Post-Ehrenfest many-body quantum interferences in ultracold atoms far-out-of-equilibrium

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Out of equilibrium dynamics in cold atom systems


A single 1D quasi-condensate is phase coherently split into two parts using r.f. potentials on an atom chip.
Collapse and Revival of the Matter Wave Field of a Bose-Einstein Condensate

- one site problem
  \[ H = \frac{1}{2} U \hat{n}(\hat{n} - 1) \]
- coherent state propagation
  \[ |\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle \]
- Revival = signature of coherence (interference effect)

Overlap \[ |<\beta|\alpha(t)>|^2 \] of an arbitrary coherent state \( |\beta\rangle \) with complex amplitude \( \beta \) with the dynamically evolved quantum state \( |\alpha(t)\rangle \)
Probing the relaxation toward equilibrium in an isolated strongly correlated 1D Bose gas

(a) Concept of the experiment
(b) Even-odd resolved detection

Relaxation of the local density for different interaction strengths.

- initial state \(\equiv\) density wave \(|\Psi\rangle = |\ldots, 1, 0, 1, 0, 1 \ldots\rangle\)
- approximatively 60 lattice sites
Executive summary

- The unprecedented control that has been achieved experimentally with ultra-cold atomic systems has given rise to the exploration of many-body dynamics in isolated systems far from equilibrium.
- Very precisely defined excited states and systems can be designed, and their dynamics followed accurately. These states can be Fock states, but also coherent states.
- This has opened a new and exciting field of investigation, and poses significant challenges for theoreticians.
Simulation tools

- Exact diagonalization in Fock space
  - \( \rightarrow \) rather small systems

- Density Matrix Renormalization Group (DMRG).
  - \( \rightarrow \) low density (\( \approx 1 \) particle per site)

- What about the high density, possibly strong interaction regime?
Simulation tools

- Exact diagonalization in Fock space
  → rather small systems
- Density Matrix Renormalization Group (DMRG).
  → low density (≈ 1 particle per site)

- What about the high density, possibly strong interaction regime?
Mean Field approximation I: equilibrium (ground state / thermodynamics)

e.g.: Bose Hubbard model (1d)

\[ \hat{H} = \sum_{j=1}^{N} \left[ -J \left( \hat{a}_j^\dagger \hat{a}_{j+1} + h.c. \right) + V \hat{a}_j^\dagger \hat{a}_j + \frac{U}{2} \hat{n}_j \left( \hat{n}_j - 1 \right) \right] \]

- operators \( \hat{a}_i \rightarrow c\)-number \( \psi \Rightarrow (\text{discrete}) \) Gross-Pitaevskii Eq
  \( \Rightarrow \text{classical dynamics} \)

- “bare” mean field: coherent state “on”
  GP trajectory

- Bogoliubov - De Gennes: take into account actual linear motion around the GP trajectory
Mean Field approximation II: non-equilibrium

**Pb:** except in the neighborhood of stable periodic orbits, solutions of the GP equation tend to diverge one from each other.

→ even if the initial state is a coherent state, one cannot expect that its time evolution will be well approximated by a coherent state (even including quasi-particle coherent state).

→ there is no way that propagating a single GP solution provides enough information about the time evolution of the state.

**Fix:** propagate many solutions of the GP equation

→ truncated Wigner approximation
The Wigner transform represents the manybody density operator $\sigma$ in terms of a ``probability distribution'' over the classical field $\psi$.

$\rightarrow$ possibility to propagate these classical field with the time dependent Gross-Pitaevskii equation, and to average with the corresponding weight.

$\rightarrow$ this corresponds however to an incoherent sum of the contributions.
What about interferences?

→ There is no way the truncated Wigner approximation takes interference effects into account properly.

Our goal here is to implement the effects of interferences between mean field solutions.


