

# **BI-CRITICAL POINTS IN COMPRESSIBLE SOLIDS**

**D.E. Khmelnitskii and Fan Zhang**

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*Cavendish Laboratory  
University of Cambridge  
Cambridge  
United Kingdom*

# Outline

- Type II Phase Transition  
*(Larkin and Pikin, 1969 )*
  - Coupling with deformation
  - Gibbs Free energy
  - The Fold
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  - The Surface of Free Energy
  - Phase Diagrams
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## Type II Phase Transition

Consider a type II Phase Transition at  $T = T_c$ . Generic Hamiltonian which leads to it, has the form

$$H(\{\eta_i\}) = \sum_i \left( \frac{a}{2} \eta_i^2 + \frac{b}{4} \eta_i^4 \right) + \sum_{ij} V_{ij} (\eta_i - \eta_j)^2 \quad (1)$$

The Helmholtz Free energy  $F(T)$  at fixed volume  $V$  has a singularity

$$F(T) = F(T_c) - A|T - T_c|^{2-\alpha}, \quad \alpha > 0. \quad (2)$$

## Elastic Deformation

Elastic deformation  $\mathbf{u}(\mathbf{r})$  has the energy

$$H_{el} = \int d\mathbf{r} \left\{ \lambda \left( \frac{\partial u_\alpha}{\partial r_\alpha} \right)^2 + \mu \left( \frac{\partial u_\alpha}{\partial r_\beta} \right)^2 \right\} \quad (3)$$

Deformation also leads to a shift of critical temperature and parameter  $a$  in the Hamiltonian is replaced by

$$a \rightarrow a + q \frac{\partial \mathbf{u}}{\partial \mathbf{r}}$$

So,

$$H = \sum_i \left( \frac{a + q \operatorname{div} \mathbf{u}}{2} \eta_i^2 + \frac{b}{4} \eta_i^4 \right) + \sum_{ij} V_{ij} (\eta_i - \eta_j)^2. \quad (4)$$

## Gibbs Free Energy

So, if we want to find now the Gibbs Free Energy  $\Phi(T, \sigma_{\alpha\beta})$  at given stress  $\hat{\sigma}$ , the prescription is simple:

$$\Phi(T, \sigma_{\alpha\beta}) = -T \ln \int \prod_i d\eta_i d\mathbf{u}(\mathbf{r}_i) \exp \left[ -\frac{H(\{\eta_i\}) + H_{el} - \sigma_{\alpha\beta} \sum_i \left( \partial u_\alpha / \partial r_i^\beta \right)}{T} \right] \quad (5)$$

For this purpose, represent

$$\frac{\partial u_\alpha(\mathbf{r})}{\partial r_\beta} = u_{\alpha\beta} + \frac{1}{N} \sum_{\mathbf{k} \neq 0} i k_\beta u_\alpha(\mathbf{k}) e^{i\mathbf{kr}}$$

So, instead of integrations over  $\mathbf{u}(\mathbf{r}_i)$  we can integrate over  $\mathbf{u}(\mathbf{k})$  and  $u_{\alpha\beta}$ .

## Partition Function

One can see that integration over transverse phonons  $\mathbf{k}\mathbf{u}(\mathbf{k}) = 0$  and the shear part of deformation tensor

$$u_{\alpha\beta}^{(s)} = u_{\alpha\beta} - \frac{\delta_{\alpha\beta}}{3} u_{\gamma\gamma}$$

are reduced to the Gaussian integration.

