Post-Ehrenfest many-body quantum interferences in ultracold atoms far-out-of-equilibrium

Denis Ullmo (LPTMS, Orsay & Institut Pascal)

*[Phys. Rev. A **97**, 061606(R) (2018), and in preparation]*

Post-Ehrenfest many-body quantum interferences in ultracold atoms far-out-of-equilibrium

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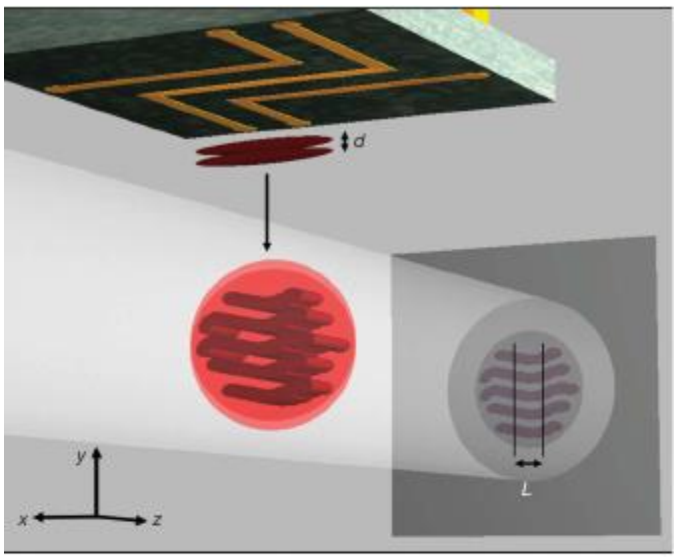
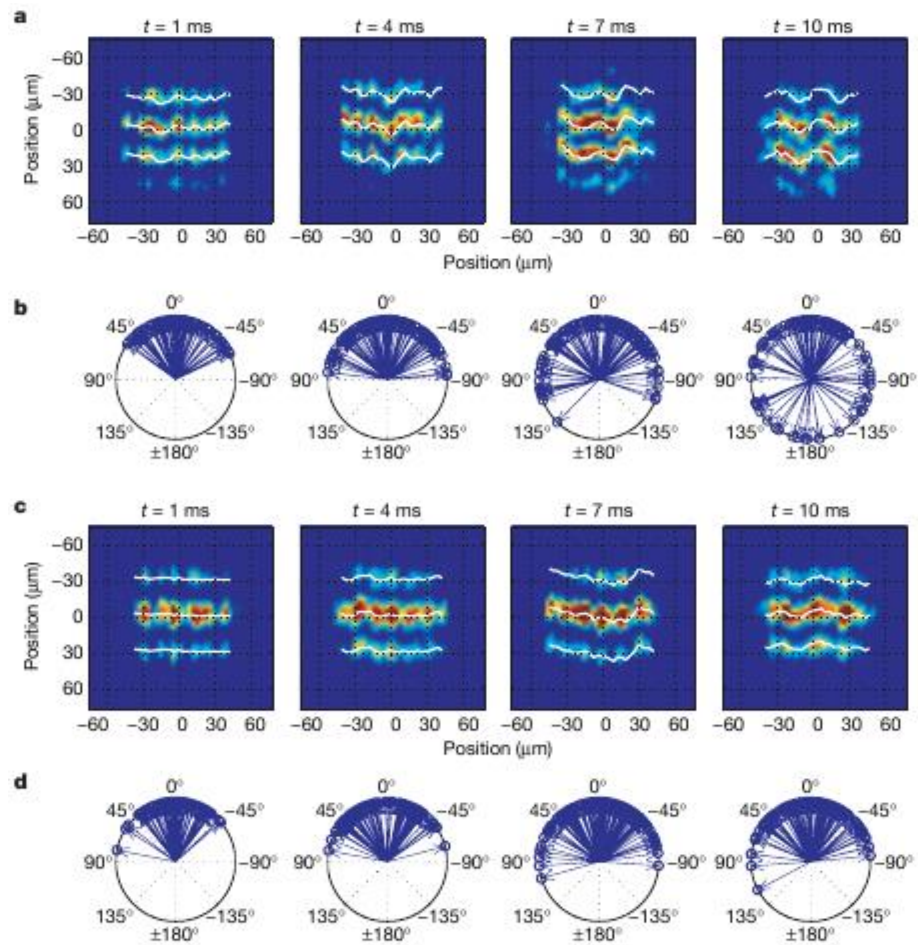
Out of equilibrium dynamics in cold atom systems

Hofferberth, et al. Nature *449*, (2007)



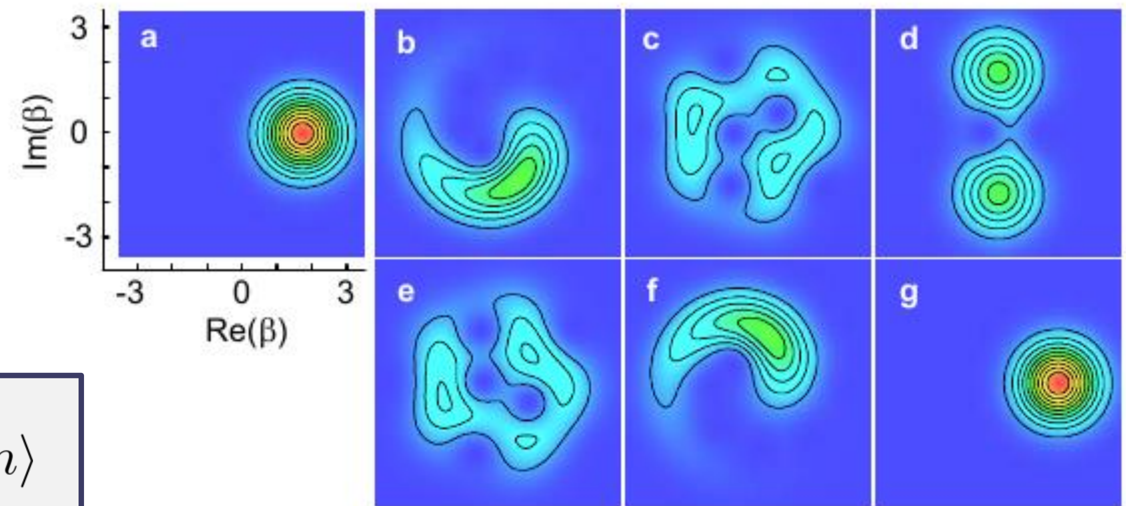


A single 1D quasi-condensate is phase coherently split into two parts using r.f. potentials on an atom chip.

M. Greiner, O. Mandel, T.W. Hänsch & I. Bloch, Nature *419*, (2002)
Collapse and Revival of the Matter Wave Field of a Bose-Einstein Condensate

- one site problem
  $H = \frac{1}{2} U \hat{n}(\hat{n}-1)$
- *coherent state* propagation

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

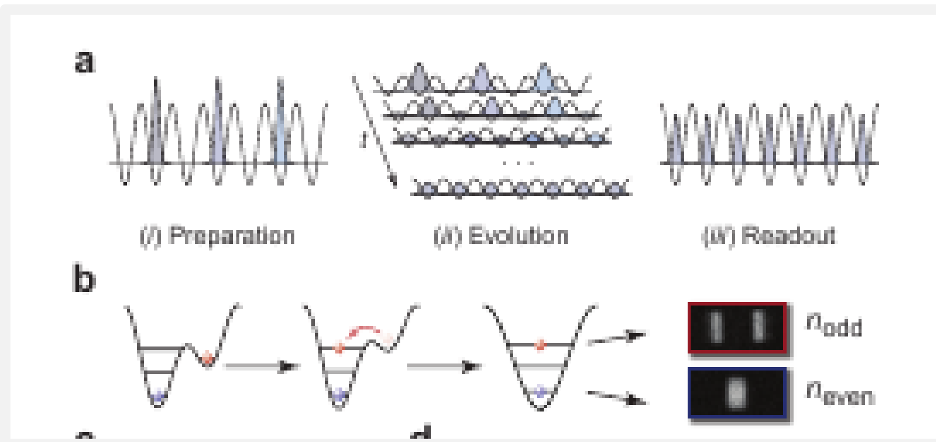


overlap $|\langle\beta|\alpha(t)\rangle|^2$ of an arbitrary coherent state $|\beta\rangle$ with complex amplitude β with the dynamically evolved quantum state $|\alpha(t)\rangle$

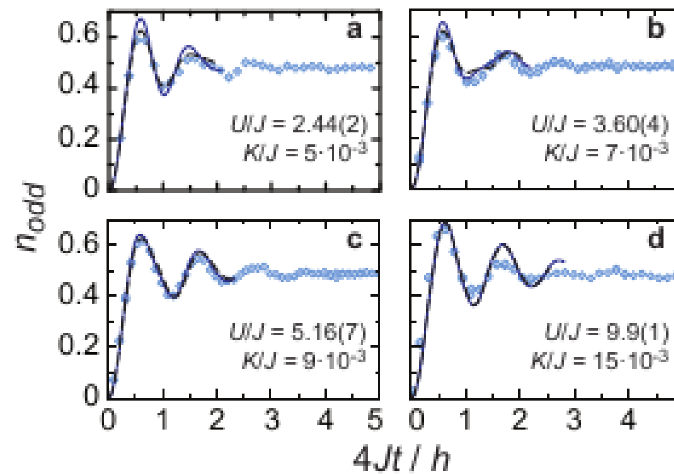
- Revival = signature of coherence (interference effect)

S. Trotzky, Y.-A. Chen, A. Flesch, I. P. McCulloch, U. Schollwöck, J. Eisert & I. Bloch, Nature Physics *8*, (2012)

Probing the relaxation toward equilibrium in an isolated strongly correlated 1D Bose gas



(a) Concept of the experiment
(b) Even-odd resolved detection



Relaxation of the local density for different interaction strengths.

- initial state $\equiv$ density wave $|\Psi\rangle = |\cdots, 1, 0, 1, 0, 1 \cdots\rangle$
- approximatively 60 lattice sites

Executive summary

- The unprecedented control that has been achieved experimentally with ultra-cold atomic systems has given rise to the exploration of many-body dynamics in isolated systems far from equilibrium.
- Very precisely defined excited states and systems can be designed, and their dynamics followed accurately. These states can be Fock states, but also ***coherent states***.
- This has opened a new and exciting field of investigation, and poses significant challenges for theoreticians.

Simulation tools

- Exact diagonalization in Fock space

  → rather small systems

- Density Matrix Renormalization Group (DMRG).

