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FLUIDS

Disclination Motion in Liquid Crystalline Films[¶]

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Abstract—We theoretically study a single disclination motion in a thin free-standing liquid crystalline film. Backflow effects and the own dynamics of the orientational degree of freedom (bond or director angle) are taken into account. We find the orientation field and the hydrodynamic velocity distribution around the moving disclination, which allows us to relate the disclination velocity to the angle gradient far from the disclination. Different cases are examined depending on the ratio of the rotational and shear viscosity coefficients. © *2002 MAIK "Nauka/Interperiodica".*

1. INTRODUCTION

The physics of thin liquid-crystalline films has been a recurrent hot topic during the past decade because of their intriguing physical properties and a wide range of applications in display devices, sensors, and for many other purposes. Hexatic, nematic, and smectic-*C* liquid-crystalline films belong to two-dimensional systems with a spontaneously broken continuous rotational symmetry. An essential role in the behavior of the films is therefore played by vortexlike excitations (disclinations). Defects are almost necessarily present in liquid crystals, and their dynamics plays a crucial role in the overall pattern organization. Early studies of defects focused on classifying the static properties of the defects and their interactions [1, 2]. More recently, the focus has shifted to examining the dynamics of defects (see, e.g., [3] and references therein). We note that, although defects are undesirable in most practical applications of liquid crystals, such as traditional display devices, because they destroy an optical adjustment, there are novel display designs (bistable, multidomain liquid-crystalline structures) exploiting defect properties.

Although experimental dynamic studies are likely to be more fruitful than static ones, theoretical research of the film dynamics is in a rather primitive stage. This is largely accounted for by a complexity of dynamic phenomena in films, and a complete and unifying description of the problem is still unavailable. Moreover, some papers devoted to this problem (dynamics of defects) claim contradicting results. These contradictions come mainly from the fact that different authors take different microscopic dissipation mechanisms into account, but partially the source of controversy is related to semantics, because different definitions of the forces acting on defects are used (see, e.g., the discussion in [4]). We believe that such problems are irrelevant if the macroscopic (phenomenological) approach to the film dynamics is used.

In this paper, we theoretically examine the disclination dynamics in free-standing liquid-crystalline films at scales that are much larger than the film thickness, where the films can be treated as 2D objects. Our investigation is devoted to the first (but compulsory) step of defect dynamics studies: a single point disclination in a liquid-crystalline film. A number of theoretical efforts [5–9] deal with similar problems. Our justification for adding one more paper to the subject is the fact that, in the literature, we did not see a full investigation of the problem with the hydrodynamic backflow effects taken into account. Evidently, these effects can drastically modify the dynamics of defects. The goal of this work is to study the disclination motion in free-standing liquid crystalline films on the basis of hydrodynamic equations containing some phenomenological parameters (the elasticity modulus and shear and rotational viscosity coefficients).

In our approach, the disclination is assumed to be driven by a large-scale inhomogeneity in the bond or director angle, which leads to a motion of the disclination with a nonzero velocity relative to the film. As a physical realization of such a nonuniform angle field, a system of disclinations distributed with a finite density can be imagined. The inhomogeneity in the vicinity of a given disclination is then produced by fields of other disclinations. We can also think about a pair of discli-

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nations of the opposite topological charges, in which case the inhomogeneity is related to the mutual orientational distortion fields created by each disclination at the point of its counterpart. In fact, the majority of experimental and numerical studies of disclination motions in liquid crystals [10–18] is devoted to the investigation of the dynamics of two oppositely charged defects. We solve the hydrodynamic equations and find the bond (director) angle and the flow velocity distributions around the moving disclination. The results enable us to relate the disclination velocity and the gradient of the angle far from the disclination.

An obvious context where our results can be applied is the film dynamics near the Berezinskii–Kosterlitz–Thouless phase transition. The static properties of the films near the transition have been investigated in a great number of papers starting from the famous papers by Berezinskii [19] and Kosterlitz and Thouless [20]. There are several works discussing the theory of dynamic phenomena associated with vortexlike excitations in condensed matter physics: vortices in type-II superconductors (see, e.g., [21]), vortices in superfluid ^{4}He and ^{3}He (see, e.g., [22, 23]), dislocations in 2D crystals, and disclinations (and other topological defects) in liquid crystals (see [10–14, 24–27]). But most of the theoretical works on the subject start from phenomenological equations of motion of the defects, and our aim is to derive the equations and to verify their validity.

The structure of our paper is as follows. Section 2 contains basic hydrodynamic equations for liquid-crystalline films necessary for our investigation. In Section 3, we find the bond (director) angle and the flow velocity around the uniformly moving disclination, which allows us to relate the disclination velocity to the angle gradient far from the disclination. Different cases, depending on the ratio of the rotational and shear viscosity coefficients, are examined in Section 4. Section 5 contains a summary and discussion. The appendices are devoted to the details of calculations of the velocity and bond angle fields around the moving disclination. Those readers who are not very interested in mathematical derivations can skip these appendices, finding all the essential physical results in the main text of the paper.

2. BASIC RELATIONS FOR LIQUID-CRYSTALLINE FILMS

We formulate the basic relations needed to describe a disclination motion in thin liquid-crystalline films. Here, we investigate freely suspended hexatic, nematic, and smectic-*C* films that can be pulled from 3D (bulk) smectics [3]. We examine scales larger than the film thickness, where the films can be treated as two-dimensional objects and can be described in terms of a macroscopic approach containing some phenomenological parameters.

Liquid crystalline films with the in-plane orientational ordering of different types (hexatic, nematic, and smectic-*C*) are observed experimentally. In these films, as in 3D nematic liquid crystals, the rotational symmetry is spontaneously broken. The general analysis of their symmetry can be found in [28]. The smectic-*C* films are characterized by the director that is tilted with respect to the normal to the film, which defines a preferred direction in the plane of the film. The ordering of this type can be described by a vector Q_α (the subscripts denoted by Greek letters take two values, because we treat the films as 2D objects). The nematic films have higher symmetry D_2, which corresponds to the 2D nematic phase. The order parameter of the nematic phase is the irreducible (traceless) symmetric tensor of the second rank $Q_{\alpha\beta}$. In the hexatic films (pulled from smectics-*B*), molecules are locally arranged in a triangular lattice, but the lattice is not an ideal one. The positional order does not extend over distances larger than several molecular sizes. Nevertheless, the bond order extends over macroscopic distances. The phase is therefore characterized by the D_{6h} point group symmetry, and hence, the order parameter for the case is the sixth-rank symmetric irreducible tensor $Q_{\alpha\beta\gamma\delta\mu\nu}$. In liquid crystalline films of all the types enumerated above, the order parameter Q has two independent components (e.g., Q_{xx} and Q_{xy} for the 2D nematics). We note that the order can be readily observed in the smectic-*C* or nematic films by looking for in-plane anisotropies in quantities such as the dielectric permeability tensor. Because of its intrinsic sixfold rotational symmetry, the hexatic orientational order is hardly observable. But it can be detected, e.g., as a sixfold pattern of spots in the in-plane monodomain *X*-ray structure factor, proportional to $Q_{\alpha\beta\gamma\delta\mu\nu}$(see, e.g., [3] and references therein).

In accordance with the Goldstone theorem, in films of all types with a broken rotational symmetry, the only degree of freedom of the order parameter that is relevant at large scales is an angle φ (like the phase of the order parameter for the superfluid ^{4}He). In hexatics, it is the bond angle, whereas, in 2D nematics and in smectic-*C* films, it is an angle related to the director. It is convenient to express a variation of the order parameter in terms of a variation of the angle φ. For the smectic-*C* films, the relation is

$$\delta Q_\alpha = -\delta\varphi\epsilon_{\alpha\mu}Q_\mu, \tag{2.1}$$

where $\epsilon_{\alpha\mu}$ is the two-dimensional antisymmetric tensor. For an orientational order with a higher symmetry, the relation has a similar form. For example, for hexatic films,

$$\delta Q_{\alpha\beta\gamma\delta\mu\nu} = -\delta\varphi\epsilon_{\alpha\rho}Q_{\rho\beta\gamma\delta\mu\nu} + \ldots, \tag{2.2}$$

where the dots represent the sum of all other possible combinations of the same structure. Therefore, for films of all types, the order parameter can be characterized by its absolute value $|Q|$ and the phase φ, which are traditionally represented as a complex quantity Ψ (see, e.g.,

[24]). The quantity is written as $\Psi = |Q|\exp(6i\varphi)$ for hexatic films, as $\Psi = |Q|\exp(2i\varphi)$ for 2D nematic films, and as $\Psi = |Q|\exp(i\varphi)$ for smectic-*C* films.

The angle φ should be included in the set of macroscopic variables of the films. A convenient starting point of the consideration is the energy density (per unit area) $\rho v^2/2 + \varepsilon$, where ρ is the 2D mass density, $\mathbf{v}$ is the film velocity, and ε is the internal energy density. The latter is a function of the mass density ρ, the specific entropy σ, and the angle φ. In fact, ε depends on $\nabla\varphi$, because any homogeneous shift of the angle φ does not affect the energy. For hexatic films, the leading terms of the energy expansion over gradients of φ are

$$\varepsilon = \varepsilon_0(\rho, \sigma) + \frac{K}{2}(\nabla\varphi)^2, \tag{2.3}$$

where K is the only (because of the hexagonal symmetry) orientational elastic module of the film. For low-symmetry films (2D nematic or smectic-*C* films), two orientational elastic modules are introduced, the longitudinal and transversal ones with respect to the specific in-plane direction (characterized by the so-called **c** director). But fluctuations of the director lead to a renormalization of the modules, and isotropization of the smectic-*C* or 2D nematic films [29] occurs at large scales. The same isotropic expression (2.3) for the elastic energy can therefore be used at large scales.