[cf also Simon & Strunz, Phys Rev A 89 (2014)]
Outline

A. Warming: the one mode case
   ▪ *Path integral approach*
   ▪ *Semiclassics “à la Maslov”*

B. The multimode case
   ▪ How to explore a large phase space
   ▪ Specificity of coherent states: going complex or not going complex

C. A case study

D. Symmetries
A. Warmup : The one-mode case


\[ \hat{H} = \hat{H}(\hat{a}^\dagger, \hat{a}, t) \rightarrow \text{e.g.: } \hat{H} = V(t)\hat{a}^\dagger \hat{a} + \frac{U(t)}{2} \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \]

coherent state:  
\[ |z\rangle = e^{-\frac{1}{2}|z|^2} e^{z\hat{a}^\dagger} |0\rangle \]

propagator: 
\[ K(z'', t; z', 0) = \langle z'' | \hat{T} \exp[-\frac{i}{\hbar} \int_0^t \hat{H}(t') dt'] |z'\rangle \]

path integral representation:

\[ K(z'', t; z', 0) = \int \mathcal{D}[\bar{z}_t^* z_t] \exp \left[ \int_0^t ds \frac{1}{2} (\partial_t \bar{z}_s^* z_s - z_s^* \partial_t \bar{z}_s^*) + \frac{i}{\hbar} \mathcal{H}(z_s^*, z_s) \right] \]
A. Warmup : The one-mode case

1. **Path Integral approach** [Baranger et al, J. Phys A 34 (2001)]

\[ \hat{H} = \hat{H}(\hat{a}^\dagger, \hat{a}, t) \rightarrow \text{e.g.: } \hat{H} = V(t)\hat{a}^\dagger \hat{a} + \frac{U(t)}{2}\hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \]

coherent state: \[ |z\rangle = e^{-\frac{1}{2}|z|^2} e^{z\hat{a}^\dagger}|0\rangle \]

propagator: \[ K(z'', t; z', 0) = \langle z'' | \hat{T} \exp[-\frac{i}{\hbar} \int_0^t \hat{H}(t')dt'] | z' \rangle \]

path integral representation:

\[ K(z'', t; z', 0) = \int \mathcal{D}[\hat{z}_s^* \hat{z}_t] \exp \left[ \int_0^t ds \frac{1}{2} (\partial_t \hat{z}_s^* \hat{z}_s - \hat{z}_s^* \partial_t \hat{z}_s^*) + \frac{i}{\hbar} \mathcal{H}(\hat{z}_s^*, \hat{z}_s) \right] \]

Classical Hamiltonian
Stationary phase approximation (Baranger et al.):

- $\delta S = 0 \rightarrow$ classical evolution under $\mathcal{H}(z^*, z)$

- “trajectories” $z_t' = u_t' + iv_t'$ such that
  \[
  \begin{align*}
  u_0 - iv_0 & \equiv u' - iv' = z'^* \\
  u_t + iv_t & \equiv u'' + iv'' = z''
  \end{align*}
  \]

\[
K(z'', t; z', 0) = \sum_{u' + iv' = z', u'' + iv'' = z''} \sqrt{\frac{\partial v'}{\partial v''}} \exp \left\{ \frac{i}{2\hbar} \int_0^t dt' \frac{\partial^2 \mathcal{H}}{\partial u \partial v} \right\} \times \exp \left\{ \int_0^t dt' \left[ \frac{1}{2} (\dot{v}u - \dot{u}v) + \frac{i}{\hbar} \mathcal{H} \right] + \frac{1}{2} (v''u'' + v'u') - \frac{1}{2} (|z'|^2 + |z''|^2) \right\}
\]

- classical action
- initial and final states
Stationary phase approximation (Baranger et al.):