  → low density (≈ 1 particle per site)

- **What about the high density, possibly strong interaction regime ?**

Simulation tools

- Exact diagonalization in Fock space

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- **What about the high density, possibly strong interaction regime ?**

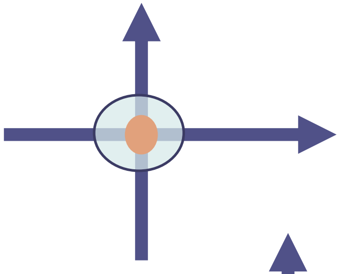
Mean Field

Mean Field approximation I : equilibrium (ground state / thermodynamics)

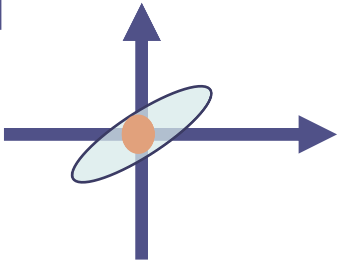
e.g.: Bose Hubbard model (1d)

$$\hat{H} = \sum_{j=1}^{N} \left[-J \left(\hat{a}_j^\dagger \hat{a}_{j+1} + h.c. \right) + V_j \hat{a}_j^\dagger \hat{a}_j + \frac{U}{2} \hat{n}_j \left(\hat{n}_j - 1 \right) \right]$$

- operators $\hat{a}_i \rightarrow c$-number ψ $\Rightarrow$ (discrete) Gross-Pitaevskii Eq $\Rightarrow$ *classical* dynamics



- "bare" mean field : coherent state "on" GP trajectory

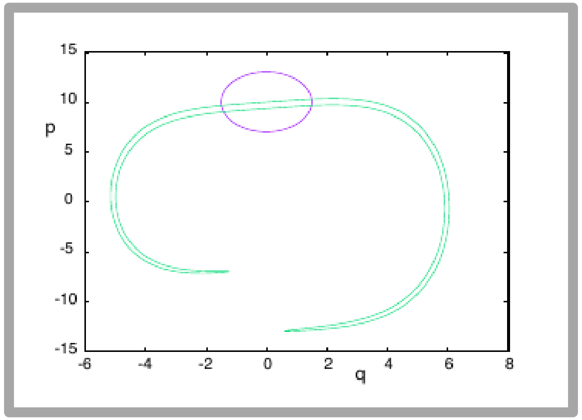


- Bogoliubov - De Gennes : take into account actual linear motion around the GP trajectory

Mean Field approximation II : non-equilibrium

Pb : except in the neighborhood of stable periodic orbits, solutions of the GP equation tend to diverge one from each other.

→ even if the initial state is a coherent state, one cannot expect that its time evolution will be well approximated by a coherent state (even including quasi-particle coherent state).



1-d quartic oscillator

→ there is no way that propagating a single GP solution provides enough information about the time evolution of the state.

Fix : propagate many solutions of the GP equation

→ **truncated Wigner approximation**

Truncated Wigner approximation

*[Steel et al., Phys. Rev. A **58** (1998), Sinatra et al. J. Phys. B **35** (2002), Dujardin et al. Ann. Phys. **527** (2015)]*

classical field

$$W[\mathbf{\Psi}] \equiv \int \prod_i \frac{d\gamma_i^R d\gamma_i^I}{\pi^2} \chi[\gamma] \exp\left[\sum_i \gamma_i^* \psi_i - \gamma_i \psi_i^*\right]$$

$$\chi[\gamma] \equiv \mathrm{Tr}\left[\hat{\sigma} \exp\left[\sum_i \gamma_i \hat{a}_i^\dagger - \gamma_i^* \hat{a}_i\right]\right]$$

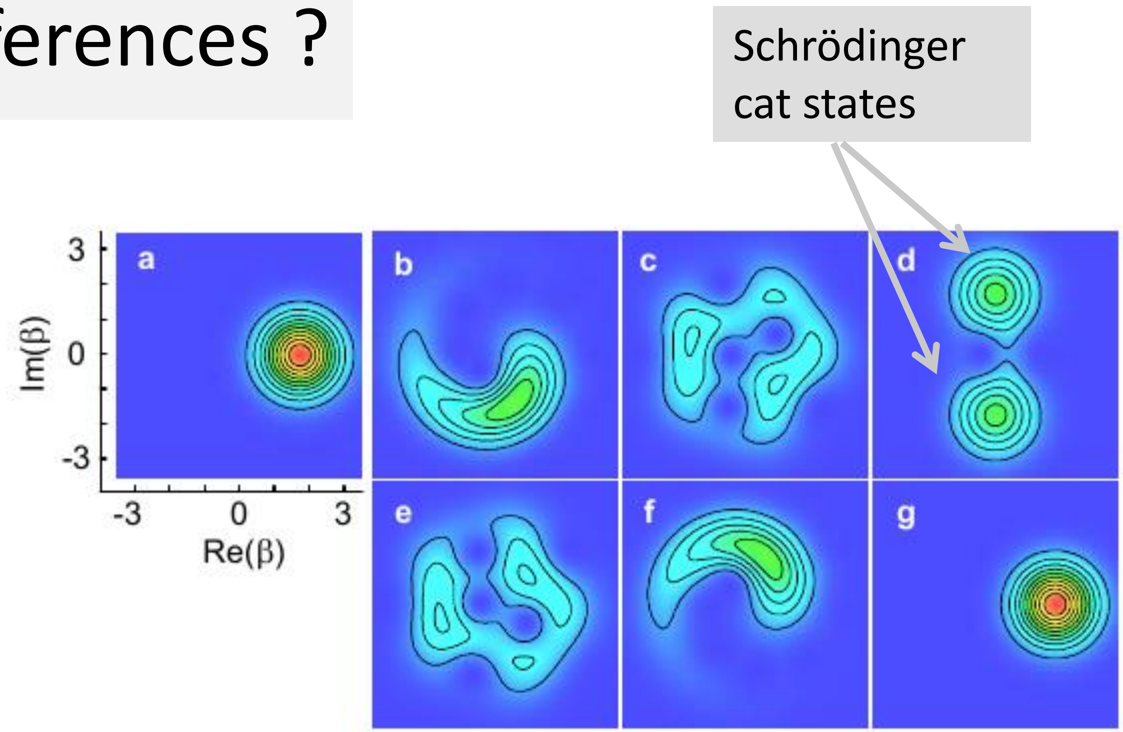
manybody density operator

The Wigner transform represent the manybody density operator σ in term of a ``probability distribution” over the classical field $\boldsymbol{\psi}$

→ possibility to propagate these classical field with the time dependent Gross-Pitaevskii equation, and to average with the corresponding weight.

→ this corresponds however to an ***incoherent sum*** of the contributions.

What about interferences ?



→ there is no way the truncated Wigner approximation takes interference effects into account properly.

[M. Greiner, *et al.* Nature (2002)]

Our goal here is to implement the effects of interferences between mean field solutions

*[cf also Simon & Strunz, Phys Rev A **89** (2014)]*

<u>Outline</u>

A. Warming : the one mode case
   - *Path integral approach*
   - *Semiclassics “à la Maslov”*

B. The multimode case
   - How to explore a large phase space
   - Specificity of coherent states : going complex or not going complex

C. A case study

D. Symmetries

A. Warmup : The one-mode case

1. Path Integral approach *[Baranger et al, J. Phys A **34** (2001)]*

$$\hat{H} = \hat{H}(\hat{a}^\dagger, \hat{a}, t) \;\rightarrow\; \text{e.g.:} \quad \hat{H} = V(t)\hat{a}^\dagger\hat{a} + \frac{U(t)}{2}\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}$$

coherent state: $\quad |z\rangle = e^{-\frac{1}{2}|z|^2} e^{z\hat{a}^\dagger}|0\rangle$

propagator: $\quad K(z", t; z', 0) = \langle z"|\hat{T}\exp[-\frac{i}{\hbar}\int_0^t \hat{H}(t')dt']|z'\rangle$

path integral representation:

Classical Hamiltonian

$$K(z", t; z', 0) = \int \mathcal{D}[z_t^* z_t] \exp\left[\int_0^t ds \frac{1}{2}\left(\partial_t z_s^* z_s - z_s^* \partial_t z_s^*\right) + \frac{i}{\hbar}\mathcal{H}(z_s^*, z_s)\right]$$

A. Warmup : The one-mode case

1. <u>Path Integral approach</u> *[Baranger et al, J. Phys A **34** (2001)]*

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Classical Hamiltonian

N

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Stationnary phase approximation (Baranger et al.):

- $\delta S = 0 \rightarrow$ classical evolution under $\mathcal{H}(z^*, z)$
- "trajectories" $z_{t'} = u_{t'} + iv_{t'}$ such that $\begin{cases} u_0 - iv_0 \equiv \; u' - iv' = z'^* \\ u_t + iv_t \equiv u" + iv" = z" \end{cases}$

$$K(z", t; z', 0) = \sum_{\substack{u'+iv'=z' \\ u"+iv"=z"}} \sqrt{\frac{\partial v'}{\partial v"}} \exp\left\{ \overbrace{\frac{i}{2\hbar}\int_0^t dt' \frac{\partial^2 \mathcal{H}}{\partial u \partial v}} \right\} \times$$

$$\exp\left\{ \underbrace{\left[\int_0^t dt' \frac{1}{2}\left(\dot{v}u - \dot{u}v\right) + \frac{i}{\hbar}\mathcal{H}\right]}_{\text{classical action}} + \underbrace{\frac{1}{2}(v"u" + v'u') - \frac{1}{2}(|z'|^2 + |z"|^2)}_{\text{initial and final states}} \right\}$$

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$$\exp\left\{ \underbrace{\left[\int_0^t dt' \frac{1}{2} (\dot{v}u - \dot{u}v) + \frac{i}{\hbar} \mathcal{H} \right]}_{\text{classical action}} + \underbrace{\frac{1}{2}(v"u" + v'u') - \frac{1}{2}(|z'|^2 + |z"|^2)}_{\text{initial and final states}} \right\}$$

- $u_t, v_t \equiv$ **complex numbers** $\Rightarrow \begin{cases} u' + iv' \neq z' \\ u" - iv" \neq z"^* \end{cases}$

2. Semiclassics "à la Maslov" $\left(N \simeq \frac{1}{\hbar}\right)$

- use the "quadratures" $(\hat{p}, \hat{q})$

$$\begin{cases} \hat{a} \equiv \frac{1}{\sqrt{2}}(\hat{q} + i\hat{p}) \\ \hat{a}^\dagger \equiv \frac{1}{\sqrt{2}}(\hat{q} - i\hat{p}) \end{cases}$$

$\Rightarrow$ e.g.: $\hat{H} = V(t)\hat{a}^\dagger\hat{a} + \frac{U(t)}{2}\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}$