Integration over longitudinal phonons  $\mathbf{k}\mathbf{u}(\mathbf{k}) = ku$  results in appearance of the new term

$$\sim q^2 \sum_{\mathbf{k} \neq 0} \sum_{ij} \frac{\eta_i^2 \eta_j^2}{K_0 + 4\mu/3} e^{i\mathbf{k}(\mathbf{r}_i - \mathbf{r}_j)}$$

In order to add the missing term with  $\mathbf{k} = 0$ , make the substitution

$$u_{\alpha\alpha} = \frac{v}{K_0} \sqrt{\frac{3K_0 + 4\mu}{4\mu}} + \frac{1}{K_0} \left( 1 - \sqrt{\frac{4\mu}{3K_0 + 4\mu}} \right) \sum_i \eta_i^2 - \frac{p}{K_0} \quad (6)$$

## Partition Function

This results in

$$\Phi = \frac{p^2}{K_0} - T \ln \int dv \exp \left\{ -\frac{1}{T} \left[ \frac{3K_0 + 4\mu}{8\mu} v^2 + F \left( a + q \frac{v-p}{K_0}, b - \frac{3q^2}{3K_0 + 4\mu} \right) \right] \right\}$$

Macroscopic limit corresponds to the steepest descent in this integral. Therefore,

$$\Phi = \frac{p^2}{K_0} + \frac{3K_0 + 4\mu}{8\mu} v^2 + F \left( a + q \frac{v-p}{K_0}, b - \frac{3q^2}{3K_0 + 4\mu} \right)$$

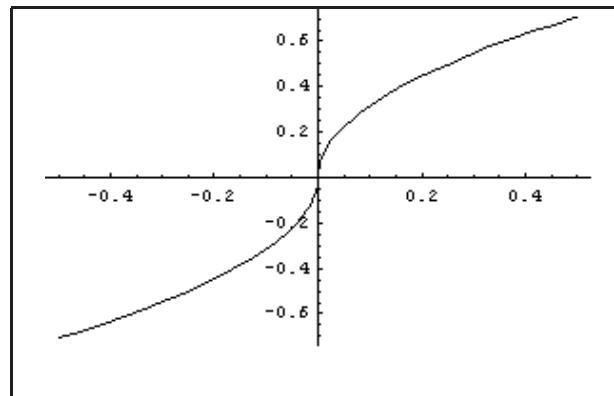
$$\frac{\partial \Phi}{\partial v} = 0$$