- $\delta S = 0 \rightarrow$ classical evolution under $\mathcal{H}(z^*, z)$

- "trajectories" $z_{t'} = u_{t'} + iv_{t'}$ such that
  \[
  \begin{align*}
  u_0 - iv_0 & \equiv u' - iv' = z'^* \\
  u_t + iv_t & \equiv u'' + iv'' = z''
  \end{align*}
  \]

\[
K(z'', t; z', 0) = \sum_{u' + iv' = z'} \sqrt{\frac{\partial v'}{\partial v''}} \exp \left\{ \frac{i}{2\hbar} \int_0^t dt' \frac{\partial^2 \mathcal{H}}{\partial u \partial v} \right\} \times \exp \left\{ \frac{1}{2} \left( v'' u'' + v' u' \right) - \frac{1}{2} (|z'|^2 + |z''|^2) \right\}
\]

- $u_t, v_t \equiv$ complex numbers $\Rightarrow\left\{ \begin{align*}
  u' + iv' & \neq z' \\
  u'' - iv'' & \neq z''^*
  \end{align*} \right.$
2. **Semiclassics “à la Maslov”**  
\( (N \simeq \frac{1}{\hbar}) \)

- use the “quadratures” \((\hat{p}, \hat{q})\)

\[
\begin{align*}
\hat{a} & \equiv \frac{1}{\sqrt{2}}(\hat{q} + i\hat{p}) \\
\hat{a}^\dagger & \equiv \frac{1}{\sqrt{2}}(\hat{q} - i\hat{p})
\end{align*}
\]

\( \Rightarrow \quad \text{e.g.:} \quad \hat{H} = V(t)\hat{a}^\dagger \hat{a} + \frac{U(t)}{2} \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \)

\[
\rightarrow \quad \frac{V(t)}{2}(\hat{p}^2 + \hat{q}^2) + \frac{U(t)}{4}(\hat{p}^2 + \hat{q}^2)^2 + O(\hbar)
\]

- to any initial wave function in the semiclassical form

\[
\psi_0(q) \equiv \varphi(q) \exp \left[ \frac{i}{\hbar} S_0(q) \right]
\]

\( \rightarrow \) associate a **Lagrangian manifold** (just a curve here)

\[
\mathcal{L}_0 \equiv \{(q_0, p_0(q_0) = \partial_q S_0)\}
\]
The quantum evolution is based on the classical evolution of the Lagrangian manifold

\[ \Psi(q, t) = \sum_{r=(q,p) \in \mathcal{L}_t} \sqrt{\frac{\partial q_0}{\partial q}} \varphi(q_0) \exp \left[ \frac{i}{\hbar} \left( \int_{r_0}^{r} pdq - \mathcal{H}dt \right) \right] \]

- for a correlation function \( \langle \alpha'' | \hat{T} \exp[-\frac{i}{\hbar} \int_{0}^{t} \hat{H}(t')dt'] | \beta' \rangle \)
  
  \( \rightarrow \) intersection of \( \mathcal{L}_t(\beta') \) and \( \mathcal{L}_0(\alpha'') \) (saddle trajectory)

[Maslov & Fedoriuk (1981)]
Extra phase: \[ \exp \left\{ \frac{i}{2} \int_0^t dt' \frac{\partial^2 \mathcal{H}}{\partial p \partial q} \right\} \]

\[ \Rightarrow \text{associated with the ordering of the operators} \]

Various ways to "quantize" a classical \( H(p, q) \)

1. \( \hat{H}^{(1)}(\hat{p}, \hat{q}) \quad \rightarrow \quad [...] \exp \left[ \frac{i}{\hbar} \left( \int_{r_0}^r pdq - \mathcal{H}dt \right) \right] \cdot \exp \left[ \frac{1}{2} \int_0^t \frac{\partial^2 H}{\partial q \partial p} dt \right] \]

2. \( \hat{H}^{(2)}(\hat{p}, \hat{q}) \quad \rightarrow \quad [...] \exp \left[ \frac{i}{\hbar} \left( \int_{r_0}^r pdq - \mathcal{H}dt \right) \right] \cdot \exp \left[ -\frac{1}{2} \int_0^t \frac{\partial^2 H}{\partial q \partial p} dt \right] \]