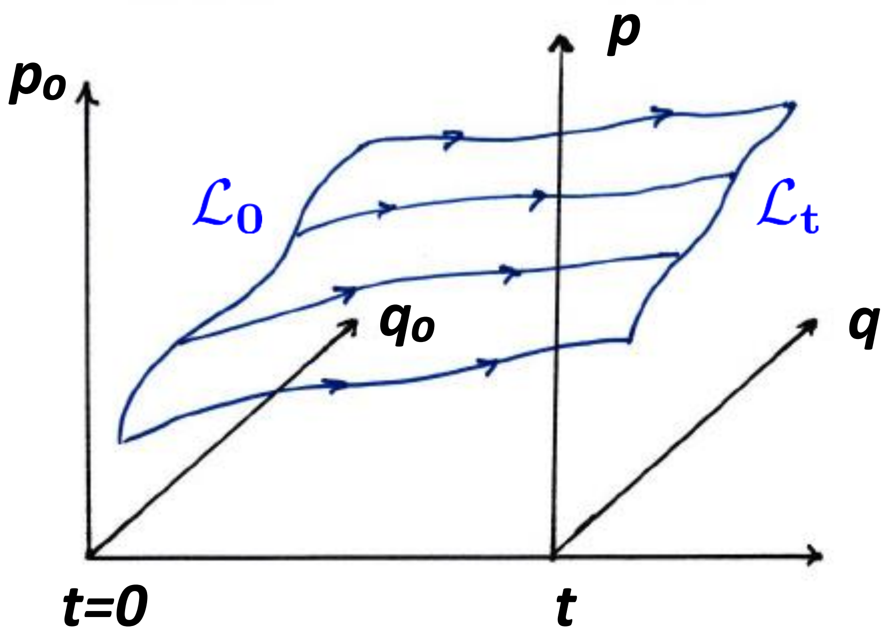
$$\to \quad \frac{V(t)}{2}(\hat{p}^2 + \hat{q}^2) + \frac{U(t)}{4}(\hat{p}^2 + \hat{q}^2)^2 \quad + \quad O(\hbar)$$

- to any intial wave function in the semiclassical form

$$\psi_0(q) \equiv \varphi(q) \exp\left[\frac{i}{\hbar}S_0(q)\right]$$

$\to$ associate a Lagrangian manifold (just a curve here)

$$\mathcal{L}_0 \equiv \{(q_0, p_0(q_0) = \partial_q S_0)\}$$



The quantum evolution is based on the classical evolution of the Lagrangian manifold

[Maslov & Fedoriuk (1981)]

$$\Psi(q,t) = \sum_{\mathbf{r}=(q,p)\in\mathcal{L}_t} \sqrt{\frac{\partial q_0}{\partial q}}\varphi(q_0)\exp\left[\frac{i}{\hbar}\left(\int_{\mathbf{r}_0}^{\mathbf{r}} pdq - \mathcal{H}dt\right)\right]$$

- for a correlation function $\langle\alpha"|\hat{T}\exp[-\frac{i}{\hbar}\int_0^t \hat{H}(t')dt']|\beta'\rangle$

$\rightarrow$ intersection of $\mathcal{L}_t(\beta')$ and $\mathcal{L}_0(\alpha")$ (saddle trajectory)

Extra phase: $\exp\left\{\frac{i}{2}\int_0^t dt' \frac{\partial^2 \mathcal{H}}{\partial p \partial q}\right\}$

$\Rightarrow$ associated with the ordering of the operators

Various ways to "quantize" a classical $H(p,q)$

- $\hat{H}(\overset{(1)}{\hat{p}},\overset{(2)}{\hat{q}}) \longrightarrow [\ldots]\exp\left[\frac{i}{\hbar}\left(\int_{\mathbf{r}_0}^{\mathbf{r}} pdq - \mathcal{H}dt\right)\right]\cdot\exp\left[+\frac{1}{2}\int_0^t \frac{\partial^2 H}{\partial q \partial p}dt\right]$
- $\hat{H}(\overset{(2)}{\hat{p}},\overset{(1)}{\hat{q}}) \longrightarrow [\ldots]\exp\left[\frac{i}{\hbar}\left(\int_{\mathbf{r}_0}^{\mathbf{r}} pdq - \mathcal{H}dt\right)\right]\cdot\exp\left[-\frac{1}{2}\int_0^t \frac{\partial^2 H}{\partial q \partial p}dt\right]$
- $\frac{1}{2}\left(\hat{H}(\overset{(1)}{\hat{p}},\overset{(2)}{\hat{q}})+\hat{H}(\overset{(2)}{\hat{p}},\overset{(1)}{\hat{q}})\right) \longrightarrow [\ldots]\exp\left[\frac{i}{\hbar}\left(\int_{\mathbf{r}_0}^{\mathbf{r}} pdq - \mathcal{H}dt\right)\right]$

Manybody physics most often prefers normal ordering $\hat{H}(\overset{(2)}{\hat{a}^\dagger},\overset{(1)}{\hat{a}})$

$$\Longrightarrow \quad [\ldots]\exp\left[\frac{i}{\hbar}\left(\int_{\mathbf{r}_0}^{\mathbf{r}} pdq - \mathcal{H}dt\right)\right]\cdot\exp\left[\frac{i}{2}\int_0^t \frac{\partial^2 H}{\partial q \partial p}dt'\right]$$

complexification of phase space

$\Rightarrow$ associated with the choice of coherent states

coherent state $|z\rangle$: $\quad (z = q_z + ip_z)$

$$\langle q|z\rangle = \frac{1}{(\pi\hbar)^{1/4}} \exp\left[-\frac{(q-q_z)^2}{2\hbar}\right] \exp\left[\frac{i}{\hbar} p_z (q - \frac{1}{2} q_z)\right]$$

semiclassical form $\propto \exp\left[\frac{i}{\hbar} S_0(q)\right]$

$$\rightarrow \quad S_0(q) = p_z (q - \frac{1}{2} q_z) + \frac{i}{2}(q - q_z)^2$$

$$p_0(q) = \partial_q S_0 = p_z + i(q - q_z)$$

$\Rightarrow$ complexe manifold

B. The many-mode case

- Once the question of ordering of the operator is taken into account, no difference between the path integral approach and the traditional time dependent WKB.
- More generally there is no conceptual difference between usual semiclassics and many-bosons semiclassics, even if the small parameter is not the same ($\hbar$ vs N^{-1})
- WKB "à la Maslov" is however much simpler, and thus more "versatile" than the path integral approach

  → no particular difficulty to generalize the formalism to the many-mode case and to a large class of initial states (including for example Fock states)
- There are of course practical difficulties …

  → complexification of phase space

  → large phase space

Case study : propagation of coherent state density wave

- Hamiltonian = 1-d Bose Hubbard

$$\hat{H} = \sum_{j=1}^{N} \left[-J \left(\hat{a}_j^\dagger \hat{a}_{j+1} + h.c. \right) + \frac{U}{2} \hat{n}_j \left(\hat{n}_j - 1 \right) \right]$$

- Initial state = coherent state density wave

$$|\vec{n}\rangle = \prod_{j=1}^{N} \exp\left(-\frac{|z_j|^2}{2} + z_j a_j^\dagger \right) |\vec{0}\rangle$$

(here $z_i = \sqrt{n_i}$
$\rightarrow$ no initial dephasing)

e.g.: $|\vec{n}\rangle = |n, 0, n, 0, ..., n, 0\rangle$.

- Observable = autocorrelation function

$$\mathcal{C}(t) = |\mathcal{A}(t)|^2 \qquad \mathcal{A}(t) = \langle \vec{n} \left| \hat{U}(t) \right| \vec{n} \rangle$$

$$(\hat{U}(t) \equiv \exp[-\frac{i}{\hbar}\hat{H}t])$$

- Intial and final manifold

$$\begin{cases} p_0^j(\mathbf{q_0}) = +i(q_0^j - \sqrt{2n_j}) & \text{initial} \\ p_f^j(\mathbf{q_0}) = -i(q_0^j - \sqrt{2n_j}) & \text{final} \end{cases} \qquad (j = 1, \cdots, N)$$

- Autocorrelation function $\quad C(t) = \left| \sum_\gamma D_\gamma^{1/2}(t) \exp\left[\frac{i}{\hbar} \phi_\gamma(t) \right] \right|^2$

$$\frac{i}{\hbar}\phi_\gamma(t) = \frac{i}{\hbar} \underbrace{S(\vec{q}_t^\gamma, \vec{q}_0^\gamma; t)}_{\text{classical action}} - i \underbrace{\nu_\gamma}_{\text{Maslov Index}} \frac{\pi}{2} + F_0^{\gamma -} + F_t^{\gamma,+}$$

$$F_{0,t}^{\gamma,\pm} = \frac{i}{\hbar^2} \vec{p}_{0,t}^R \cdot \vec{p}_{0,t}^I - \frac{1}{2\hbar^2} \vec{p}_{0,t}^I \cdot \vec{p}_{0,t}^I - \frac{1}{2} \vec{q}_{0,t}^I \cdot \vec{q}_{0,t}^I \pm \frac{1}{\hbar} \vec{p}_{0,t}^R \cdot \vec{q}_{0,t}^I$$

$$D_\gamma(t) = \mathrm{Det}\left[\frac{1}{2} (\mathbf{M_{11}^\gamma} + \mathbf{M_{22}^\gamma} + i\hbar \mathbf{M_{21}^\gamma} - \frac{i}{\hbar} \mathbf{M_{12}^\gamma}) \right]$$

$$\mathbf{M_{11}^\gamma} = \frac{\partial \vec{q}_t}{\partial \vec{q}_0}, \ \mathbf{M_{22}^\gamma} = \frac{\partial \vec{p}_t}{\partial \vec{p}_0}, \ \mathbf{M_{12}^\gamma} = \frac{\partial \vec{q}_t}{\partial \vec{p}_0}, \ \mathbf{M_{21}^\gamma} = \frac{\partial \vec{p}_t}{\partial \vec{q}_0}.$$

[cf. H. Pal, M. Vyas, and S. Tomsovic, Phys. Rev. E 93, 012213 (2016).]