The complete dynamic equations for the freely suspended liquid-crystalline films, valid at scales larger than the film thickness, can be found in [30]. We consider a quasistationary motion of the disclination. Then, hard degrees of freedom are not excited. In other words, we can accept incompressibility and neglect bending deformations (which are suppressed by the presence of the surface tension in freely suspended films). Similarly, the thermodiffusive mode is not excited for the quasistationary disclination motion, which implies the isothermal condition. For freely suspended films, such effects as the substrate friction (relevant, e.g., for Langmuir films) are absent. In describing the disclination motion, we can therefore consider the system of equations for only the velocity $\mathbf{v}$ and the angle φ. The equations have to be formulated under the conditions $\rho = \text{const}$, $T = \text{const}$ (where T is the temperature), and $\nabla\mathbf{v} = 0$.

The equation for the velocity follows from the momentum density $\mathbf{j} = \rho\mathbf{v}$ conservation law,

$$\partial_t j_\alpha = -\nabla_\beta[T_{\alpha\beta} - \eta(\nabla_\alpha v_\beta + \nabla_\beta v_\alpha)], \tag{2.4}$$

where $T_{\alpha\beta}$ is the reactive (nondissipative) stress tensor and η is the 2D shear viscosity coefficient of the film. For two-dimensional hexatics, the reactive stress tensor is (see [30], Chapter 6)

$$\begin{aligned} T_{\alpha\beta} &= \rho v_\beta v_\alpha - \varsigma\delta_{\alpha\beta} + K\nabla_\alpha\varphi\nabla_\beta\varphi \\ &\quad - \frac{K}{2}\epsilon_{\alpha\gamma}\nabla_\gamma\nabla_\beta\varphi - \frac{K}{2}\epsilon_{\beta\gamma}\nabla_\gamma\nabla_\alpha\varphi, \end{aligned} \tag{2.5}$$

where $\varsigma = \varepsilon - \rho\partial\varepsilon/\partial\rho$ is the surface tension. We note that the ratio $K\rho/\eta^2$ is a dimensionless parameter that can be estimated by substituting 3D quantities instead of 2D ones (because all the 2D quantities can be estimated as the corresponding 3D quantities times the film thickness, and the latter drops from the ratio). For all known liquid crystals, the ratio is 10^{-3}–10^{-4}(see, e.g., [1–3, 31]) and can therefore be treated as a small parameter of the theory.

The second dynamic equation, the equation for the bond angle, is

$$\partial_t\varphi = \frac{1}{2}\epsilon_{\alpha\beta}\nabla_\alpha v_\beta - v_\alpha\nabla_\alpha\varphi + \frac{K}{\gamma}\nabla^2\varphi, \tag{2.6}$$

where γ is the so-called 2D rotational viscosity coefficient. We did not find the values of the coefficient γ for thin liquid-crystalline films in the literature. For bulk liquid crystals (see, e.g., [1–3, 31]), the 3D rotational viscosity coefficient is usually several times larger than the 3D shear viscosity coefficient. We can therefore expect that $\gamma > \eta$. But in order to span a wide range of possibilities, we treat the dimensionless ratio $\Gamma = \gamma/\eta$ as an arbitrary parameter in what follows.

If disclinations are present in the film, it is no longer possible to define a single-valued continuous bond-angle variable φ. But the order parameter is a well-defined function of coordinates that goes to zero at the disclination position. The gradient of $\varphi(t, \mathbf{r})$ is a single-valued function of $\mathbf{r}$ and is analytic everywhere except at an isolated point, the position of the disclination. The phase acquires a certain finite increment at each rotation around the disclination,

$$\oint dr_\alpha\nabla_\alpha\varphi = 2\pi s, \tag{2.7}$$

where the integration contour is a closed counterclockwise loop around the disclination position and s is the topological charge of the disclination: $s = (1/6)n$ for the hexatic ordering, $s = (1/2)n$ for the 2D nematic symmetry, and $s = n$ for the smectic-*C* films, where n is an integer. We can restrict ourselves to disclinations with the unitary charge $n = \pm1$ only, because disclinations with larger $|s|$ possess a higher energy than the set of unitary disclinations with the same net topological charge, and defects with larger charges are therefore unstable with respect to the dissociation to the unitary ones. Therefore, disclinations with the charges $|n| > 1$ do not play an essential role in the physics of films [1–3, 31]. To write the expressions given below in a compact form, we keep the notation s for the topological charge, with the respective values $|s| = 1, 1/2, 1/6$ for the smectic-*C*, nematic, and hexatic films.

The static bond angle is determined by the stationary condition $\delta E/\delta\varphi = 0$, where

$$E = \int d^2r\left(\frac{\rho}{2}v^2 + \varepsilon\right)$$

is the energy of the film. For the energy density in Eq. (2.3), the condition is reduced to the Laplace equation $\nabla^2\varphi = 0$. For an isolated static disclination, there is a symmetric solution to this equation φ_0 that satisfies Eq. (2.7) and whose gradient is given by

$$\nabla_\alpha\varphi_0 = -s\epsilon_{\alpha\beta}\frac{r_\beta - R_\beta}{(\mathbf{r}-\mathbf{R})^2}, \tag{2.8}$$

where $\mathbf{R}$ is the position of the disclination. If the origin of the reference system is placed at this point, we can write $\varphi_0 = s\arctan(y/x)$, where x and y are coordinates of the observation point $\mathbf{r}$. In dynamics, distribution (2.8) is disturbed as φ varies in time. It is also perturbed because of the presence of an angular distortion related to boundaries or other disclinations.

In what follows, we have in mind a case where a system of a large number of disclinations (with an uncompensated topological charge) is created. For 3D nematics, this can be done rather easily [1–3], because the energies of positive and negative defects are different due to the intrinsic elastic anisotropy. We are unaware of experimental or theoretical studies of defect nucleation mechanisms in free-standing films. Hopefully, the situation with a finite 2D density of defects can also be realized for films (for instance, the defects could even appear spontaneously as a mechanism to relieve frustrations in chiral smectic or hexatic films, similarly to the formation of the Abrikosov vortex lattice in superconductors [32]). Examining the motion of a disclination in this case, we investigate a vicinity of the disclination of the order of the interdisclination distance. Far from the disclination, the bond angle φ can then be written as const + $\mathbf{ur}$, where $\mathbf{u}$ is much larger than the inverse interdisclination distance (because the number of disclinations is large). Near the disclination position, the bond angle φ can be approximated by expression (2.8). Our main problem is to establish a general coordinate dependence of φ and $\mathbf{v}$, which, in particular, allows relating the bond (director) angle gradient $\mathbf{u}$ and the velocity of the disclination.

3. FLOW AND ANGULAR FIELDS AROUND A UNIFORMLY MOVING DISCLINATION

Here, we proceed to the main subject of our study, a single disclination driven by a large-scale inhomogeneity in the bond (director) angle φ. The disclination velocity is determined by an interplay of the hydrodynamic back-flow and the intrinsic dynamics of the angle φ. To find the disclination velocity, one has to solve the system of equations (2.4), (2.5), and (2.6) with constraint (2.7) ensuring a suitable asymptotic behavior. As we explained in the previous section, the angle φ is supposed to behave as const + $\mathbf{ur}$ at large distances from the disclination. We work in the reference system where the film as a whole is at rest. This means that the flow velocity excited by the disclination must tend to zero far from the disclination.

We consider the situation where the disclination moves with a constant velocity $\mathbf{V}$. The angle φ and the flow velocity are then functions of $\mathbf{r} - \mathbf{V}t$ (where $\mathbf{R} = \mathbf{V}t$ is the disclination position). Equation (2.4) for the velocity can then be written as

$$\begin{aligned} &\rho(V_\beta - v_\beta)\nabla_\beta v_\alpha + \eta\nabla^2 v_\alpha + \frac{K}{2}\epsilon_{\alpha\beta}\nabla_\beta\nabla^2\varphi \\ &\quad - K\nabla_\alpha\varphi\nabla^2\varphi + \nabla_\alpha\left[\varsigma - \frac{K}{2}(\nabla\varphi)^2\right] = 0. \end{aligned} \tag{3.1}$$

We can omit the first (inertial) term on the left-hand side of (3.1), which is small because of the smallness of the parameter $K\rho/\eta^2$. It then follows from Eqs. (2.4) and (2.5) that

$$\nabla^2 v_\alpha + \frac{K}{2\eta}\epsilon_{\alpha\beta}\nabla_\beta\nabla^2\varphi - \frac{K}{\eta}\nabla_\alpha\varphi\nabla^2\varphi + \nabla_\alpha\varpi = 0, \tag{3.2}$$

where $\varpi = \eta^{-1}[\varsigma - (K/2)(\nabla\varphi)^2]$. Under the same conditions, the equation for the angle φ following from Eq. (2.6) is

$$\nabla^2\varphi + \frac{\gamma}{K}V_\alpha\nabla_\alpha\varphi = \frac{\gamma}{K}v_\alpha\nabla_\alpha\varphi - \frac{\gamma}{2K}\epsilon_{\alpha\beta}\nabla_\alpha v_\beta. \tag{3.3}$$

We seek a solution characterized by the asymptotic behavior that the velocity $\mathbf{v}$ vanishes and $\nabla\varphi$ tends to a constant vector $\mathbf{u}$ as $r \longrightarrow \infty$. It is clear from the symmetry of the problem that the gradient $\mathbf{u}$ of the bond angle is directed along the Y axis if the velocity is directed along the X axis. Therefore, $\varphi \longrightarrow uy$ as $r \longrightarrow \infty$. Our problem is to find a relation between V and u, that is, between the disclination velocity and the bond angle gradient far from the disclination. There are two different regions: the region of large distances $r \gg u^{-1}$ and the region near the disclination $r \ll u^{-1}$. At large distances, corrections to the leading behavior $\varphi \approx uy$ are small and the problem can be treated in the linear approximation with respect to these corrections. In the region near the disclination, φ is close to static value (2.8) and the flow velocity $\mathbf{v}$ is close to the disclination velocity $\mathbf{V}$ (the special case where the ratio γ/η is extremely small is discussed in Subsection 4C). In what follows, these two regions are examined separately. The relation between u and V can be found by matching the asymptotics at $r \sim u^{-1}$. As a result, we obtain

$$V = \frac{K}{\eta}Cu, \tag{3.4}$$

where C is a dimensionless factor depending on the dimensionless ratio $\Gamma = \gamma/\eta$. This factor C is on the order of unity if $\Gamma \sim 1$. We are interested in the asymptotic behavior of C at small and large Γ.