3. \( \frac{1}{2} \left( \hat{H}^{(1)}(\hat{p}, \hat{q}) + \hat{H}^{(2)}(\hat{p}, \hat{q}) \right) \quad \rightarrow \quad [...] \exp \left[ \frac{i}{\hbar} \left( \int_{r_0}^r pdq - \mathcal{H}dt \right) \right] \]

Manybody physics most often prefers normal ordering \( \hat{H}(\hat{a}^{\dagger}, \hat{a}^{\dagger}) \)

\[ \Rightarrow \quad [...] \exp \left[ \frac{i}{\hbar} \left( \int_{r_0}^r pdq - \mathcal{H}dt \right) \right] \cdot \exp \left[ \frac{i}{2} \int_0^t \frac{\partial^2 H}{\partial q \partial p} dt' \right] \]
complexification of phase space

⇒ associated with the choice of coherent states

coherent state $|z\rangle$: $z = q_z + ip_z$

$$\langle q|z \rangle = \frac{1}{(\pi\hbar)^{1/4}} \exp \left[-\frac{(q - q_z)^2}{2\hbar}\right] \exp \left[\frac{i}{\hbar}p_z(q - \frac{1}{2}q_z)\right]$$

semiclassical form $\propto \exp \left[\frac{i}{\hbar}S_0(q)\right]$

$S_0(q) = p_z(q - \frac{1}{2}q_z) + \frac{i}{2}(q - q_z)^2$

$p_0(q) = \partial_q S_0 = p_z + i(q - q_z)$

⇒ complexe manifold
B. The many-mode case

- Once the question of ordering of the operator is taken into account, no difference between the path integral approach and the traditional time dependent WKB.
- More generally there is no conceptual difference between usual semiclassics and many-bosons semiclassics, even if the small parameter is not the same (\(\hbar\) vs \(N^{-1}\))
- WKB “à la Maslov” is however much simpler, and thus more “versatile” than the path integral approach
  \(\rightarrow\) no particular difficulty to generalize the formalism to the many-mode case and to a large class of initial states (including for example Fock states)
- There are of course practical difficulties ...
  \(\rightarrow\) complexification of phase space
  \(\rightarrow\) large phase space
Case study: propagation of coherent state density wave

- Hamiltonian = 1-d Bose Hubbard

\[
\hat{H} = \sum_{j=1}^{N} \left[ -J \left( \hat{a}_j^\dagger \hat{a}_{j+1} + h.c. \right) + \frac{U}{2} \hat{n}_j \left( \hat{n}_j - 1 \right) \right]
\]

- Initial state = coherent state density wave

\[
|\vec{n}\rangle = \prod_{j=1}^{N} \exp \left( -\frac{|z_j|^2}{2} + z_j \hat{a}_j^\dagger \right) |0\rangle
\]

(e.g.: \( |\vec{n}\rangle = |n, 0, n, 0, ..., n, 0\rangle \).  

(here \( z_i = \sqrt{n_i} \rightarrow \) no initial dephasing)

- Observable = autocorrelation function

\[
C(t) = |A(t)|^2 \quad A(t) = \langle \vec{n} | \hat{U}(t) | \vec{n}\rangle
\]

\( (\hat{U}(t) \equiv \exp[-\frac{i}{\hbar} \hat{H} t]) \)
- Initial and final manifold

\[
\begin{cases}
  p_0^j(q_0) = +i(q_0^j - \sqrt{2n_j}) & \text{initial} \\
  p_f^j(q_0) = -i(q_0^j - \sqrt{2n_j}) & \text{final} \\
\end{cases}
\quad (j = 1, \ldots, N)
\]