Practical issues

1. <u>Exploring a large phase space</u>

Problem :

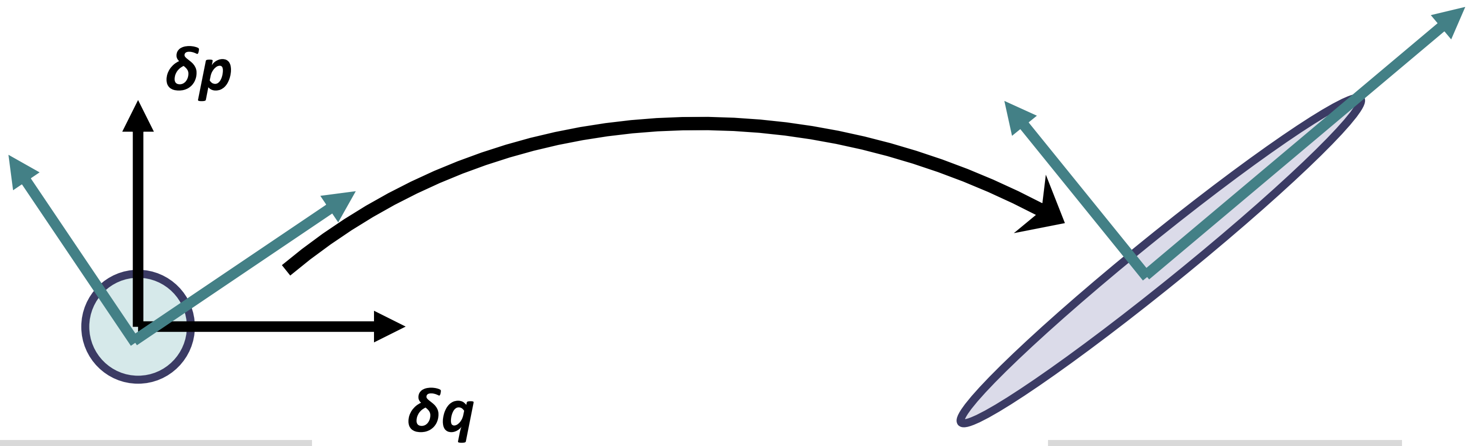
- Quantities like the autocorrelation function *C(*t*)* involve a mixed intial/final value “shooting”-problem ($r_0 \rightarrow$ initial manifold, $r_t \rightarrow$ final manifold)
- Phase space is a big place, especially if the dimensionality (number of modes is large).

→ limit the time for which the semiclassical approach can be used.

→favor system close to integrability (eg large *U*).

To mitigate this : explore predominantly unstable directions

$$\begin{pmatrix} \delta\mathbf{p}_t \\ \delta\mathbf{q}_t \end{pmatrix} = \begin{pmatrix} \mathbf{M}_t^{11} & \mathbf{M}_t^{12} \\ \mathbf{M}_t^{21} & \mathbf{M}_t^{22} \end{pmatrix} \begin{pmatrix} \delta\mathbf{p}_0 \\ \delta\mathbf{q}_0 \end{pmatrix}$$



phase space representation of initial state

classical evolution

$$(\delta\mathbf{p}_0, \delta\mathbf{q}_0) \cdot \begin{pmatrix} \delta\mathbf{p}_0 \\ \delta\mathbf{q}_0 \end{pmatrix} = \text{const.} \quad \Rightarrow \quad = (\delta\mathbf{p}_t, \delta\mathbf{q}_t) \cdot \underbrace{\mathbf{M}^{-1^T}\mathbf{M}^{-1}}_{\mathcal{M}} \cdot \begin{pmatrix} \delta\mathbf{p}_0 \\ \delta\mathbf{q}_0 \end{pmatrix}$$

Large eigenvalues of $\mathcal{M}$ $\rightarrow$ First directions to explore

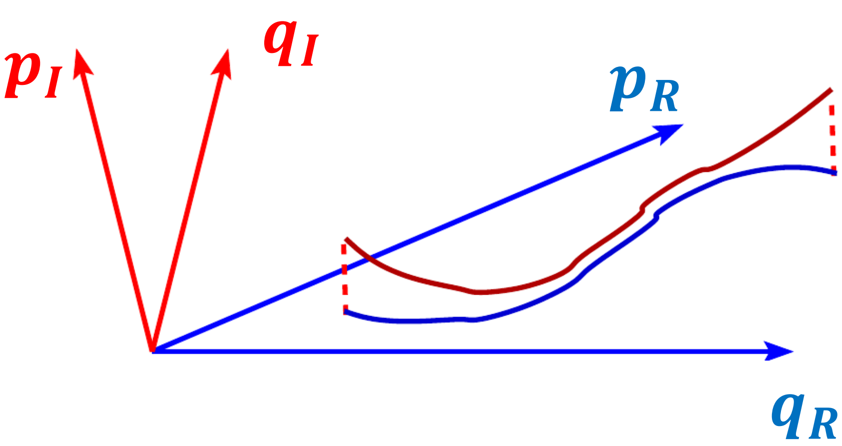
2. Complexification of the classical dynamics

Problem :

- Going complex doubles the number of variables
  → this is manageable (just more work)
- Motion in complexified phase space is non-compact
  → this makes the search of saddle trajectories completely impractical

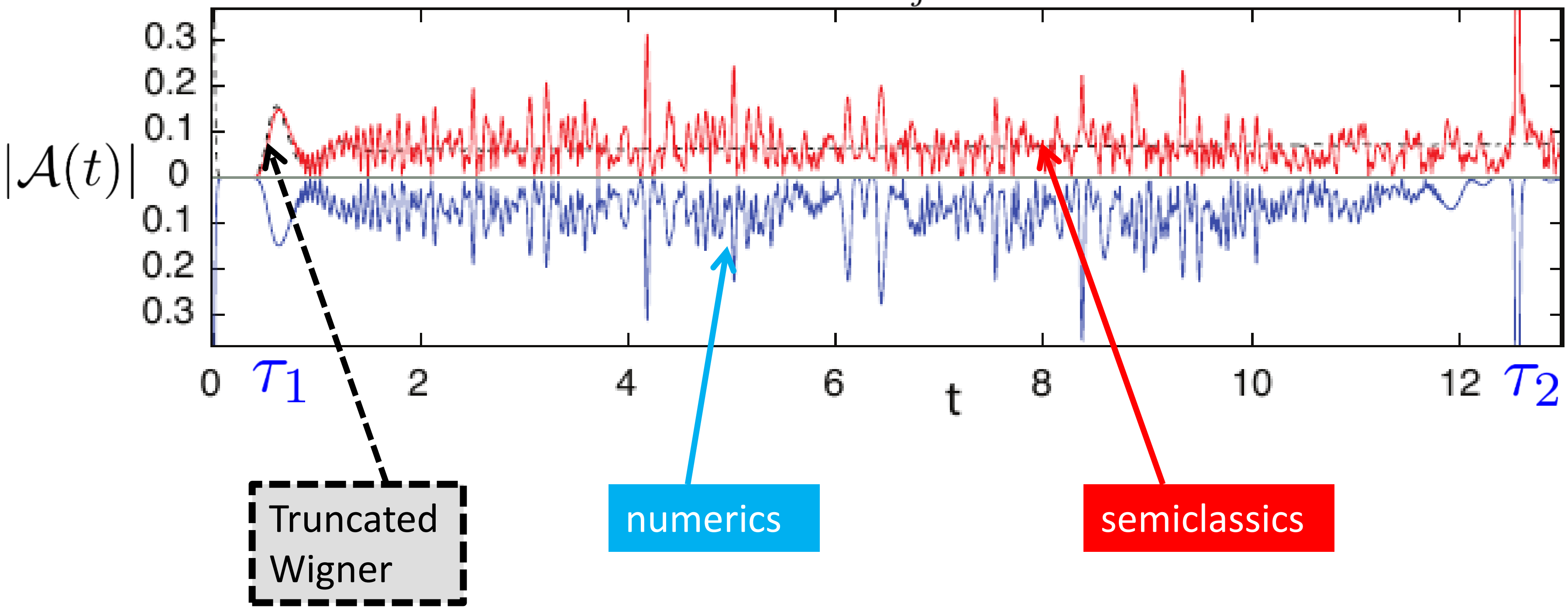
$$\mathrm{Proba}[\mathbf{r}_t \in \mathcal{L}_t] \propto \frac{1}{\mathrm{Vol}}$$

fix: Use "ghost" real trajectory and converge to true trajectory with Newton-Raphson



C. Case study

- ring with four sites
- initial coherent state density wave $|\vec{n}\rangle = |20, 0, 20, 0\rangle$
- $J = 0.2,\ U = 0.5 \quad \Rightarrow \quad \tau_1 = \dfrac{2\pi\hbar}{Un_j} \simeq 0.63 \quad \tau_2 = \dfrac{2\pi\hbar}{U} \simeq 12.57$



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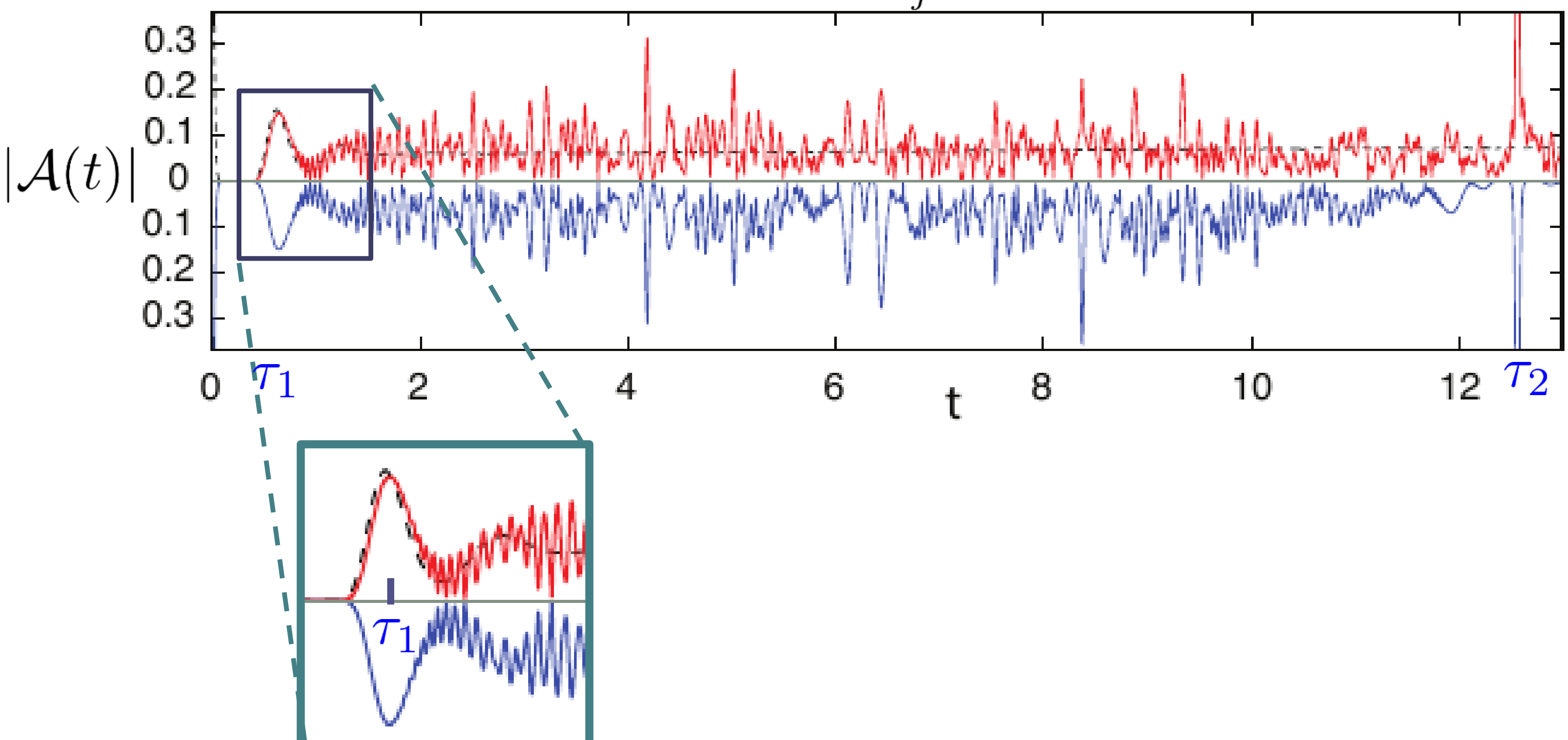