## Fold

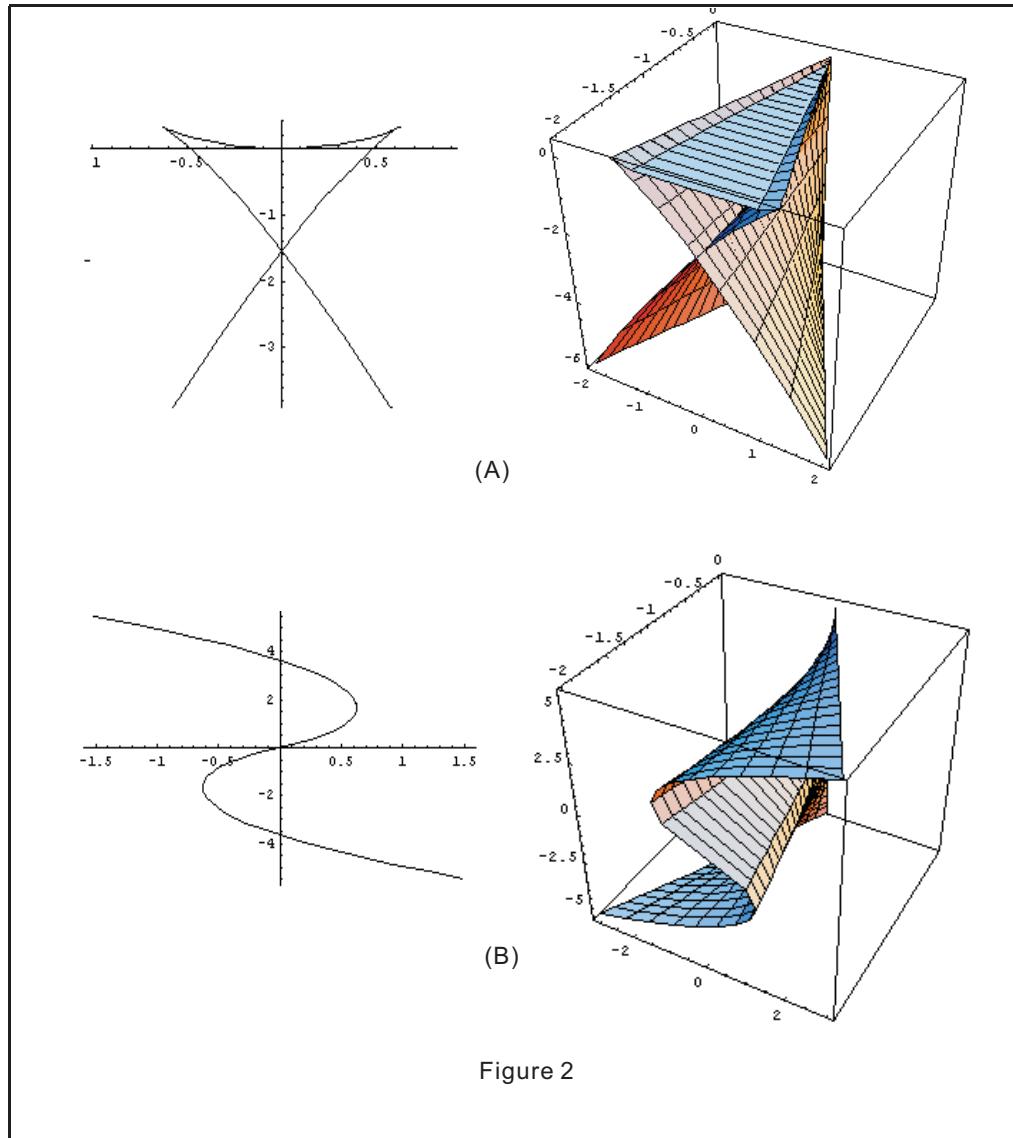
since  $F(a) \propto a^{2-\alpha}$ , condition