A. The region near the disclination

We consider the region $r \ll u^{-1}$. Here, we can write

$$\varphi = \varphi_0(\mathbf{r}-\mathbf{R}) + \varphi_1(\mathbf{r}-\mathbf{R}), \tag{3.5}$$

where $\mathbf{R} = \mathbf{V}t$ is the disclination position, φ_0 is the static bond (director) angle with gradient (2.8), and φ_1 is a small correction to φ_0. The gradients of φ_0 are determined by Eq. (2.8).

Linearizing Eqs. (3.2) and (3.3) with respect to φ_1, we obtain

$$\begin{aligned} &\eta\nabla^2 v_\alpha + \frac{K}{2}\epsilon_{\alpha\beta}\nabla_\beta\nabla^2\varphi_1 - K\nabla_\alpha\varphi_0\nabla^2\varphi_1 \\ &+ \nabla_\alpha\left[\varsigma - \frac{K}{2}(\nabla\varphi)^2\right] = 0, \end{aligned} \tag{3.6}$$

$$\nabla^2\varphi_1 - \frac{\gamma}{K}v_\alpha\nabla_\alpha\varphi_0 + \frac{\gamma}{2K}\epsilon_{\alpha\beta}\nabla_\alpha v_\beta = -\frac{\gamma}{K}V_\alpha\nabla_\alpha\varphi_0. \tag{3.7}$$

Introducing a new variable $\chi = (K/\eta)\nabla^2\varphi_1$, we rewrite Eqs. (3.6) and (3.7) as

$$\nabla^2 v_\alpha + \frac{1}{2}\epsilon_{\alpha\beta}\nabla_\beta\chi - \nabla_\alpha\varphi_0\chi + \nabla_\alpha\varpi = 0, \tag{3.8}$$

$$\chi - \Gamma v_\alpha\nabla_\alpha\varphi_0 + \frac{\Gamma}{2}\epsilon_{\alpha\beta}\nabla_\alpha v_\beta = -\Gamma V_\alpha\nabla_\alpha\varphi_0, \tag{3.9}$$

where $\Gamma = \gamma/\eta$, as above, and $\varpi = \eta^{-1}[\varsigma - K/2(\nabla\varphi)^2]$. It follows from Eq. (3.8) and $\nabla_\alpha v_\alpha = 0$ that $\nabla^2\varpi = \nabla_\alpha\varphi_0\nabla_\alpha\chi$. A solution of the system in Eqs. (3.8) and (3.9) can be written as

$$v_\alpha = V_\alpha + \epsilon_{\alpha\beta}\nabla_\beta\Omega, \tag{3.10}$$

where V_α is the obvious (because of the Galilean invariance) forced solution and the stream function Ω describes a zero mode of system (3.8) and (3.9). The system is homogeneous in r, and Ω is therefore a sum of contributions that are powerlike functions of r.

Taking the curl of Eq. (3.8), we obtain

$$-\nabla^4\Omega - \frac{1}{2}\nabla^2\chi - \epsilon_{\gamma\alpha}\nabla_\alpha\varphi_0\nabla_\gamma\chi = 0. \tag{3.11}$$

Substituting χ expressed in terms of $\mathbf{v}$ from Eq. (3.9) into Eq. (3.11) and using explicit expressions (2.8) for the derivatives of φ_0, we obtain

$$\begin{aligned} &\left(1 + \frac{\Gamma}{4}\right)\nabla^4\Omega \\ &+ s\Gamma\left(\frac{2}{r^2}\partial_r^2\Omega - \frac{1}{r^2}\nabla^2\Omega - s\frac{1}{r^2}\partial_r^2\Omega + s\frac{1}{r^3}\partial_r\Omega\right) = 0 \end{aligned} \tag{3.12}$$

in the polar coordinates (r, ϕ). Solutions to Eq. (3.12) are superpositions of the terms $\propto r^{\alpha+1}\exp(im\phi)$. Substituting this r, ϕ dependence into Eq. (3.12), we obtain an equation for α that has the roots

$$\begin{aligned} \alpha = \pm\frac{1}{\sqrt{2}}[&2 + 2m^2 - s(1-s)\tilde{\Gamma} \\ &\pm\{(2 + 2m^2 - s(1-s)\tilde{\Gamma})^2 \\ &- 4s\tilde{\Gamma}(m^2 - 1 + s) - 4(m^2-1)^2\}^{1/2}]^{1/2}, \end{aligned} \tag{3.13}$$

where $\tilde{\Gamma} = \Gamma(1 + \Gamma/4)^{-1}$. Hence, $0 < \tilde{\Gamma} < 4$ for any γ and η. Evidently, all the roots in Eq. (3.13) are real. We emphasize that there is no solution $\alpha = 0$ (corresponding to a logarithmic behavior of the velocity in r) among the set (3.13). The first angular harmonic with $|m| = 1$ is of particular interest because $\varphi_1 = ur\sin\phi$ and $\Omega = -Vr\sin\phi$ far from the disclination. If Γ is small, there is a pair of small solutions among (3.13),

$$\alpha = \pm\alpha_1, \quad \alpha_1 = s\sqrt{\Gamma}/2, \tag{3.14}$$

for $m = \pm1$. Otherwise, for any other relevant m, solutions (3.13) have no special smallness (terms with $m = 0$ are forbidden because of the symmetry).

We established that Ω is a superposition of the terms $\propto r^{\alpha+1}\exp(im\phi)$ with the exponents α determined by Eq. (3.13). The velocity can then be found from Eq. (3.10). To avoid a singularity in the velocity at small r, one should keep contributions with positive α only. In other words, the velocity field contains contributions with all powers α given by (3.13), but the factors at the terms with negative α are formed at $r \sim a$ (where a is the disclination core radius), and the corresponding contributions to the velocity are therefore negligible at $r \gg a$ (this statement must be clarified and refined for small negative exponents $-\alpha_1$ in the limit of small Γ; see Subsection 4C). We conclude that the correction to $\mathbf{V}$ in the flow velocity $\mathbf{v}$ related to Ω in Eq. (3.10) is negligible at $r \sim a$. We thus arrive at the nonslipping condition for the disclination motion: the disclination velocity $\mathbf{V}$ coincides with the flow velocity $\mathbf{v}$ at the disclination position.

Next, to find φ, one should solve the equation $(K/\eta)\nabla^2\varphi = \chi$, where χ is determined from Eq. (3.9). In addition to the part determined by the velocity, φ_1 can then involve zero modes of the Laplacian. The most dangerous zero mode is Uy, because it produces a nonzero momentum flux to the disclination core (and the Magnus force associated to it),

$$\oint dr_\alpha\epsilon_{\alpha\beta}T_{\beta\gamma} \sim KU. \tag{3.15}$$

But because of the condition $\alpha \neq 0$, all the contributions to the velocity correspond to zero viscous momentum flux to the origin. Consequently, it is impossible to compensate the Magnus force by other terms. The above reasoning leads us to the conclusion that the factor U (and therefore, the Magnus force) must be zero. Thus, φ_1 contains only terms proportional to $r^{\alpha+1}$ with

$\alpha > 0$. This conclusion is related to the fact that, for free-standing liquid-crystalline films, any distortion of the bond angle unavoidably produces hydrodynamic backflow motions (i.e., $\mathbf{v} \neq 0$). For liquid-crystalline films on substrates (Langmuir films), in contrast to free-standing films, hydrodynamic motions (backflows) are strongly suppressed by the substrate, and the situation where the backflow is irrelevant for the disclination motion can be realized.