- Autocorrelation function

\[
C(t) = \left| \sum_{\gamma} D_{\gamma}^{1/2}(t) \exp \left[ \frac{i}{\hbar} \phi_{\gamma}(t) \right] \right|^2
\]

\[
\frac{i}{\hbar} \phi_{\gamma}(t) = \frac{i}{\hbar} S(\tilde{q}_t^\gamma, \tilde{q}_0^\gamma; t) - i\nu \gamma \frac{\pi}{2} + F_{0,\gamma}^- + F_{t,\gamma}^+ + F_{0,t,\gamma}^{\pm}
\]

\[
D_{\gamma}(t) = \text{Det} \left[ \frac{1}{2} (M_{11}^\gamma + M_{22}^\gamma + i\hbar M_{21}^\gamma - \frac{i}{\hbar} M_{12}^\gamma) \right]
\]

\[
M_{11}^\gamma = \frac{\partial \tilde{q}_t^\gamma}{\partial q_0}, \quad M_{22}^\gamma = \frac{\partial \tilde{p}_t^\gamma}{\partial p_0}, \quad M_{12}^\gamma = \frac{\partial \tilde{q}_t^\gamma}{\partial p_0}, \quad M_{21}^\gamma = \frac{\partial \tilde{p}_t^\gamma}{\partial q_0}.
\]

1. Exploring a large phase space

Problem:
- Quantities like the autocorrelation function $C(t)$ involve a mixed initial/final value “shooting”-problem ($r_0 \rightarrow$ initial manifold, $r_t \rightarrow$ final manifold).
- Phase space is a big place, especially if the dimensionality (number of modes is large).

→ limit the time for which the semiclassical approach can be used.

→ favor system close to integrability (eg large $U$).

To mitigate this: explore predominantly unstable directions
\[
\begin{pmatrix}
\delta p_t \\
\delta q_t \\
\end{pmatrix}
= 
\begin{pmatrix}
M_t^{11} & M_t^{12} \\
M_t^{21} & M_t^{22} \\
\end{pmatrix}
\begin{pmatrix}
\delta p_0 \\
\delta q_0 \\
\end{pmatrix}
\]

phase space representation of initial state

classical evolution

\[
(\delta p_0, \delta q_0) \cdot \begin{pmatrix}
\delta p_0 \\
\delta q_0 \\
\end{pmatrix} = \text{const.} \quad \Rightarrow \quad (\delta p_t, \delta q_t) \cdot \underbrace{M^{-1T}}_{M} \underbrace{M^{-1}}_{M} \cdot \begin{pmatrix}
\delta p_0 \\
\delta q_0 \\
\end{pmatrix}
\]

Large eigenvalues of \( M \rightarrow \) First directions to explore
2. **Complexification of the classical dynamics**

**Problem:**

- Going complex doubles the number of variables → this is manageable (just more work)
- Motion in complexified phase space is non-compact → this makes the search of saddle trajectories completely impractical

\[
\text{Proba}[r_t \in \mathcal{L}_t] \propto \frac{1}{\text{Vol}}
\]

**fix:** Use “ghost” real trajectory and converge to true trajectory with Newton-Raphson
C. Case study

- ring with four sites
- initial coherent state density wave $|\vec{n}\rangle = |20, 0, 20, 0\rangle$

$$J = 0.2, \ U = 0.5 \quad \Rightarrow \quad \tau_1 = \frac{2\pi\hbar}{U n_j} \simeq 0.63 \quad \tau_2 = \frac{2\pi\hbar}{U} \simeq 12.57$$
C. Case study

- ring with four sites
- initial coherent state density wave \(|20, 0, 20, 0\rangle\)
- \(J = 0.2, U = 0.5 \implies \tau_1 = \frac{2\pi \hbar}{Un_j} \simeq 0.63 \quad \tau_2 = \frac{2\pi \hbar}{U} \simeq 12.57\)
C. Case study

- ring with four sites
- initial coherent state density wave $|20, 0, 20, 0\rangle$

$J = 0.2, U = 0.5 \Rightarrow \tau_1 = \frac{2\pi \hbar}{Un_j} \simeq 0.63 \quad \tau_2 = \frac{2\pi \hbar}{U} \simeq 12.57$
Spectrum

\[ \mathcal{S}\mathcal{P}(E) = \sum_{\alpha} |\langle \xi_\alpha | \vec{n} \rangle |^2 \delta(E - E_\alpha) \propto \int dt e^{iEt/\hbar} A(t) \]