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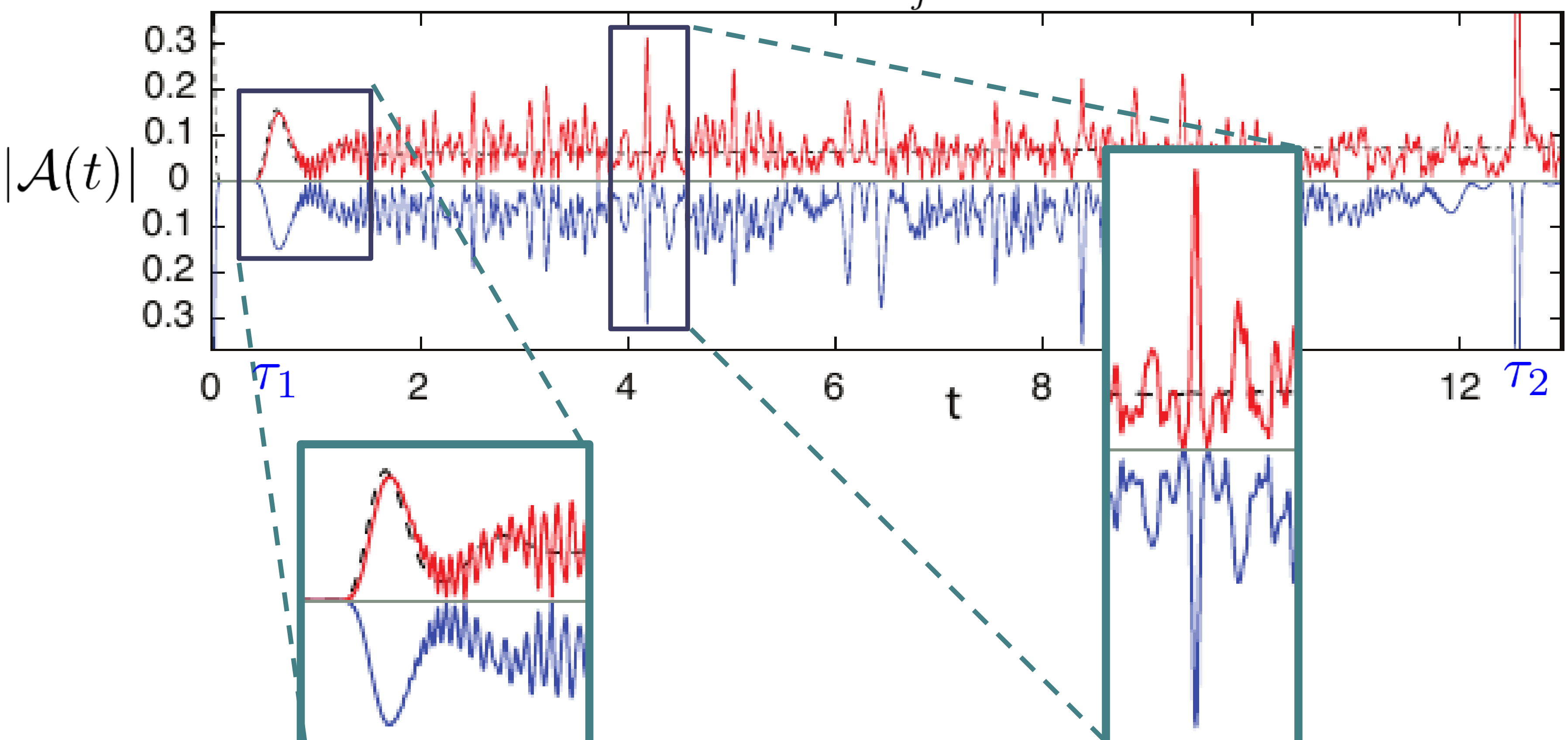
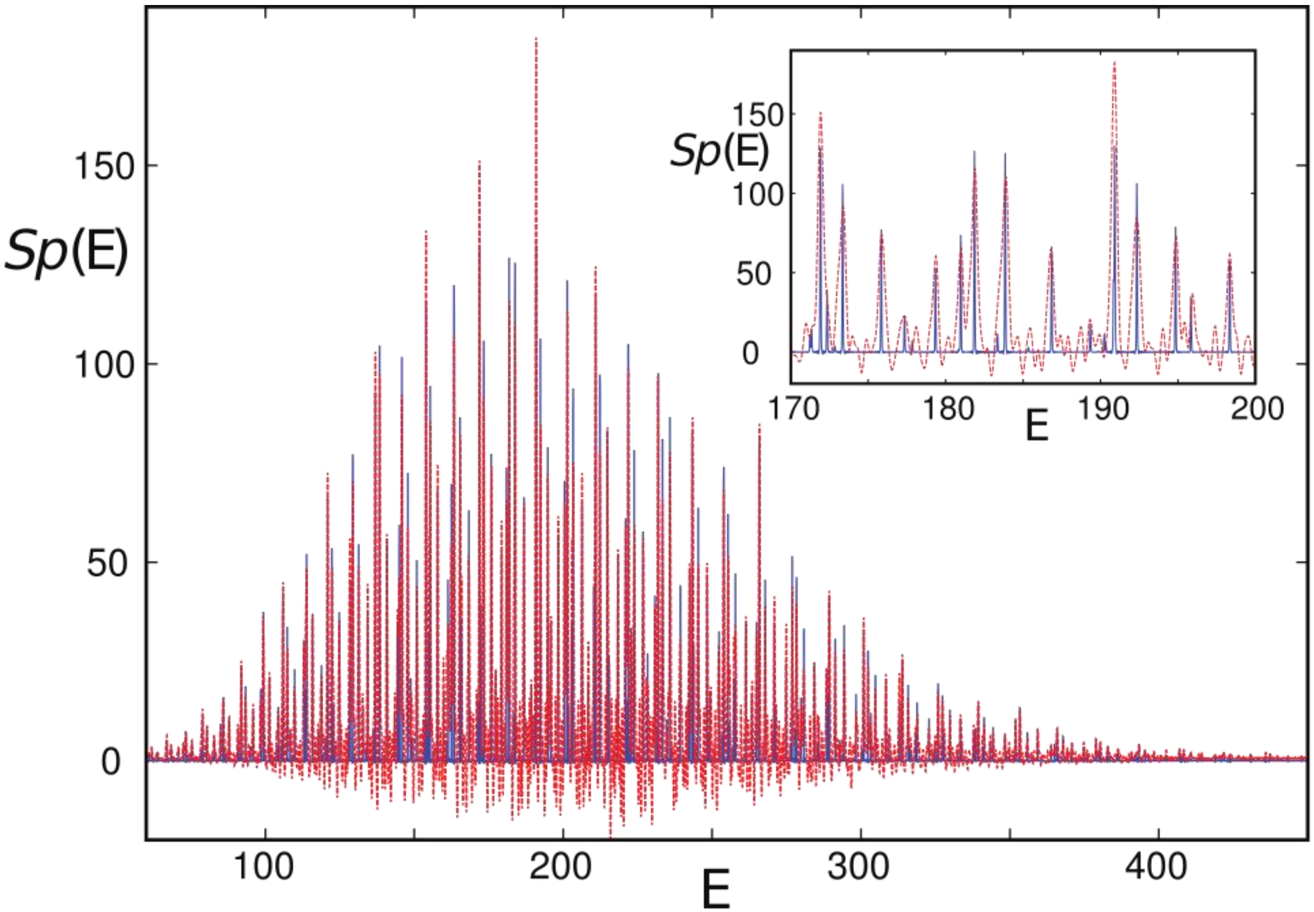


Spectrum

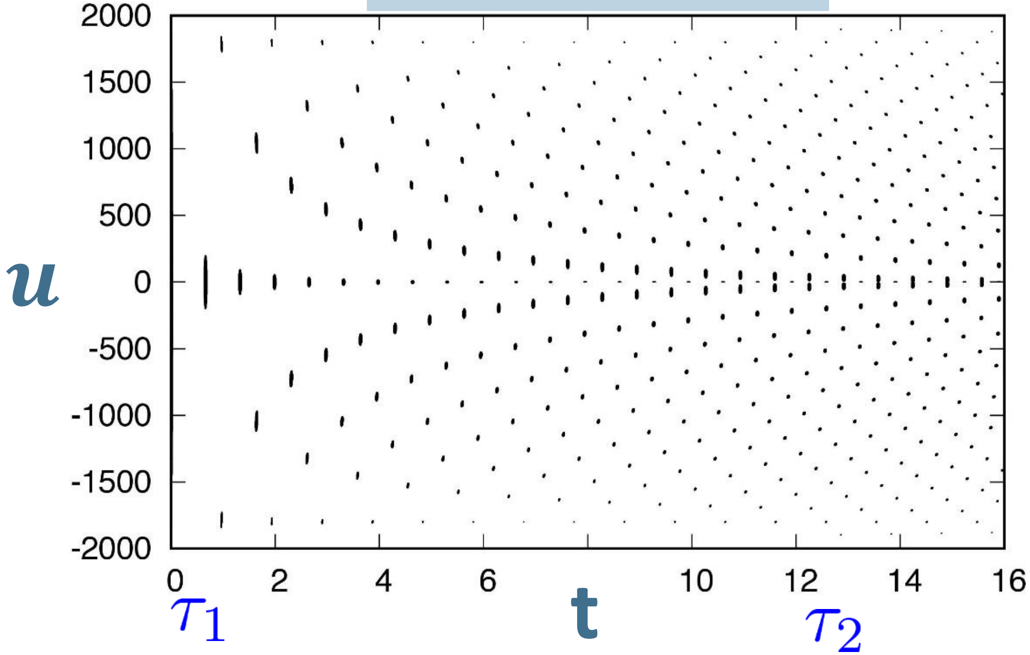
$$\mathcal{SP}(E) = \sum_\alpha |\langle \xi_\alpha | \vec{n} \rangle|^2 \delta(E - E_\alpha) \propto \int dt e^{iEt/\hbar} A(t)$$