B. The remote region

Let us consider the region $r \gg u^{-1}$, where we can write $\varphi = uy + \tilde{\varphi}$ and linearize the system of equations (3.2) and (3.3) with respect to $\tilde{\varphi}$. We then obtain the system of linear equations for $\mathbf{v}$ and $\tilde{\varphi}$,

$$\begin{aligned} &\nabla^2 v_\alpha + \frac{K}{2\eta}(\epsilon_{\alpha\beta}\nabla_\beta\nabla^2\tilde{\varphi} - 2u_\alpha\nabla^2\tilde{\varphi}) + \nabla_\alpha\varpi = 0, \\ &(\nabla^2 + 2p\partial_x)\tilde{\varphi} + \frac{\gamma}{2K}(\epsilon_{\alpha\beta}\nabla_\alpha v_\beta - 2uv_y) = 0, \end{aligned} \tag{3.16}$$

where $p = V\gamma/2K$. Taking the curl of the first equation and eliminating the Laplacian, we obtain

$$\epsilon_{\beta\alpha}\nabla_\beta v_\alpha = \frac{K}{2\eta}[(\nabla^2 + 2u\partial_x)\tilde{\varphi} + \Phi], \tag{3.17}$$

where Φ is a harmonic function. In terms of Φ, system (3.16) is reduced to

$$\left[\left(1 + \frac{\Gamma}{4}\right)\nabla^4 + 2p\nabla^2\partial_x - \Gamma u^2\partial_x^2\right]\tilde{\varphi} = \frac{\Gamma}{2}u\partial_x\Phi. \tag{3.18}$$

Equation (3.18) can be written as

$$(\nabla^2 + 2k_1\partial_x)(\nabla^2 - 2k_2\partial_x)\tilde{\varphi} = \frac{\tilde{\Gamma}}{2}u\partial_x\Phi, \tag{3.19}$$

$$k_{1,2} = \frac{1}{2(1+\Gamma/4)}\left(\sqrt{p^2 + \Gamma\left(1 + \frac{\Gamma}{4}\right)u^2} \pm p\right). \tag{3.20}$$

The quantities k_1 and k_2 have the meaning of characteristic wave vectors. We conclude from Eq. (3.19) that zero modes of the operator on the left-hand side of the equation are proportional to

$$\exp(-k_1 r - k_1 x), \quad \exp(-k_2 r + k_2 x),$$

that is, they are exponentially small everywhere outside narrow angular regions near the X axis. The behavior of the zero modes inside the regions is powerlike in r. In addition, there is a contribution to $\tilde{\varphi}$ related to the harmonic function Φ. It contains a part that decays as a power of r (the leading term is $\propto r^{-1}$) at $r \gg u^{-1}$. This solution is examined in more detail in Appendix A.

4. DIFFERENT REGIMES GOVERNED BY Γ

The behavior of the velocity and the bond (director) angle fields around the moving disclination is sensitive to the ratio of the rotational and the shear viscosity coefficients $\Gamma = \gamma/\eta$. In this section, we examine different cases depending on the Γ value.

A. The case where $\Gamma \gtrsim 1$

We start analyzing different mobility regimes with the most probable case where $\Gamma \gtrsim 1$. If $\Gamma \sim 1$, then the factor C in Eq. (3.4) is on the order of 1 and $u \sim p$. It then follows from Eqs. (3.20) that k_1, $k_2 \sim u$. This is a manifestation of the fact that there is a unique characteristic scale in this case, given by u^{-1}. We can then estimate $\tilde{\varphi}$ by matching the solutions in the regions near the disclination and far from it at $r \sim u^{-1}$. We conclude that it is a function of the dimensionless parameter ur; the function is on the order of unity when its argument ur is on the order of unity.

For large Γ, there remains a unique characteristic scale u^{-1}, and consequently, $C \sim 1$ in this case. To prove this statement, we first treat small distances $r \ll u^{-1}$. As shown in Section 3A, the respective corrections φ_1 and $\delta\mathbf{v}$ to φ_0 and $\mathbf{V}$ are expanded in a series over the zero modes characterized by exponents (3.13). In particular, for $m = 1$, we can write $\varphi_1 \sim uy(ur)^\alpha$. In the large-$\Gamma$ limit, the exponents α given by (3.13) are regular because $\tilde{\Gamma} \longrightarrow 4$. From (3.13), we have $\alpha_1 \sim 1$, and in this case,

$$\chi \sim \frac{K}{\eta r^2}uy(ur)^{\alpha_1}.$$

Comparing Eqs. (3.8) and (3.9), we conclude that, for large Γ, the term involving χ can be omitted in Eq. (3.9), and the equation therefore becomes a constraint imposed on the velocity. Equation (3.8) then gives

$$|\delta v| \sim \frac{K}{\eta r}uy(ur)^{\alpha_1}.$$

The disclination velocity can now be found from the relation $V \sim |\delta v|$ at the scale u^{-1}, that is, $p \sim \Gamma u$, or $C \sim 1$. The complete analysis also covers the remote region. With the condition $p \sim \Gamma u$, it follows that $k_{1,2} \sim u^{-1}$. Using the procedure given in Appendix A, we can then prove that the solutions in the two regions can be matched at $r \sim u^{-1}$, and therefore, there are no new characteristic scales, indeed. We also note that the rotational viscosity γ drops from the hydrodynamic equations at large Γ. Although this is not true inside the disclination core (see Appendix D), the boundary conditions for $\mathbf{v}$ and φ on the core boundary reveal no dramatic changes in behavior. Consequently, it is the shear viscosity alone that determines the disclination mobility, which implies that $C \sim 1$.

We can therefore say that, in the limit as $\Gamma \longrightarrow \infty$, no additional features appear compared to $\Gamma \sim 1$. But this is not the case for small Γ, because $u \gg p$ for $\Gamma \ll 1$. We study this case in the next subsection.

B. Small Γ

Here, we consider the case where $\Gamma \ll 1$. This limit is physically attained at anomalously large η, with $K\rho/\eta^2$ still treated as the smallest dimensionless parameter. This justifies the use of the same equations (3.2) and (3.3) as in the previous subsections.

For $r \ll u^{-1}$, the analysis in Section A is correct. As we noted, the contributions to $\mathbf{v}$ and φ_1 related to the modes with negative α should not be taken into account there. For $\Gamma \ll 1$, the leading role is played by the mode with the smallest exponent ($\alpha_1 = s\sqrt{\Gamma}/2$), because the presence of modes with positive exponents $\alpha \sim 1$ would contradict the condition of smooth matching at $r \sim u^{-1}$. Strictly speaking, neglecting a small negative exponent $-\alpha_1$ is correct under the condition $\alpha_1|\ln(ua)| \gg 1$, where a is the core radius of the disclination. This is what is considered in this subsection. The opposite case, which we call the extremely small-Γ limit, is analyzed in Section 4C. At $r \ll u^{-1}$, we can therefore write

$$\varphi_1 \sim uy(ur)^{\alpha_1}, \quad V - v_x \sim \alpha_1 u\frac{K}{\gamma}(ur)^{\alpha_1}, \tag{4.1}$$

with the coefficient at $y(ur)^{\alpha_1}$ determined from matching at $r \sim u^{-1}$, where $\nabla\varphi \sim 1/r$. Similarly, matching $V - v_x \sim V$ at $r \sim u^{-1}$ gives $V \sim \alpha_1 uK/\gamma$. The relation can be rewritten as $p \sim \alpha_1 u \ll u$, and we therefore conclude that $C \sim 1/\sqrt{\Gamma}$.

In accordance with Eq. (3.20), the relation $p \sim \sqrt{\Gamma}u$ leads to $k_{1,2} \sim p \ll u$. In other words, a new scale p^{-1} (different from u^{-1}) appears in the problem. A detailed investigation of the remote region $r \gg u^{-1}$ is therefore needed to establish the $\mathbf{r}$ dependences of the both angle φ and the velocity field $\mathbf{v}$ there. This investigation can be based on the equations formulated in Section 3B, which are correct irrespective of the value of pr.

Explicit expressions describing the velocity and the angle are presented in Appendix A. They contain three dimensionless functions $\zeta_1(\nabla/u)$, $c_1(\nabla/u)$, and $c_2(\nabla/u)$. At $ur \gg 1$, only zero terms of the expansions of these functions in the Taylor series can be kept. Only one of these three coefficients is independent (see Eq. (A.10)). The general solution can therefore be expressed in terms of a single parameter, which we choose as $\zeta \equiv \zeta_1(0)$. The procedure corresponds to the following construction of the solutions to equations of motion (3.16) in the region $ur \gg 1$. We have to match the solutions in the outer and the inner regions (far from and close to the disclination respectively) at $ur \sim 1$. Technically, the matching is equivalent to the appropriate boundary conditions for the outer problem at $ur \sim 1$, and these boundary conditions can be formally replaced by the local source terms in the equations, acting at $ur \sim 1$. We can expand these sources in the standard multipolar series. We thus arrive at the expansion with respect to the gradients of the δ function. The gradients scale as u, and therefore, ζ, c_1, and c_2 are dimensionless functions of the dimensionless ratio ∇/u.

To find the asymptotic behavior of the angle φ and of the velocity $\mathbf{v}$, we first consider the region $u^{-1} \ll r \ll p^{-1}$. From Eqs. (A.4), (A.5), and (A.10), we then derive

$$v_x = \frac{K(2s-\zeta)}{\gamma u}k_1k_2\ln(pr), \tag{4.2}$$

where we keep only the leading logarithmic contribution of the zero harmonic in v_x. Matching the velocity derivatives determined by Eqs. (4.1) and (4.2) at $r \sim u^{-1}$, we find that $\zeta \sim 1$ (we imply that $s \sim 1$). Using Exps. (A.2), (A.5), and (A. 10), we obtain

$$\varphi = \varphi_0 + uy + spy\ln(pr) \tag{4.3}$$

in the region $u^{-1} \ll r \ll p^{-1}$. We see that there is only a small correction to the simple expression $\varphi_0 + uy$ in that region, because $p \ll u$.