(cutoff \( \tau_c = 40 \))
Trajectory search

- \( u \equiv \) distance along the direction associated with the largest eigenvalue of \( \mathcal{M} = \mathbf{M}^{-1}T \mathbf{M}^{-1} \)

- trajectory kept if
  \[
  (|\mathbf{q}_0 - \bar{\mathbf{q}}|^2 + |\mathbf{p}_0 - \bar{\mathbf{p}}|^2) + (|\mathbf{q}_f - \bar{\mathbf{q}}|^2 + |\mathbf{p}_f - \bar{\mathbf{p}}|^2) \leq \text{const.}
  \]

- for each "seed", converge to the saddle through Newton-Raphson
D. Symmetries

Exact quantum

Truncated Wigner
D. Symmetries

- Truncated Wigner
- Exact quantum
- Smoothed quantum \((\sigma = 0.5)\)

Graph showing the evolution of \(C(t)\) with time \(t\) from 0 to 5, comparing different quantum models.
D. Symmetries

- Exact quantum
- Smoothed quantum ($\sigma = 0.5$)
- “Saddle TWA”
- Truncated Wigner
Diagonal approximation

- Autocorrelation function:

\[ C(t) = \left| \sum_{\gamma} D_{\gamma}^{1/2}(t) \exp \left[ \frac{i}{\hbar} \phi_{\gamma}(t) \right] \right|^2 \]

\[ \gamma \equiv \text{trajectories } p_0^j(q_0) = +i(q_0^j - \sqrt{2n_j}) \rightarrow p_f^j(q_f) = -i(q_f^j - \sqrt{2n_j}) \]

\[ \frac{i}{\hbar} \phi_{\gamma}(t) = \frac{i}{\hbar} S(q_t^\gamma, q_0^\gamma; t) - i\nu_{\gamma} \frac{\pi}{2} + F_0^{\gamma} + F_t^{\gamma_0} \]

\[ D_{\gamma}(t) = \text{Det} \left[ \frac{1}{2} (M_{11}^{\gamma} + M_{22}^{\gamma} + i\hbar M_{21}^{\gamma} - \frac{i}{\hbar} M_{12}^{\gamma}) \right] \]

- Diagonal approximation

\[ C(t) = \sum_{\gamma, \gamma'} D_{\gamma}^{1/2} D_{\gamma'}^{1/2} \exp \left[ \frac{i}{\hbar} (\phi_{\gamma} - \phi_{\gamma'}) \right] \rightarrow C_{\text{diag}}(t) = \sum_{\gamma} D_{\gamma} \]

\[ C_{\text{diag}}(t) \equiv C_{\text{TWA}}(t) \]
Diagonal approximation

- Autocorrelation function:
  \[ C(t) = \left| \sum_{\gamma} D_{\gamma}^{1/2}(t) \exp \left[ \frac{i}{\hbar} \phi_{\gamma}(t) \right] \right|^2 \]

  \[ \gamma \equiv \text{trajectories } p_0^j(q_0) = +i(q_0^j - \sqrt{2n_j}) \rightarrow p_f^j(q_f) = -i(q_f^j - \sqrt{2n_j}) \]

  \[ \frac{i}{\hbar} \phi_{\gamma}(t) = \frac{i}{\hbar} S(q_t^\gamma, q_0^\gamma; t) - i\nu_{\gamma} \frac{\pi}{2} + F_0^\gamma - F_t^\gamma,^+ \]

  \[ D_{\gamma}(t) = \text{Det} \left[ \frac{1}{2} (M_{11}^\gamma + M_{22}^\gamma + i\hbar M_{21}^\gamma - \frac{i}{\hbar} M_{12}^\gamma) \right] \]