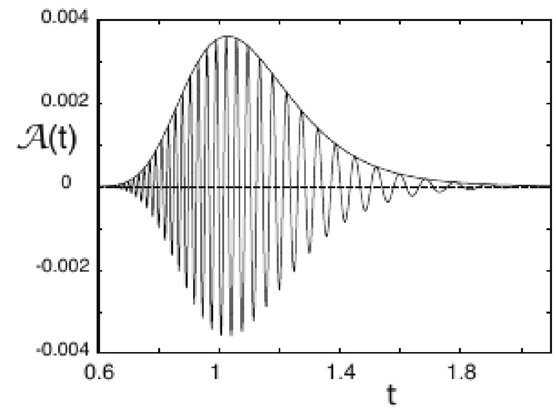


(cutoff $\tau_c = 40$)

Trajectory search

"seed" trajectories



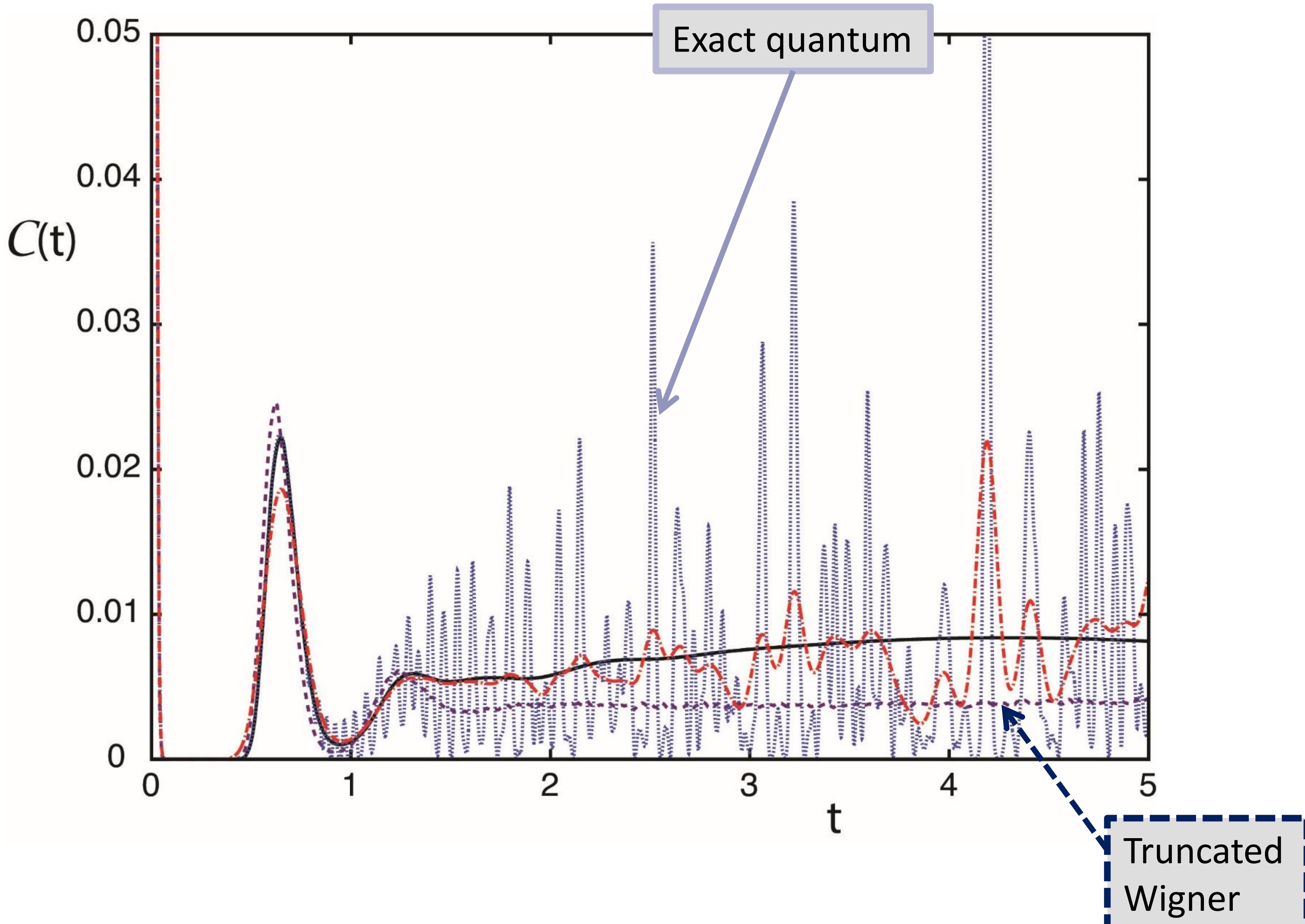


Each seed → one "chirp
(here second saddle traj.)

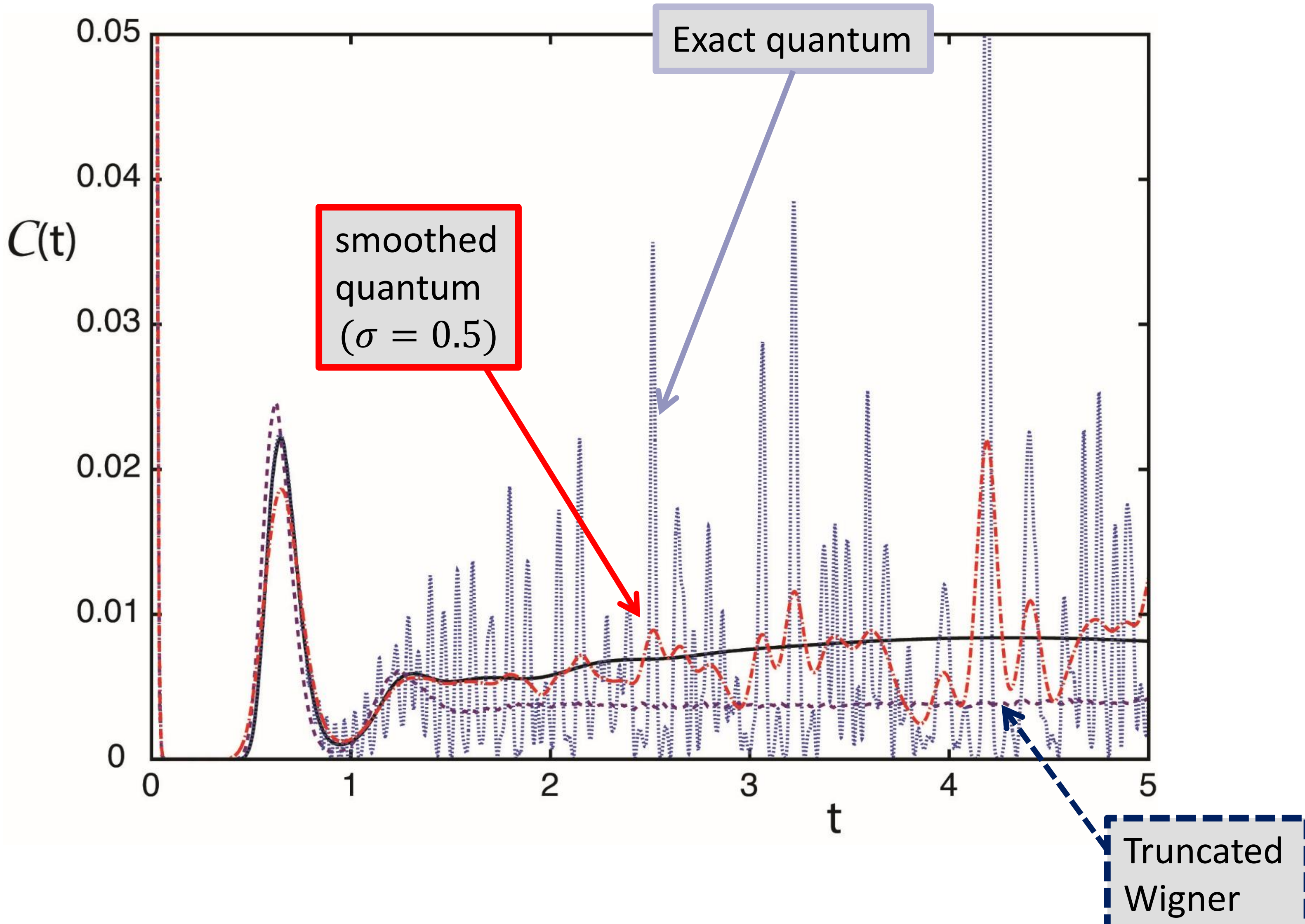
$\tau_2/3 \to 60$ saddles
$\tau_2 \to 600$ saddles

- $u \equiv$ distance along the direction associated with the largest eigen-value of $\mathcal{M} = \mathbf{M}^{-1^T}\mathbf{M}^{-1}$
- trajectory kept if $(|\mathbf{q}_0 - \bar{\mathbf{q}}|^2 + |\mathbf{p}_0 - \bar{\mathbf{p}}|^2) + (|\mathbf{q}_f - \bar{\mathbf{q}}|^2 + |\mathbf{p}_f - \bar{\mathbf{p}}|^2) \leq \text{const.}$
- for each "seed", converge to the saddle through Newton-Raphson