In the region $pr \gg 1$, the expressions for the angle φ and the velocity $\mathbf{v}$ are more complicated. Using Eqs. (A.2)–(A.5), we obtain

$$\begin{aligned}\partial_x\varphi = -s\sqrt{\frac{\pi}{2}}[c_1\sqrt{k_1}\exp(-k_1r-k_1x) \\ + c_2\sqrt{k_2}\exp(-k_2r+k_2x)]\frac{y}{r^{3/2}} - \frac{\zeta}{2}\frac{y}{r^2},\end{aligned} \tag{4.4}$$

$$\begin{aligned}\partial_y\varphi = u - 2s\sqrt{\frac{\pi}{2}}[c_1\sqrt{k_1}\exp(-k_1r-k_1x) \\ - c_2\sqrt{k_2}\exp(-k_2r+k_2x)]\frac{y}{r^{1/2}} - \frac{\zeta}{2}\frac{x}{r^2},\end{aligned} \tag{4.5}$$

$$\begin{aligned}v_y = \frac{K}{\gamma u}\Bigg\{2s\sqrt{\frac{\pi}{2}}[c_1k_2\sqrt{k_1}\exp(-k_1r-k_1x) \\ - c_2k_1\sqrt{k_2}\exp(-k_2r+k_2x)]\frac{y}{r^{3/2}} - p\zeta\frac{y}{r^2}\Bigg\},\end{aligned} \tag{4.6}$$

$$\begin{aligned}v_x = -\frac{K}{\gamma u}\Bigg\{\sqrt{\frac{\pi}{2}}s\Gamma u^2\Bigg[\frac{c_1}{\sqrt{k_1r}}\exp(-k_1r-k_1x) \\ + \frac{c_2}{\sqrt{k_2r}}\exp(-k_2r+k_2x)\Bigg] + p\zeta\frac{x}{r^2}\Bigg\},\end{aligned} \tag{4.7}$$

where $c_1 \sim 1$ and $c_2 \sim 1$ are determined by Eq. (A.10) (we omitted the argument 0 to simplify the notation). Expressions (4.4)–(4.7) contain terms of two types, isotropic and anisotropic ones. The anisotropic contributions are essential only in the narrow angular regions near the *X* axis, where they dominate. It is worth noting a very nontrivial structure of the flow, in which the isotropic flux to the origin is compensated by the anisotropic terms.

The expressions found in this subsection generalize the famous Lamb solution for the hydrodynamic flow around a hard cylinder (see, e.g., [33–35]), where the velocity field is exponentially small everywhere far from the cylinder except for the wake of the corps, i.e., in a very narrow angular sector ("tail"). Disclination motion in liquid-crystalline films can be regarded as the motion of a cylinder framed by a "soft" (i.e., deformable) orientational field φ. Because of the additional degree of freedom (compared to the classical Lamb problem), our solution has two tails around the moving disclination: a wake beyond the disclination and a precursor in front of it. In fact, both degrees of freedom (the flow velocity and the bond angle) are relevant.

C. Extremely small Γ

In the above analysis, we implied the condition $\alpha_1|\ln(ua)| \gg 1$ (we recall that $\alpha_1 = s\sqrt{\Gamma}/2$ at small Γ), imposing a restriction from below on Γ at a given u. If $\alpha_1|\ln(ua)| \ll 1$, the terms with both $\alpha = \pm\alpha_1$ determined by Eq. (3.14) must be taken into account near the disclination, which leads to a logarithmic behavior of the correction φ_1 to φ_0 in that region,

$$\varphi_1 \sim uy\ln\left(\frac{r}{a}\right)|\ln(au)|^{-1}, \tag{4.8}$$

instead of Eq. (4.1). Matching the derivatives of expressions (4.3) and (4.8) at $r \sim u^{-1}$ gives $p \sim u|\ln(au)|^{-1}$. In other words, $C \sim [\Gamma\ln(au)]^{-1}$. This case formally corresponds to the limit $\eta \longrightarrow \infty$ in our equations, where we can drop the backflow hydrodynamic velocity in the equation for the bond angle. The situation was examined in [6–9]. We present the simple analysis of the case in Appendix B. We also note that there is no crossover at $r \sim u^{-1}$ in the bond angle behavior in this situation.

We now clarify the question regarding the Magnus force in this case. In accordance with Eq. (4.8), the reactive momentum flux to the disclination core is

$$\oint dr_\alpha \epsilon_{\alpha\beta} T_{\beta\gamma} \sim Ku\ln\left(\frac{r}{a}\right)|\ln(au)|^{-1}.$$

The flux is therefore r-dependent, tending to zero as $r \longrightarrow a$. This reactive momentum flux is compensated by the viscous momentum flux (related to derivatives of the flow velocity $\mathbf{v}$), which is nonzero in this case because of the logarithmic behavior of the flow velocity in r near the disclination. The flow velocity can be found from Eqs. (3.6) and (4.8) as

$$v_\alpha \sim \frac{Ku}{\eta|\ln(au)|}\epsilon_{\alpha\beta}\nabla_\beta\left[y\ln^2\left(\frac{r}{a}\right)\right],$$

which is a generalization of the Stokes–Lamb solution [33, 34]. But unlike in the Lamb problem (a hard cylinder moving in a viscous liquid), $|\mathbf{V} - \mathbf{v}(\mathbf{r} = \mathbf{a})| \sim V$ in our case; i.e., we have a slipping of the core of the moving disclination. This slipping seems natural in the limit of extremely small values of Γ, corresponding to the limit $\eta \longrightarrow \infty$, that is, to a strongly suppressed hydrodynamic flow. Physically, this property implies that the disclination cannot be understood as a hard impenetrable object. It is also worth noting that the logarithmic behavior found above is similar to the general feature of two-dimensional hydrodynamic motion that comes from the well-known fact (see, e.g., [33–35]) that nonlinear terms cannot be neglected in a two-dimensional laminar flow even for a small Reynolds number; these terms become relevant for sufficiently large distances. But in our case, these nonlinear terms do not come from the convective hydrodynamic nonlinearity; they come from the terms in stress tensor (2.4) that are nonlinear in φ.

An explicit expression for φ and its asymptotic forms corresponding to the considered case are given in Appendix B. An expression for the flow velocity field induced by the disclination motion at extremely small Γ is derived in Appendix C.

5. CONCLUSIONS

We now summarize the results of our paper. To understand the physics underlying the freely suspended film dynamics, we studied the ground case—a single disclination motion in a thin hexatic, smectic-*C*, or nematic liquid-crystalline film, driven by an inhomogeneity in the bond (or director) angle. We investigated the uniform motion (the one with a constant velocity). In this case, we derived and solved the equations of motion and found the bond angle and hydrodynamic velocity distributions around the disclination. This allows us to relate the velocity of the disclination V to the bond angle gradient $u = |\nabla\varphi|$ in the region far from the disclination. So much effort is needed because the full set of equations must be solved everywhere, not only locally. We established the proportionality coefficient C (see Eq. (3.4)) in this nonlocal relationship; it has the meaning of an effective mobility coefficient. The coefficient C depends on the dimensionless ratio Γ of rotational (γ) and shear viscosity (η) coefficients.

There is little experimental knowledge of the values of the coefficients γ and η in liquid-crystalline films. It is generally believed that the corresponding values in a film (normalized by its thickness) and in a bulk material are not very different [31, 3], in which case we are in the regime of $\Gamma \sim 1$, where the coefficient C is on the order of 1. But the case where $\Gamma \ll 1$ is not excluded from both theoretical and materials science standpoints. We found the coefficient $C \sim 1/\sqrt{\Gamma}$ in the small-Γ limit. We established a highly nontrivial behavior of the flow velocity and of the bond angle, which is powerlike in r near the disclination and extremely anisotropic far from it. Only for extremely small Γ, $\Gamma \ll 1/\ln^2(ua)$ (where a is the disclination core radius), did we find a logarithmic behavior $C \sim [\Gamma \ln(ua)]^{-1}$. The main message of our study is that the hydrodynamic motion (that is, the backflow), unavoidably accompanying any defect motion in liquid crystals, plays a significant role in the disclination mobility. Experimental evidence (see, e.g., the recent publication [36]) shows that this is indeed the case.

Our analysis can be applied to the motion of a disclination pair with the opposite topological charges. In this case, the role of the scale u^{-1} is played by the distance R between the disclinations. In accordance with Eq. (3.4), we then find that $\partial_t R \propto R^{-1}$ without a logarithm (provided the rotational viscosity coefficient γ is not anomalously small; see Section 4C for the quantitative criterion). This conclusion is confirmed by the results of numerical simulations for 2D nematics [15–18]. The authors of [15–18] consider the equations of motion in terms of the tensor order parameter, consistently taking the coupling between the disclination motion and the hydrodynamic flow into account. They simulated dynamics of the disclination pair annihilation and found that the distance R between the disclinations scales depends on time t as $t^{1/2}$, without logarithmic corrections (as follows from our theoretical analysis) for all values of the parameter Γ except extremely small ones. Unfortunately, we did not find in [16–18] the magnitudes of the shear viscosity that were used in the simulations. Lacking sufficient data on the values of γ and η, we can presently discuss only the general features of the disclination dynamics. For instance, the authors of [18] numerically found an asymmetry of the disclination dynamics with respect to the sign of the topological charge ($s = \pm 1/2$) in the one-constant approximation. In our approach, the asymmetry naturally appears from nonlinear terms in stress tensor (2.5) and from the first term on the right-hand side of Eq. (2.6) responsible for the different couplings of orientational and hydrodynamic flow patterns for positive and negative disclinations. This results in the fact that, for each m, the smaller positive exponents in Eq. (3.13) (corresponding to the minus in the brackets) are larger for $s = 1/2$ than for $s = -1/2$. The disclination with $s = 1/2$ therefore exerts a stronger influence on the flow velocity; this conclusion was qualitatively obtained in [18].