- Diagonal approximation

  \[ C(t) = \sum_{\gamma, \gamma'} D_{\gamma}^{1/2} D_{\gamma'}^{1/2} \exp \left[ \frac{i}{\hbar} (\phi_{\gamma} - \phi_{\gamma'}) \right] \rightarrow C_{\text{diag}}(t) = \sum_{\gamma} D_{\gamma} \]

  \[ C_{\text{diag}}(t) \equiv C_{\text{TWA}}(t) \]
Intermezzo: derivation of the “diagonal” approximation

1. The semiclassical (Van-Vleck) propagator (in \(q\)-quadrature basis)

\[
K(q_f, q_i, t) = \langle q_f | e^{-i \frac{t \hat{H}}{\hbar}} | q_i \rangle = \sum_\gamma A_\gamma(q_f, q_i) e^{\frac{i}{\hbar} R_\gamma(q_f, q_i)}
\]

\(\gamma\) solution of the GP equation

\[
i\hbar \frac{\partial}{\partial t} \psi_l(t) = \sum_{l'=1}^L H_{l,l'} \psi_{l'}(t) + U_l(|\psi_l(t)|^2 - 1) \psi_l(t)
\]

with boundary cond: \(\text{Re}[\psi^l(0)] = \frac{q_i^l}{\sqrt{2}}, \text{Re}[\psi^l(t)] = \frac{q_f^l}{\sqrt{2}}\).

\[
R_\gamma = \int_{\gamma: q_i \to q_f} p dq - \mathcal{H} dt \quad \text{(action)}
\]

\[
A_\gamma(q_f, q_i) = \frac{e^{-i \kappa_\gamma \pi/2}}{(2i \pi \hbar)^{L/2}} \left| \det \left( -\frac{\partial^2 R_\gamma}{\partial q_f^l \partial q_i^{l'}}(q_f, q_i) \right) \right|^{1/2}
\]
2. **Time evolution of the mean value of an operator**

Initial manybody state $|\Phi_0\rangle \rightarrow$ density $\hat{\rho}(t) \equiv |\hat{U}(t)\Phi_0\rangle\langle\hat{U}(t)\Phi_0|$

Arbitrary operator $\hat{O}$ \quad $\rightarrow \langle O \rangle(t) = \text{Tr} \left[ \hat{\rho}(t)\hat{O} \right]$

In $q$-quadrature basis

$$\langle \hat{O} \rangle(t) = \int dq_i' dq_i'' dq_f' dq_f'' \Phi_0^*(q_i')\Phi_0(q_i'') \langle q_f' | \hat{O} | q_f'' \rangle$$

$$\times \sum_{\gamma',\gamma''} A^*_{\gamma'}(q_f', q_i') A_{\gamma''}(q_f'', q_i'') e^{\frac{i}{\hbar} (R_{\gamma''}(q_f'', q_i'') - R_{\gamma'}(q_f', q_i'))}$$
3. Diagonal approximation

assumptions: (i) only diagonal terms $\gamma' = \gamma''$ survive averaging

(ii) only short chords $q_i' \simeq q_i''$, $q_f' \simeq q_f''$ are relevant

$\rightarrow$ introduce $Q_{f,i} \equiv \frac{1}{2}(q''_{f,i} + q'_{f,i})$ and $\delta q_{f,i} \equiv (q''_{f,i} - q'_{f,i})$

and expand in $\delta q_{f,i}$, using $p_f^{(\gamma)} = \frac{\partial R_{\gamma}}{\partial q_f}$; $p_i^{(\gamma)} = -\frac{\partial R_{\gamma}}{\partial q_i}$

$$\langle \hat{O} \rangle(t)_{\text{diag}} = \int dQ_i dQ_f d\delta q_f d\delta q_i \langle q_i'' | \hat{\rho}_0 | q_i' \rangle \langle q_f' | \hat{O} | q_f'' \rangle \sum_{\gamma} |A_{\gamma}(Q_f Q_i)|^2 \exp \left[ \frac{i}{\hbar} \left( p_f^{(\gamma)} \delta q_f - p_i^{(\gamma)} \delta q_i \right) \right]$$