D. Symmetries



D. Symmetries



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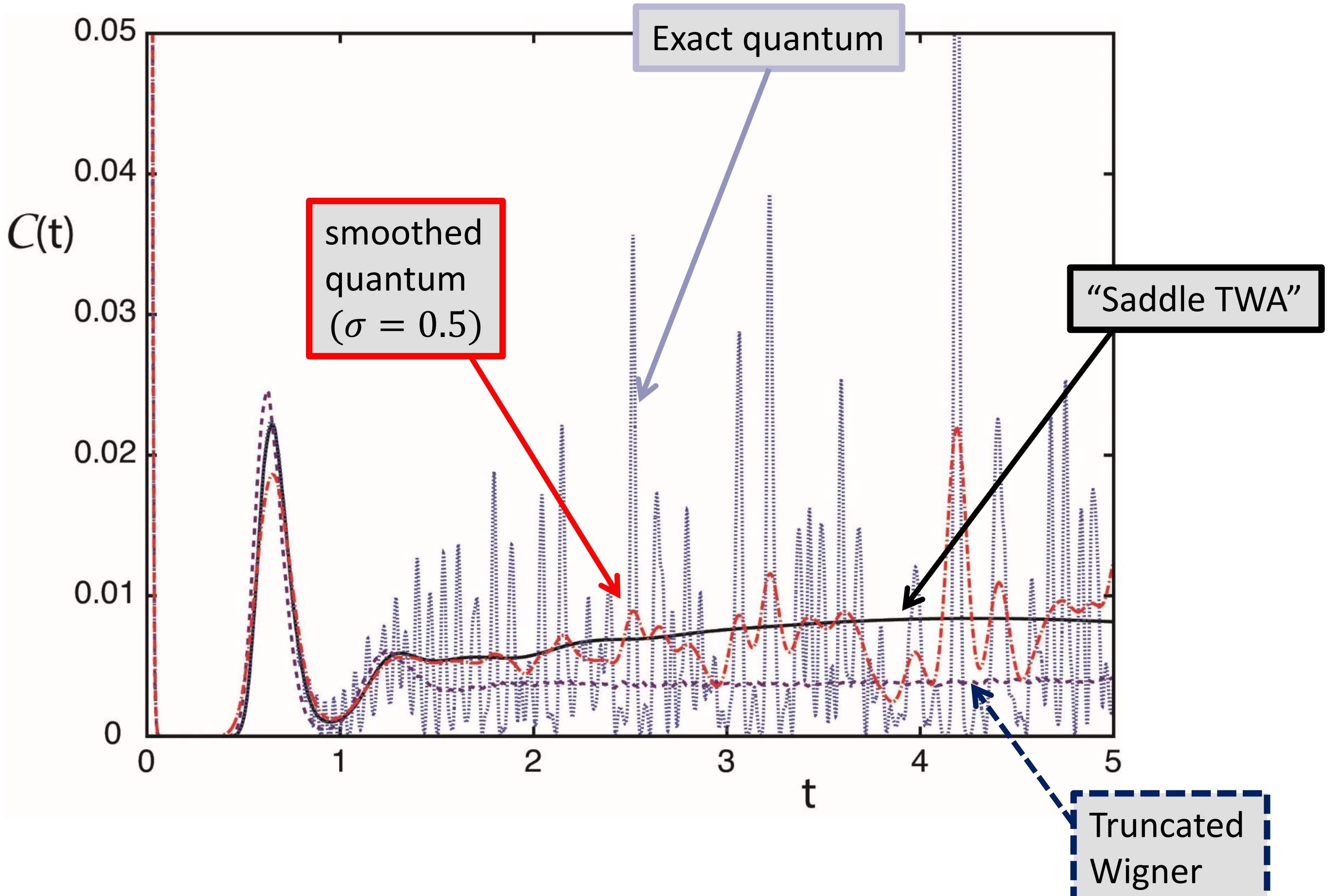

Diagonal approximation

- Autocorrelation function: $$C(t) = \left| \sum_{\gamma} D_{\gamma}^{1/2}(t) \exp\left[\frac{i}{\hbar} \phi_{\gamma}(t) \right] \right|^2$$

$$\gamma \equiv \text{trajectories } p_0^j(\mathbf{q_0}) = +i(q_0^j - \sqrt{2n_j}) \rightarrow p_f^j(\mathbf{q_f}) = -i(q_f^j - \sqrt{2n_j})$$

$$\frac{i}{\hbar}\phi_{\gamma}(t) = \frac{i}{\hbar} S(\vec{q}_t^{\gamma}, \vec{q}_0^{\gamma}; t) - i\nu_{\gamma}\frac{\pi}{2} + F_0^{\gamma -} + F_t^{\gamma, +}$$

$$D_{\gamma}(t) = \text{Det}\left[\frac{1}{2}(\mathbf{M}_{\mathbf{11}}^{\gamma} + \mathbf{M}_{\mathbf{22}}^{\gamma} + i\hbar \mathbf{M}_{\mathbf{21}}^{\gamma} - \frac{i}{\hbar}\mathbf{M}_{\mathbf{12}}^{\gamma}) \right]$$

- Diagonal approximation

$$C(t) = \sum_{\gamma, \gamma'} D_{\gamma}^{1/2} D_{\gamma'}^{1/2} \exp\left[\frac{i}{\hbar}(\phi_{\gamma} - \phi_{\gamma'}) \right] \quad \longrightarrow \quad C_{\text{diag}}(t) = \sum_{\gamma} D_{\gamma}$$

Diagonal approximation

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$$D_\gamma(t) = \mathrm{Det}\left[\frac{1}{2}(\mathbf{M}_{\mathbf{11}}^\gamma + \mathbf{M}_{\mathbf{22}}^\gamma + i\hbar \mathbf{M}_{\mathbf{21}}^\gamma - \frac{i}{\hbar}\mathbf{M}_{\mathbf{12}}^\gamma)\right]$$

- Diagonal approximation

$$C(t) = \sum_{\gamma,\gamma'} D_\gamma^{1/2} D_{\gamma'}^{1/2} \exp\left[\frac{i}{\hbar}(\phi_\gamma - \phi_{\gamma'})\right] \quad \longrightarrow \quad C_{\mathrm{diag}}(t) = \sum_\gamma D_\gamma$$

$$\boxed{C_{\mathrm{diag}}(t) \equiv C_{\mathrm{TWA}}(t)}$$

Intermezzo : derivation of the "diagonal" approximation

1. The semiclassical (Van-Vleck) propagator (in q-quadrature basis)

$$K(\mathbf{q}_f, \mathbf{q}_i, t) = \langle \mathbf{q}_f | \underbrace{e^{-it\hat{H}/\hbar}}_{\hat{U}(t)} | \mathbf{q}_i \rangle = \sum_\gamma A_\gamma(\mathbf{q}_f, \mathbf{q}_i) e^{\frac{i}{\hbar} R_\gamma(\mathbf{q}_f, \mathbf{q}_i)}$$

γ solution of the GP equation

$$i\hbar \frac{\partial}{\partial t} \psi_l(t) = \sum_{l'=1}^{L} H_{l,l'} \psi_{l'}(t) + U_l(|\psi_l(t)|^2 - 1)\psi_l(t)$$

$$\text{with boundary cond : } \mathrm{Re}[\psi^l(0)] = \frac{q_i^l}{\sqrt{2}}, \ \mathrm{Re}[\psi^l(t)] = \frac{q_f^l}{\sqrt{2}}.$$

$$R_\gamma = \int_{\gamma : \mathbf{q}_i \to \mathbf{q}_f} \mathbf{p} d\mathbf{q} - \mathcal{H} dt \qquad \text{(action)}$$

$$A_\gamma(\mathbf{q}_f, \mathbf{q}_i) = \frac{e^{-i\kappa_\gamma \pi/2}}{(2i\pi\hbar)^{L/2}} \left| \det \left(-\frac{\partial^2 R_\gamma}{\partial q_f^l \partial q_i^{l'}} (\mathbf{q}_f, \mathbf{q}_i) \right) \right|^{1/2}$$

2. Time evolution of the mean value of an operator

initial manybody state $|\Phi_0\rangle$ $\rightarrow$ density $\hat{\rho}(t) \equiv |\hat{U}(t)\Phi_0\rangle\langle\hat{U}(t)\Phi_0|$

arbitrary operator $\hat{O}$ $\rightarrow \langle O\rangle(t) = \mathrm{Tr}\left[\hat{\rho}(t)\hat{O}\right]$

in q-quadrature basis

$$\langle\hat{O}\rangle(t) = \int d\mathbf{q}_i' d\mathbf{q}_i'' d\mathbf{q}_f' d\mathbf{q}_f'' \Phi_0^*(\mathbf{q}_i')\Phi_0(\mathbf{q}_i'')\langle\mathbf{q}_f'|\hat{O}|\mathbf{q}_f''\rangle$$
$$\times \sum_{\gamma',\gamma''} A_{\gamma'}^*(\mathbf{q}_f',\mathbf{q}_i') A_{\gamma''}(\mathbf{q}_f'',\mathbf{q}_i'')\, e^{\frac{i}{\hbar}(R_{\gamma''}(\mathbf{q}_f'',\mathbf{q}_i'')-R_{\gamma'}(\mathbf{q}_f',\mathbf{q}_i'))}$$

3. Diagonal approximation

assumptions : (i) only diagonal terms $\gamma' = \gamma''$ survive averaging

(ii) only short chords $\mathbf{q}_i' \simeq \mathbf{q}_i''$, $\mathbf{q}_f' \simeq \mathbf{q}_f''$ are relevant

$\rightarrow$ introduce $\mathbf{Q}_{f,i} \equiv \frac{1}{2}(\mathbf{q}''_{f,i} + \mathbf{q}'_{f,i})$ and $\delta\mathbf{q}_{f,i} \equiv (\mathbf{q}''_{f,i} - \mathbf{q}'_{f,i})$

and expand in $\delta\mathbf{q}_{f,i}$, using $\mathbf{p}_f^{(\gamma)} = \dfrac{\partial R_\gamma}{\partial \mathbf{q}_f}$; $\mathbf{p}_i^{(\gamma)} = -\dfrac{\partial R_\gamma}{\partial \mathbf{q}_i}$

$$\begin{aligned}\langle \hat{O} \rangle (t)_{\text{diag}} \quad &= \quad \int d\mathbf{Q}_i d\mathbf{Q}_f d\delta\mathbf{q}_f d\delta\mathbf{q}_i \, \langle \mathbf{q}_i'' | \hat{\rho}_0 | \mathbf{q}_i' \rangle \, \langle \mathbf{q}_f' | \hat{O} | \mathbf{q}_f'' \rangle \\ &\times \quad \sum_\gamma |A_\gamma(\mathbf{Q}_f \mathbf{Q}_i)|^2 \exp\left[\frac{i}{\hbar}\left(\mathbf{p}_f^{(\gamma)}\delta\mathbf{q}_f - \mathbf{p}_i^{(\gamma)}\delta\mathbf{q}_i\right)\right]\end{aligned}$$

Wigner $\qquad [O]_W(\mathbf{Q}, \mathbf{P}) \equiv \int d\delta\mathbf{q} e^{(i/\hbar)\mathbf{P}\cdot\delta\mathbf{q}} \langle \mathbf{Q} + \frac{\delta\mathbf{q}}{2} | \hat{f} | \mathbf{Q} - \frac{\delta\mathbf{q}}{2} \rangle$

$$\rightarrow \qquad \langle\hat{O}\rangle(t)_{\mathrm{diag}} = \int d\mathbf{Q}_i d\mathbf{Q}_f \sum_\gamma |A_\gamma(\mathbf{Q}_f\mathbf{Q}_i)|^2[\rho_0]_W(\mathbf{Q}_i,\mathbf{p}_i^{(\gamma)})[O]_W(\mathbf{Q}_f,\mathbf{p}_f^{(\gamma)})$$

