p-q$, thus one finds

$$\langle\mathcal{P}(z+\zeta)\Lambda(z)\rangle = -\frac{\pi^3D^2}{9\times 2^7}\mathrm{Re}\int dk\int_0^\infty dq\,e^{i(k^2-q^2)\zeta} \times\frac{k(k-q)(1+k^2+q^2+kq)(1+q^2)(5+3q^2)}{\cosh[\pi k/2]\cosh^3[\pi q/2]\sinh[\pi(k-q)/2]}, \quad \text{(B34)}$$

which becomes the following expression at $\zeta\to 0$:

$$\langle\mathcal{P}(z+0)\Lambda(z)\rangle = -\frac{\pi^3D^2}{9\times 5\times 2^7}\int_0^\infty dq\frac{q(1+q^2)^2(5+3q^2)(7+3q^2)}{\cosh^4[\pi q/2]} \approx -0.068D^2. \quad \text{(B35)}$$

We also find from Eq. (B34)

$$\left\langle\mathcal{P}(z)\int_0^z dz'\Lambda(z')\right\rangle = -\frac{\pi^3D^2}{9\times 2^7}\mathrm{Re}\int dk\,k\int_0^\infty dq\frac{e^{i(k^2-q^2)z}-1}{i(k^2-q^2)}\frac{(k-q)(1+k^2+q^2+kq)(1+q^2)(5+3q^2)}{\cosh[\pi k/2]\cosh^3[\pi q/2]\sinh[\pi(k-q)/2]}$$
$$= -\frac{\pi^3D^2}{9\times 2^7}\left\{\int_0^\infty dq\frac{(1+3q^2)(1+q^2)(5+3q^2)}{\cosh^4(\pi q/2)} - \pi\int_0^\infty dq\frac{q(1+q^2)^2(5+3q^2)}{\cosh^4(\pi q/2)\sinh(\pi q)}\right\} \approx -0.053D^2. \quad \text{(B36)}$$

Therefore, this cross correlation is negligible.

4. Additional impulse

An additional impulse $\mathcal{P}_1$, the last one left to be calculated, is due to the direct noise contribution $\mathcal{P}_1$ in Eq. (4.14). Expressing $v,\ v^*$ in Eq. (4.15) via ξ and performing averaging over the statistics of ξ in accordance with Eq. (2.2) one finds

$$\langle\mathcal{P}_1(z)\rangle = -\frac{D}{4}\mathrm{Im}\int \frac{dxx}{\cosh^4 x}\frac{dkdq e^{ikx-iqx}}{\cosh(\pi k/2)\cosh(\pi q/2)} \times\frac{\exp[i(k^2-q^2)z]-1}{i(k^2-q^2)}\left\{(q-i)^2+\frac{2iqe^{-x}}{\cosh x} + \frac{1}{\cosh^2 x}\right\}\left[1+\left(\frac{k-i}{k+i}\right)^2 e^{-iky-i\alpha}\right] \times\left[1+\left(\frac{q+i}{q-i}\right)^2 e^{iqy+i\alpha}\right].$$

Integrating the resulting expression over x, one derives

$$\langle\mathcal{P}_1(z)\rangle = \frac{\pi D}{192}\int \frac{dkdq(k-q)^2(k+q)}{\cosh(\pi k/2)\cosh(\pi q/2)}\frac{1}{i(k^2-q^2)} \times\left[1+\left(\frac{k-i}{k+i}\right)^2 e^{-iky-i\alpha}\right] \times\left[1+\left(\frac{q+i}{q-i}\right)^2 e^{iqy+i\alpha}\right]\frac{\exp[i(k^2-q^2)z]-1}{\sinh^2[\pi(k-q)/2]} \times\left\{-4(k-q)\sinh\left[\frac{\pi(k-q)}{2}\right]+\pi(4+k^2-2kq +q^2)\cosh\left[\frac{\pi(k-q)}{2}\right]\right\}. \tag{B37}$$

The main contribution to the integral comes from k close to q. Simplifying the expression and keeping only the main terms in $k-q$, one can then take the integrals over k and q, thus deriving $\langle\mathcal{P}_1(z)\rangle = D/3$. This contribution should be taken into account on an equal footing with Eq. (B12). That gives a systematic drift $2Dz/3$ for y_1.

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