Wigner $[O]_W(Q, P) \equiv \int d\delta q e^{(i/\hbar)P \cdot \delta q} \langle Q + \frac{\delta q}{2} | f | Q - \frac{\delta q}{2} \rangle$
\[
\langle \hat{O} \rangle (t)_{\text{diag}} = \int dQ_i dQ_f \sum_{\gamma} |A_\gamma(Q_f, Q_i)|^2 [\rho_0] W(\mathbf{Q}_i, \mathbf{p}_i^{(\gamma)}) [O] W(\mathbf{Q}_f, \mathbf{p}_f^{(\gamma)})
\]

3. Initial value representation:

\[
- \frac{\partial^2 R_\gamma}{\partial Q_f \partial Q_i} = \frac{\partial P_i}{\partial Q_f^\gamma}
\]

\[|A|^2 \equiv \text{Jacobian of } Q_f \rightarrow P_i \Rightarrow \sum_{\gamma} \int dQ_f |A_\gamma|^2 \rightarrow \int \frac{dP_i}{(2\pi \hbar)^L}
\]

\[
\langle \hat{O} \rangle_{\text{diag}}(t) = \int \frac{dQ_i dP_i}{(2\pi \hbar)^L} [\rho_0] W(\mathbf{Q}_i, \mathbf{P}_i) [O] W(\mathbf{Q}_f, \mathbf{P}_f)
\]

(NB: for autocorrelation, \(\hat{O} \equiv \hat{\rho}_0\))
Symmetries

- Hamiltonian
  \[ \hat{H} = \sum_{j=1}^{4} \left[ -J \left( \hat{a}_j^\dagger \hat{a}_{j+1} + h.c. \right) + \frac{U}{2} \hat{n}_j \left( \hat{n}_j - 1 \right) \right] \]

- Initial (and final) state
  \[ |\vec{n}\rangle = |n, 0, n, 0\rangle. \]

Both \( H \) and \( |n\rangle \) symmetric under:
- time reversal symmetry
- \( (1 \to 3, 2 \to 4, 3 \to 1, 4 \to 2) \)

\[ \rightarrow \text{ short trajectories } \equiv \text{ own symmetric } \quad (g_\gamma = 1) \]
\[ \rightarrow \text{ longer trajectories have a symmetric partner } \quad (g_\gamma = 2) \]

\[ C_{\text{smooth}}(t) \simeq C_{\text{STWA}}(t) \equiv \sum_{\gamma} g_\gamma^2 D_\gamma \]
\( N = 6, \ |\vec{n}\rangle = |10, 0, 10, 0, 10, 0\rangle \).

\[ J = 0.2, \ U = 1 \]
(nearly integrable regime)

\((u_1, u_2) \equiv \text{directions associated with the 2 largest eigenvalues of} \)

\[ M = M^{-1}T M^{-1} \]

“Seed trajectories” in the search plan
There is no conceptual difference between the $N \to \infty$ limit of bosonic mean field approximation and the $\hbar \to 0$ limit of few body systems.

This imply that usual semiclassics and many-bosons semiclassics, are formally the same theory, even if the small parameter is not the same ($\hbar$ vs $N^{-1}$).

The tool from quantum chaos which have been developed in the former context can be used in the latter, and are particularly well adapted to tackle interference effects (NB: trying to start from path integral on the other hand is uselessly complicated).

The complexification of phase space associated with the use of coherent states is a technical difficulty that can be overcome.
The size of the phase space to explore is however a game changer for any practical implementation

→ need to design techniques making it possible to explore large phase space.
→ even in this way, one will be limited to either
  • small systems
  • system sufficiently close to integrability that the size of the phase space to explore is $O(1)$ as the number of modes $\rightarrow \infty$
  • fairly short times

However:

→ this will always beat truncated Wigner (which can be shown to be equivalent to just neglecting interference terms in the WKB approach).
→ it is presumably the correct framework to think about the mean field approximation for many-boson out of equilibrium systems (eg: effects of symmetries, etc..).