3. Initial value representation : $-\frac{\partial^2 R_\gamma}{\partial\mathbf{Q}_f\partial\mathbf{Q}_i} = \frac{\partial\mathbf{P}_i}{\partial\mathbf{Q}_f}_\gamma$

$|A|^2 \equiv \text{Jacobian of } \mathbf{Q}_f \rightarrow \mathbf{P}_i \quad \Rightarrow \quad \sum_\gamma \int d\mathbf{Q}_f|A_\gamma|^2 \mapsto \int \frac{d\mathbf{P}_i}{(2\pi\hbar)^L}$

$$\boxed{\langle\hat{O}\rangle_{\mathrm{diag}}(t) = \int \frac{d\mathbf{Q}_i d\mathbf{P}_i}{(2\pi\hbar)^L}[\rho_0]_W(\mathbf{Q}_i,\mathbf{P}_i)\,[O]_W(\mathbf{Q}_f,\mathbf{P}_f)}$$

(NB: for autocorrelation, $\hat{O} \equiv \hat{\rho}_0$)

Symmetries

- Hamiltonian $$\hat{H} = \sum_{j=1}^{4} \left[-J \left(\hat{a}_j^\dagger \hat{a}_{j+1} + h.c. \right) + \frac{U}{2} \hat{n}_j \left(\hat{n}_j - 1 \right) \right]$$
- Initial (and final) state $|\vec{n}\rangle = |n, 0, n, 0\rangle.$

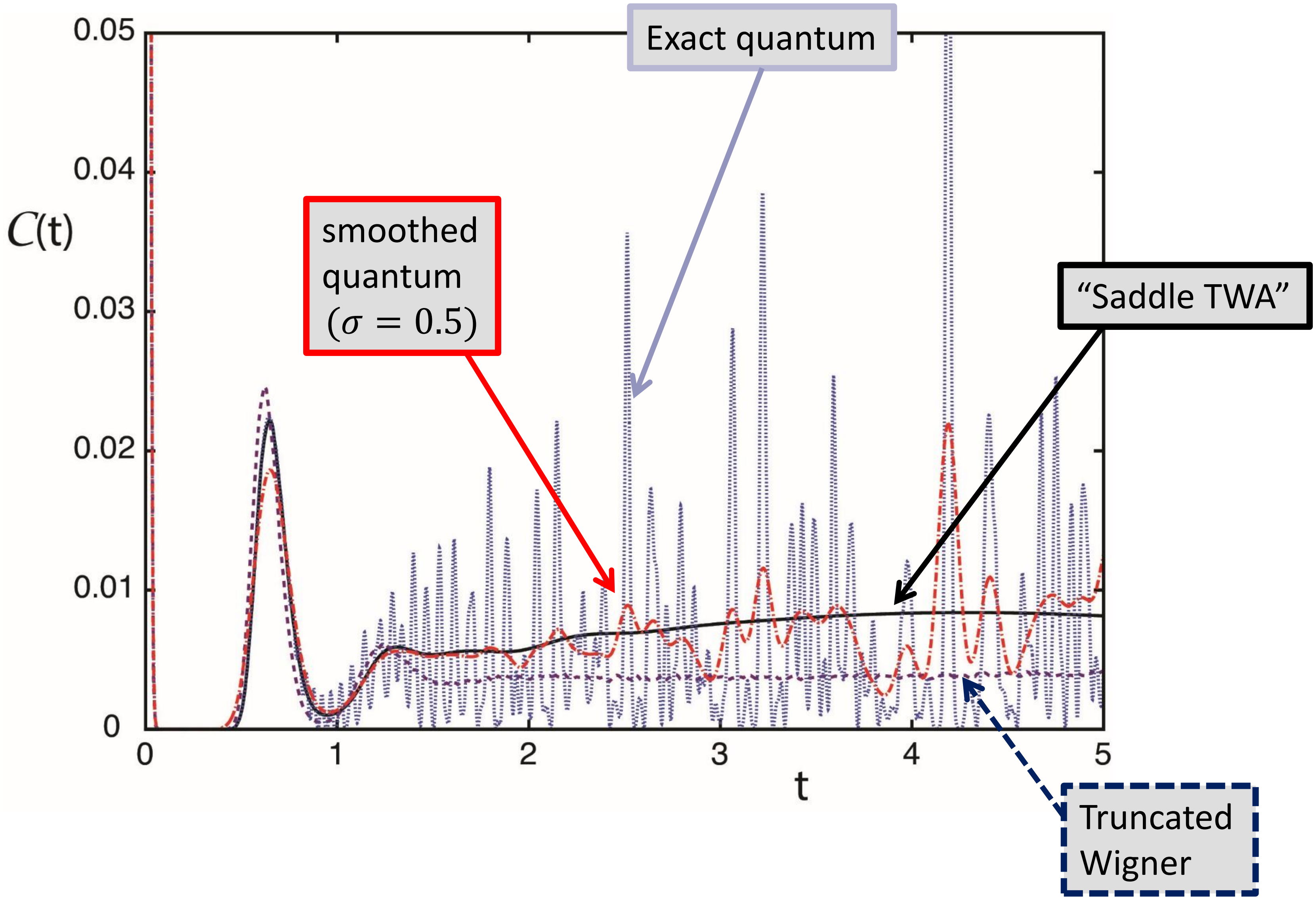
Both H and $|n\rangle$ symmetric under

- time reversal symmetry
- $(1 \to 3, 2 \to 4, 3 \to 1, 4 \to 2)$

$\longrightarrow$ short trajectories $\equiv$ own symmetric $(g_\gamma = 1)$

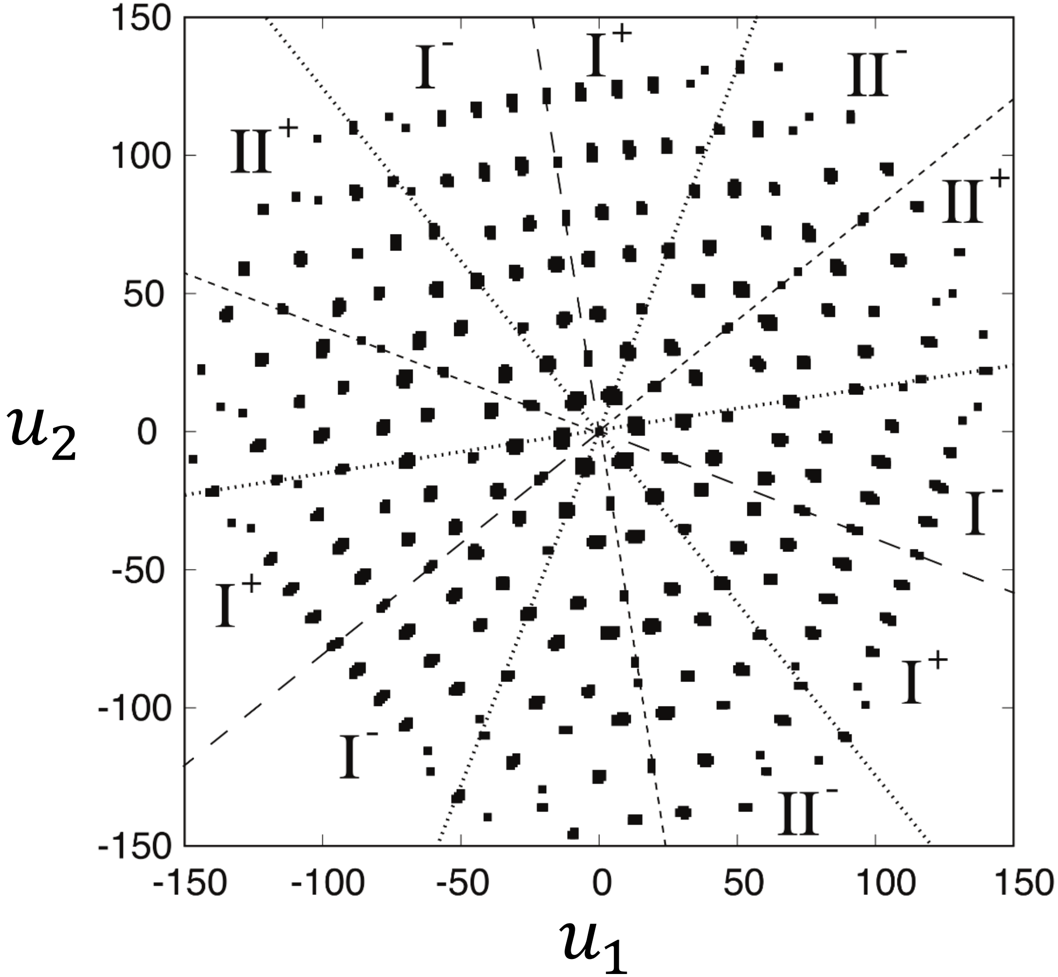
$\longrightarrow$ longer trajectories have a symmetric partner $(g_\gamma = 2)$

$$C_{\mathrm{smooth}}(t) \simeq C_{\mathrm{STWA}}(t) \equiv \sum_\gamma g_\gamma^2 D_\gamma$$



Exact quantum
smoothed quantum ($\sigma = 0.5$)
"Saddle TWA"
Truncated Wigner
C(t)
t
0.05
0.04
0.03
0.02
0.01
0
0
1
2
3
4
5

$\mathbf{N = 6},\ |\ \vec{\mathbf{n}}\rangle = |\mathbf{10, 0, 10, 0, 10, 0}\rangle.$



$J = 0.2$, $U = 1$ (nearly integrable regime)

$(u_1, u_2) \equiv$ directions associated with the 2 largest eigenvalues of $\mathcal{M} = \mathbf{M}^{-1^T}\mathbf{M}^{-1}$

“Seed trajectories” in the search plan

Conclusion- I

- There is no conceptual difference between the $N \to \infty$ limit of bosonic mean field approximation and the $\hbar \to 0$ limit of few body systems.
- This imply that usual semiclassics and many-bosons semiclassics, are formally the same theory, even if the small parameter is not the same ($\hbar$ vs N^{-1})
- The tool from quantum chaos which have been developed in the former context can be used in the latter, and are particularly well adapted to tackle interference effects
  (NB: trying to start from path integral on the other hand is uselessly complicated).
- The complexification of phase space associated with the use of coherent states is a technical difficulty that can be overcome

Conclusion- II

The size of the phase space to explore is however a game changer for any practical implementation

→ need to design techniques making it possible to explore large phase space.

→ even in this way, one will be limited to either

- small systems
- system sufficiently close to integrability that the size of the phase space to explore is 0(1) as the number of modes $\to\infty$
- fairly short times

However :

→ this will always beat truncated Wigner (which can be shown to be equivalent to just neglecting interference terms in the WKB approach).

→ it is presumably the correct framework to think about the mean field approximation for many-boson out of equilibrium systems (eg: effects of symmetries, etc..).