Although the theory presented in this paper is valid for free-standing liquid-crystalline films, the general scheme can be applied to liquid-crystalline films on solid or liquid substrates. Because such a film is arranged on the substrate surface, any of its hydrodynamic motions are accompanied by the substrate motion. For solid substrates, the situation where the hydrodynamic backflow is irrelevant for the disclination dynamics can therefore be realistic. In Section 4C (also see Appendix B), we examine this limit and reproduce the results in [6–9], where the hydrodynamic backflow was neglected from the very beginning. The case of the films on a liquid substrate requires a special investigation, but the approach and the main ideas of our paper could be useful there as well.

Our results can be directly tested by comparing with the experimental data for smectic-C or nematic films. The hexatic order parameter, which has a sixfold local symmetry, is not coupled to the light in any simple way (and, therefore, ideal hexatic disclinations are hardly observed in optics). But it is possible to observe the core splitting of the disclinations in tilted hexatic smectic films [26]. Indeed, because of discontinuity of the tilt direction (which is locked to the bond direction), the hexatic order and hexatic disclinations can be observed indirectly. The second possibility of detecting the defects of hexatic ordering and verifying our theoretical results is classical light scattering (where the wave vectors are $q = 10^2$–10^4 cm^{-1} and the frequency is $\omega \lesssim 10^8$ s^{-1} in typical experiments). For a reasonably thick film, the power spectrum of light scattering can have some additional structure revealing the disclination properties (e.g., defects are thought to be relevant to the very low frequency noise observed in thin films). Experimental studies of this type are highly desirable.

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APPENDIX A

Distances Far from the Disclination

Here, we derive some results for the region far from the disclination. These results are used in the case of small Γ considered in Section 4B.

We examine the harmonic function Φ in Eq. (3.17). Because the function is analytic in the region $r > u^{-1}$, it can be expanded in the derivatives of $\ln r$ there. Next, because of the symmetry of the problem, Φ is an antisymmetric function of y. At least one derivative ∂_y must therefore be present in each term of the expansion, that is,

$$\Phi = u\hat{\zeta}_1 \partial_y \ln r, \tag{A.1}$$

where $\hat{\zeta}_1 = \zeta_1(\nabla/u)$ and $\zeta_1(z)$ is a series in z converging in a circle with the radius on the order of 1. The expansion coefficients in the series $\zeta_1(\nabla/u)$ are determined by matching with the inner problem at $r \sim u^{-1}$.

Because of the symmetry, the angle $\tilde{\varphi}$ can be represented as

$$\begin{gathered}\partial_x\tilde{\varphi} = \partial_y B, \quad \partial_y\tilde{\varphi} = -(H + \partial_x B),\\ \nabla^2 B + \partial_x H = 0.\end{gathered} \tag{A.2}$$

The latter equation is the condition $\epsilon_{\alpha\beta}\nabla_\alpha\nabla_\beta\tilde{\varphi} = 0$. We note that $\nabla^2\tilde{\varphi} = -\partial_y H$. In the region far from the disclination, we can use Eqs. (3.16) and (3.17). The incompressibility condition $\nabla_\alpha v_\alpha = 0$ must also be taken into account. We thus obtain expressions for the velocity in terms of B and H,

$$\begin{gathered}\mathrm{curl}\,\mathbf{v} = \frac{K}{2\eta}\partial_y[-H + 2uB + u\hat{\zeta}_1\ln(pr)],\\ v_y = \frac{K}{\gamma u}\partial_y\\ \times\left\{-H + 2pB + \frac{\Gamma}{4}[-H + 2uB + u\hat{\zeta}_1\ln(pr)]\right\},\end{gathered} \tag{A.3}$$

$$\begin{gathered}v_x = \frac{K}{\gamma u}\partial_x\\ \times\left\{-H + 2pB + \frac{\Gamma}{4}[-H + 2uB + u\hat{\zeta}_1\ln(pr)]\right\}\\ -\frac{K}{2\eta}[-H + 2uB + u\hat{\zeta}_1\ln(pr)].\end{gathered} \tag{A.4}$$

Solutions to Eq. (3.18) imply that

$$\begin{gathered}B = s[\hat{c}_1K_0(k_1r)e^{-k_1x} + \hat{c}_2K_0(k_2r)e^{k_2x}] - \frac{1}{2}\hat{\zeta}_1\ln(pr),\\ H = 2s[k_1\hat{c}_1K_0(k_1r)e^{-k_1x} - k_2\hat{c}_2K_0(k_2r)e^{k_2x}].\end{gathered} \tag{A.5}$$

Here, the particular representation in Eq. (A.1) is used and an arbitrary function of y that can contribute to H is chosen to be zero because $\nabla\tilde{\varphi} \longrightarrow 0$ (and, hence, $H \longrightarrow 0$) as $r \longrightarrow \infty$. In (A.5), $\hat{c}_1$ and $\hat{c}_2$ are dimensionless differential operators that can be represented as Taylor series in ∇/u, i.e., $c_1(\nabla/u)$ and $c_2(\nabla/u)$. These functions must scale with u because the functions must be found from matching at $r \sim u^{-1}$.

Additionally, there are two conditions for the variables in the region $ur \gg 1$. First, the correct circulation around the origin leads to the effective δ-functional term in Eq. (A.2),

$$\nabla^2 B + \partial_x H = -2\pi s\delta(\mathbf{r}). \tag{A.6}$$

The second condition is the absence of the flux to the origin,

$$\int d\phi\, v_r(r, \phi) = 0. \tag{A.7}$$

Relations (A.6) and (A.7) lead to the conditions

$$c_1(0) + c_2(0) + \frac{\zeta_1(0)}{2s} = 1, \tag{A.8}$$

$$\begin{gathered}\left(1 + \frac{\Gamma}{4}\right)[k_1c_1(0) - k_2c_2(0)] - \left(p + \frac{\Gamma u}{4}\right)\\ \times\left[c_1(0) + c_2(0) + \frac{\zeta_1(0)}{2s}\right] + \frac{\Gamma u}{8s}\zeta_1(0) = 0.\end{gathered} \tag{A.9}$$

At small Γ, the solution to Eqs. (A.8) and (A.9) is

$$\begin{gathered}\zeta_1(0) = \zeta, \quad c_1(0) = \frac{k_1 - \zeta k_2/2s}{k_1 + k_2},\\ c_2(0) = \frac{k_2 - \zeta k_1/2s}{k_1 + k_2}.\end{gathered} \tag{A.10}$$

We also assumed that $\zeta \lesssim 1$, which is justified in Subsection 4B.

APPENDIX B

Suppressed Flow

Here, we demonstrate how the disclination velocity V can be found if the hydrodynamic velocity $\mathbf{v}$ is negligible (e.g., because of substrate friction). We reproduce the results in [6–9].

In the absence of the hydrodynamic flow, the equation for the angle φ is purely diffusive,

$$\gamma\partial_t\varphi = K\nabla^2\varphi, \tag{B.1}$$

as follows from Eq. (1.6) with $\mathbf{v} = 0$. We assume that $\varphi \longrightarrow uy$ as $r \longrightarrow \infty$. The disclination motion is forced by the "external field" u. We seek a solution $\varphi(t, x, y) = \varphi(x - Vt, y)$. From Eq. (B.1), we then obtain

$$2p\partial_x\varphi + \nabla^2\varphi = 0, \quad \text{where} \quad 2p = \gamma V/K. \tag{B.2}$$

In what follows, we consider the solution corresponding to a single disclination with the circulation

$$\oint d\mathbf{r}\nabla\varphi = 2\pi s, \tag{B.3}$$

where the integral is taken along a contour encompassing the disclination counterclockwise. The quantity s in Eq. (B.3) is an arbitrary parameter (which is equal to ±1/6 for hexatic, ±1/2 for nematic, and ±1 for smectic-C ordering). For a suitable solution to Eq. (B.2) corresponding to Eq. (B.3), we have

$$\begin{aligned}\partial_x\varphi &= s\partial_y\int\frac{d^2q}{2\pi}\frac{1}{q^2 - 2ipq_x}\exp(i\mathbf{q}\cdot\mathbf{r})\\ &= s\exp(-px)\partial_yK_0(pr).\end{aligned} \tag{B.4}$$

This derivative tends to zero as $r \longrightarrow \infty$, as it should be.

Expression (B.4) does not determine φ unambiguously because $\partial_x(uy) = 0$, and we can therefore obtain a new solution by adding a term uy to a given solution. We note that uy is the zero mode of the Eq. (B.2). The solution can therefore be written as

$$\varphi = \tilde{\varphi}_L + uy,$$
$$\tilde{\varphi}_L(x', y) = -s\int_{x'}^{\infty} dx \exp(-px)\partial_y K_0(pr), \tag{B.5}$$

where $\tilde{\varphi}_L$ tends to zero as $r \longrightarrow \infty$. To relate p and u in Eq. (B.5), we must know the boundary conditions at $r \longrightarrow 0$, or, in fact, at $r \sim a$, where a is the core radius. At small r, the angle φ can be written as a series $\varphi = \varphi_0 + \varphi_1 + \ldots$, where φ_0 corresponds to the static disclination and φ_1 is the first correction to φ_0 related to the motion. Matching with the inner problem gives

$$\nabla\varphi_1(a) \sim p, \tag{B.6}$$

because the solution for the order parameter inside the core is an analytic function of r/a and the expansion in p is a regular expansion in pa (see [7] and Appendix D).

Expanding Eq. (B.4) in p, we obtain

$$\frac{1}{s}\partial_x\varphi \approx -\frac{y}{r^2} + \frac{pxy}{r^2}$$

at $pr \ll 1$. In accordance with Eq. (B.5), we then obtain with logarithmic accuracy (i.e., in the main approximation in $|\ln(pa)| \gg 1$) that

$$\varphi_1 = spy\ln(pr) + uy. \tag{B.7}$$

Using boundary condition (B.6), we now obtain

$$u = sp\ln\left(\frac{1}{pa}\right) \tag{B.8}$$

with the same logarithmic accuracy. This can be rewritten as

$$V = \frac{2Ku}{s\gamma\ln(1/pa)}. \tag{B.9}$$

The same answer (B.9) can be found from the energy dissipation balance. First of all, we can find the energy E corresponding to solution (B.5),

$$\begin{aligned} E &= \int d^2r \frac{K}{2}(\nabla\varphi)^2 \\ &= K\int d^2r\left[\frac{1}{2}u^2 + \frac{1}{2}(\nabla\tilde{\varphi}_L)^2 + u\partial_y\tilde{\varphi}_L\right], \end{aligned} \tag{B.10}$$

where the first term is the energy of the external field, the second term represents the energy of the disclination itself, and the third term is the coupling energy. Obviously, only the last cross-term depends on time. For $|x - Vt| \gg p^{-1}$,

$$\int_{-\infty}^{\infty} dy\partial_y\tilde{\varphi}_L = \begin{cases} 0 & \text{if } x > Vt, \\ -2\pi s & \text{if } x < Vt. \end{cases}$$

It then follows from Eq. (B.10) that

$$\partial_t E = -2\pi sKuV. \tag{B.11}$$

On the other hand, we can use Eq. (B.1) to obtain

$$\partial_t E = -\frac{K^2}{\gamma}\int d^2r(\nabla^2\varphi)^2. \tag{B.12}$$

Replacing $\nabla^2\varphi$ with $2p\partial_x\varphi$ here in accordance with Eq. (B.2), we obtain

$$\partial_t E = -\gamma V^2\int d^2r(\partial_x\varphi)^2.$$

The main logarithmic contribution to the integral comes from the region $a < r < p^{-1}$, where $\partial_x\varphi \approx -sy/r^2$. We thus obtain

$$\partial_t E = -\pi s^2\gamma V^2\ln\left(\frac{1}{pa}\right). \tag{B.13}$$

Comparing the expression with Eq. (B.11), we find the same answer (B.9).

APPENDIX C

Extremely Small Γ

Here, we consider the flow velocity induced by the moving disclination for extremely small Γ. The velocity is zero in the zero approximation in Γ (this case is considered in Appendix B), and we therefore examine the next, first-order, approximation in Γ. We use the same formalism and the same notation as in Appendix A.

In accordance with Appendix A, solutions to the complete set of nonlinear stationary equations can be represented as

$$\partial_x\tilde{\varphi} = \partial_y B, \quad \partial_y\tilde{\varphi} = -(H + \partial_x B), \tag{C.1}$$

$$\operatorname{curl}\mathbf{v} = \frac{K}{2\eta}[-\partial_y H + 2u\partial_y B + 2us\partial_y\ln r + \Phi'], \tag{C.2}$$

$$\begin{aligned} v_x &= -\frac{K}{2\eta}\partial_y\nabla^{-2} \\ &\times[-\partial_y H + 2u\partial_y B + 2us\partial_y\ln r + \Phi'], \end{aligned} \tag{C.3}$$

$$\begin{aligned} v_y &= \frac{K}{2\eta}\partial_x\nabla^{-2} \\ &\times[-\partial_y H + 2u\partial_y B + 2us\partial_y\ln r + \Phi'], \end{aligned} \tag{C.4}$$

where B, H, and Φ' are to be found from the equations

$$\begin{aligned} &-\partial_y H + 2p\partial_y B + \frac{\Gamma}{4}\nabla^{-2}(\nabla^2 - 2u\partial_x) \\ &\times[-\partial_y H + 2u\partial_y B + 2us\partial_y\ln r + \Phi'] \\ &= -\frac{\Gamma}{2}(\partial_y\nabla^{-2}[-\partial_y H + 2u\partial_y B + 2us\partial_y\ln r + \Phi']\partial_y B \\ &+ \partial_x\nabla^{-2}[-\partial_y H + 2u\partial_y B + 2us\partial_y\ln r + \Phi'](\partial_x B + H)), \end{aligned} \tag{C.5}$$

$$\Phi' = 2\nabla^{-2}[(\partial_x B + H)\partial_x\partial_y H + \partial_y B\partial_y^2 H], \quad (C.6)$$

$$\nabla^2 B + \partial_x H = -2\pi s\delta(\mathbf{r}). \quad (C.7)$$

If Γ is extremely small, $s^2\Gamma\ln^2(ua) \ll 1$, the solution to Eqs. (C.5)–(C.7) can be continued to the vicinity of the core. In the leading approximation, the solution for $\tilde{\varphi}$ coincides with the solution for the angle $\tilde{\varphi}_L$ in the absence of the backflow. This case, examined in [6–9], is described in Appendix B. The functions B_L and H_L corresponding to $\tilde{\varphi}_L$ are given by

$$2pB_L = H_L = 2spK_0(pr)\exp(-px). \quad (C.8)$$

This solution gives

$$\Phi' = 2s^2 p\frac{y}{r^2}\ln\left(\frac{\min\{r, p^{-1}\}}{a}\right). \quad (C.9)$$

Neglecting the nonlinear right-hand side of Eq. (C.5), we can then find

$$H(\mathbf{r}) = \frac{4\pi s}{1+\Gamma/4}\int\frac{d^2q}{(2\pi)^2}\exp(i\mathbf{q}\cdot\mathbf{r}) \quad (C.10)$$
$$\times\frac{pq^2-(s\Gamma p/4)(q^2+2iuq_x)\ln(\min\{(qa)^{-1},(pa)^{-1}\})}{(q^2-2ik_1q_x)(q^2+2ik_2q_x)}.$$

$B(\mathbf{r})$ can be found similarly. Using B and H in Eqs. (C.3) and (C.4), we calculate the flow velocity $\mathbf{v}(\mathbf{r})$ that vanishes at infinity.

For $r \gg p^{-1}$, this solution coincides with expressions (A.5), (A.8), and (A.9) with

$$\zeta_1(0) = 2s + \frac{2s^2p}{u}\ln\left(\frac{1}{pa}\right).$$

For $pr \ll 1$, expression (C.10) is reduced to (C.8) and this region produces the main contribution to Φ' in (C.9). The following expressions are obtained in the inner region ($pr \ll 1$) from the solution in Eqs. (C.1)–(C.10):

$$\varphi_1 = \left(u - sp\ln\frac{1}{pa}\right)y + spy\ln\frac{r}{a}, \quad (C.11)$$

$$\mathrm{curl}\,\mathbf{v} = \frac{Ks^2p}{\eta}\frac{y}{r^2}\ln\frac{r}{a}. \quad (C.12)$$

A relation between p and u is fixed by condition (B.6), leading to $u = sp\ln[1/(pa)]$, which is equivalent to Eq. (B.9). The flow velocity at $pr \ll 1$ and $\ln(r/a) \gg 1$ is

$$v_\alpha = -\frac{s^2\Gamma}{8}V\epsilon_{\alpha\beta}\nabla_\beta\left[y\ln^2\left(\frac{r}{a}\right)\right], \quad (C.13)$$

which corresponds to the stream function

$$\Omega = -Vy - \frac{Ks^2p}{4\eta}y\ln^2\left(\frac{r}{a}\right). \quad (C.14)$$

The expansion with respect to Γ near the disclination is regular and can be derived from Eqs. (3.8) and (3.9) with the condition $\nabla\varphi_1(a) \sim p$: $\tilde{\varphi}_L + uy$ is the zero term of the series for φ, and expression (C.14) represents the zero and the first terms for Ω.

We note that, in accordance with Eq. (C.13) in the limit as $\Gamma \longrightarrow 0$, the flow velocity tends to zero near the disclination core, $v(a)/V = O(\Gamma)$, despite the fact that the disclination itself moves with the finite velocity V; thus, there is a slipping on the disclination core in this limit.

APPENDIX D

Solution with the Complete Order Parameter

Here, we consider the dynamic equations for the coupled velocity field $\mathbf{v}$ and the complete order parameter $\Psi = Q\exp(i\varphi/|s|)$ describing the 2D orientational order in liquid-crystalline films. These equations are needed to examine the velocity field close to the disclination position. We assume that the core size a is larger than characteristic molecular scales and work in the framework of the mean field theory.

Formally, the equations can be derived using the Poisson bracket method [30, 37]. In the mean field approximation, the energy associated with the order parameter is

$$\mathcal{H}_\Psi = \frac{Ks^2}{2}\int d^2r\left(|\nabla\Psi|^2 + \frac{1}{2a^2}(1-|\Psi|^2)^2\right),$$

its density becomes the K contribution in Eq. (2.3) at large scales $r \gg a$. The only nontrivial Poisson bracket that must be added to the standard expressions is [28]

$$\{j_\alpha(r_1), \Psi(r_2)\} = -\nabla_\alpha\Psi\delta(r_1-r_2)$$
$$+\frac{i}{2|s|}\Psi(r_2)\epsilon_{\alpha\beta}\nabla_\beta\delta(r_1-r_2).$$

To be specific, we use the expressions for the energy and the Poisson bracket for hexatic films. The dynamic equations are given by

$$\rho\partial_t v_\alpha + \rho v_\beta\nabla_\beta v_\alpha = \eta\nabla^2 v_\alpha - \frac{s^2K}{2}$$
$$\times\left\{\nabla_\alpha\Psi^*\left(\nabla^2\Psi + \frac{1}{a^2}\Psi(1-|\Psi|^2)\right)\right.$$
$$\left.+\nabla_\alpha\Psi\left(\nabla^2\Psi^* + \frac{1}{a^2}\Psi^*(1-|\Psi|^2)\right)\right\} \quad (D.1)$$
$$-\frac{i|s|K}{4}\epsilon_{\alpha\beta}\nabla_\beta\{\Psi^*\nabla^2\Psi - \Psi\nabla^2\Psi^*\} + \nabla_\alpha\tilde{\zeta},$$
$$\partial_t\Psi + v_\alpha\nabla_\alpha\Psi = \frac{i}{2|s|}\Psi\epsilon_{\alpha\beta}\nabla_\alpha v_\beta$$
$$+\frac{Ks^2}{2\gamma_s}\left(\nabla^2\Psi + \frac{1}{a^2}\Psi(1-|\Psi|^2)\right),$$

the relation $\gamma_s = s^2\gamma/2$ ensures the reduction to Eq. (2.6) in the limit $|\Psi| = 1$, and the kinetic coefficients are believed to be independent of Q (otherwise, we can assume, for example, the dependence $\gamma_s = s^2\gamma|\Psi|^2/2$). The slow dynamics of a 2D liquid-crystalline system with disclinations can be described by Eqs. (D.1) with the additional incompressibility condition $\nabla\mathbf{v} = 0$ that allows excluding the passive variable $\tilde{\xi}$.

If the distance from the disclination to a boundary or other disclinations is much larger than a (i.e., the perturbation of the static solution $\Psi_0 = Q_0\exp(i\varphi_0/|s|)$ for a single defect is small), we can linearize Eqs. (D.1) with respect to the perturbation expressed in terms of the respective corrections Q_1 and φ_1 to Q_0 and φ_0,

$$\eta\nabla^2 v_\alpha - 2\gamma_s\Big\{\nabla_\alpha Q_0(v_\beta - V_\beta)\nabla_\beta Q_0 + \frac{1}{s^2}Q_0^2\nabla_\alpha\varphi_0\Big((v_\beta - V_\beta)\nabla_\beta\varphi_0 - \frac{1}{2}\epsilon_{\beta\gamma}\nabla_\beta v_\gamma\Big)\Big\} \tag{D.2}$$
$$+ 2\gamma_s\frac{1}{2s^2}\epsilon_{\alpha\beta}\nabla_\beta\Big[Q_0^2\Big((v_\mu - V_\mu)\nabla_\mu\varphi_0 - \frac{1}{2}\epsilon_{\mu\nu}\nabla_\mu v_\nu\Big)\Big] + \nabla_\alpha\tilde{\xi} = 0,$$

$$\frac{Ks^2}{2\gamma_s}\Big(\nabla^2 Q_1 - \frac{(\nabla\varphi_0)^2}{s^2}Q_1 - \frac{2\nabla_\alpha\varphi_1\nabla_\alpha\varphi_0}{s^2}Q_0 + \frac{1}{a_s^2}(1 - 3Q_0^2)Q_1\Big) = (v_\beta - V_\beta)\nabla_\beta Q_0, \tag{D.3}$$

$$\frac{Ks^2}{2\gamma_s}(\nabla^2\varphi_1 + 2Q_0^{-1}(\nabla_\alpha Q_1\nabla_\alpha\varphi_0 + \nabla_\alpha Q_0\nabla_\alpha\varphi_1)) = -\frac{1}{2}\epsilon_{\alpha\beta}\nabla_\alpha v_\beta + (v_\beta - V_\beta)\nabla_\beta\varphi_0. \tag{D.4}$$

In terms of the dimensionless quantities $L = \eta\Omega/K$, $R = r/a$, and $\Gamma = 2\gamma_s/(s^2\eta)$, Eq. (D.2) becomes (as previously, we consider a disclination with the unitary topological charge $|s|$ or $-|s|$)

$$\nabla_R^4 L + \frac{\Gamma}{4}\Big\{-4s^2\frac{(\partial_R Q_0)^2}{R^2}\partial_\phi^2 L + \Big(\nabla_R^2 + \frac{2s}{R}\partial_R\Big)\Big(Q_0^2\Big(\nabla_R^2 - \frac{2s}{R}\partial_R\Big)L\Big)\Big\} = 0, \tag{D.5}$$

where $\nabla_R^2 \equiv \partial_R^2 + \frac{1}{R}\partial_R + \frac{1}{R^2}\partial_\phi^2$ and Q_0 is found from

$$\Big(\partial_R^2 + \frac{1}{R}\partial_R - \frac{1}{R^2}\Big)Q_0 + Q_0(1 - Q_0^2) = 0,$$
$$Q_0(0) = 0, \quad Q_0(\infty) = 1.$$

If $\Gamma \gg 1$, as follows from Eq. (D.5), a new scale $R \sim 1/\sqrt{\Gamma} \ll 1$ appears inside the core, the first term in Eq. (D.5) can be neglected at larger scales, and there is no crossover at $R \sim 1$.

If $Q_0 \equiv 1$, Eq. (D.5) is reduced to Eq. (3.12). If $R \ll 1$, $Q_0 = AR$ ($A \approx 0.58$) and Eq. (D.5) can be rewritten as

$$\nabla_R^2\Big\{\nabla_R^2 L + \frac{A\Gamma}{4}(R^2\nabla_R^2 - 4s^2)L\Big\} = 0.$$

The solution to the equation is a superposition of the terms $\lambda(R)\sin(m\phi)$ with different m. After imposing the condition $\lambda(R) = 0$, two constants remain in the general solution of the ordinary differential equation for $\lambda(R)$; two partial solutions that are regular near $R = 0$ are given by

$$R^{|m|} \text{ and } R^{|m|}{}_2F_1\Big(\frac{|m| - \sqrt{m^2 + 4s^2}}{2}, \frac{|m| + \sqrt{m^2 + 4s^2}}{2}, 1 + |m|, -\frac{A^2\Gamma R^2}{4}\Big),$$

where ${}_2F_1$ is the hypergeometric function (${}_2F_1(a, b, c, z) = 1 + abz/c + \ldots$). Two constants (e.g., the derivatives $\lambda^{(|m|)}(0)$ and $\lambda^{(|m|+2)}(0)$ are chosen to ensure the slowest possible growth at $R \gg 1$ in order to eliminate the largest exponent among α in Eq. (3.13).

If $\Gamma \gg 1$, it is possible to derive a better approximation in the core region. We can expand $Q_0(R)$ in a series, seek a series solution $\lambda(R)$, and extract the terms of the highest order in Γ. For example, for $m = 1$, the series for $\lambda(R)$ begins with $l_1R + l_3R^3$, which fixes two constants in the partial solution,

$$\lambda(R) = l_1R\Big[1 + \frac{1}{A^2\Gamma s(2 - s^2)}\Big(-1 - \frac{s^2A^2\Gamma R^2}{8} + {}_2F_1\Big(\frac{1 - \sqrt{1 + 4s^2}}{2}, \frac{1 + \sqrt{1 + 4s^2}}{2}, 2, -\frac{A^2\Gamma R^2}{4}\Big)\Big)\Big]$$
$$+ l_3\frac{8}{A^2\Gamma s^2}R\Big[-1 + {}_2F_1\Big(\frac{1 - \sqrt{1 + 4s^2}}{2}, \frac{1 + \sqrt{1 + 4s^2}}{2}, 2, -\frac{A^2\Gamma R^2}{4}\Big)\Big].$$

The solutions to Eqs. (D.3) and (D.4) are given by

$$Q_1 = \vartheta(R)\partial_\phi\sin(m\phi), \quad \varphi_1 = \sigma(R)\sin(m\phi),$$

where ϑ and σ must be found from the equations

$$\vartheta'' + \frac{1}{R}\vartheta' - \frac{1+m^2}{R^2}\vartheta - \frac{2Q_0}{sR^2}\sigma + (1-3Q_0^2)\vartheta$$

$$= \Gamma\frac{1}{R}\partial_R Q_0\lambda,$$

$$\sigma'' + \frac{1}{R}\sigma' - \frac{m^2}{R^2}\sigma + \frac{2}{Q_0}\left(-\frac{sm^2}{R^2}\vartheta + \partial_R Q_0\sigma'\right)$$

$$= \frac{\Gamma}{2}\left(\lambda'' + \frac{1-2s}{R}\lambda' - \frac{m^2}{R^2}\lambda\right)$$

that generalize the expressions given in [7].

The dynamic equations with the complex order parameter demonstrate that, for all Γ, the boundary conditions for Eqs. (2.4)–(2.6) experience no significant changes on the core. The peculiarity of extremely small Γ leading to the nonslipping condition consists in a slow growth of $\nabla\Omega$ far from the disclination.

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