$$\frac{\partial \Phi}{\partial v} = 0$$

can be presented graphically as



# LP



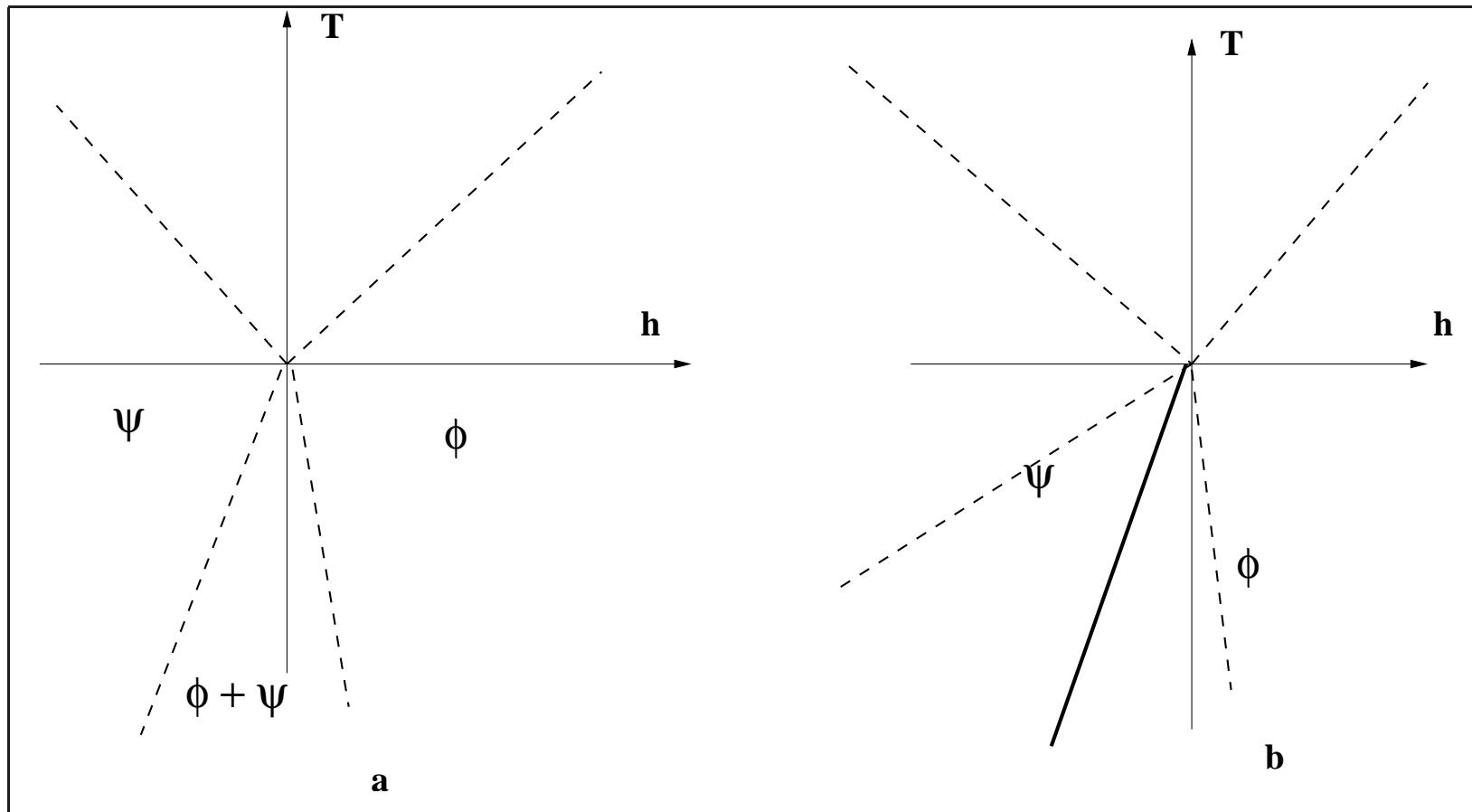
## Alternative Explanation

$$\begin{array}{c} \text{Diagram 1: } \text{A wavy line connecting two vertices equals a dashed line plus a vertical dashed line with a label } \lambda\delta(\mathbf{q}) \\ \text{Diagram 2: } \text{An oval loop equals a loop with a dashed cross plus a loop with a dashed cross plus another loop with a dashed cross} \\ \text{Diagram 3: } \text{A series of loops plus dots equals } \frac{\Pi}{1 - \lambda \Pi} \end{array}$$

The image contains three separate Feynman diagrammatic equations, each enclosed in a rectangular frame.

- Equation 1:** A wavy line connecting two vertices is shown to be equal to a dashed line connecting the same vertices plus a vertical dashed line with a label  $\lambda\delta(\mathbf{q})$ .
- Equation 2:** An oval loop is shown to be equal to a loop with a dashed cross plus a loop with a dashed cross plus another loop with a dashed cross.
- Equation 3:** A series of loops plus dots is shown to be equal to the expression  $\frac{\Pi}{1 - \lambda \Pi}$ .

## Crossing. Mean Field



$$H = \sum_i [a\psi_i^2 + \alpha\phi_i^2 + b\psi_i^4 + \beta\phi_i^4 + \gamma\phi_i^2\psi_i^2 + ..]$$

## Effect of compressibility

Assume the singularity

$$F(a, \alpha) \propto A|a|^{2-\xi} + B|\alpha|^{2-\zeta} + C|\alpha|^{1-\zeta}|a|^{1-\zeta}$$

Effect of elastic deformation

$$a \rightarrow a + q \operatorname{div} \mathbf{u}, \quad \alpha \rightarrow \alpha + J \operatorname{div} \mathbf{u}$$

All this leads to

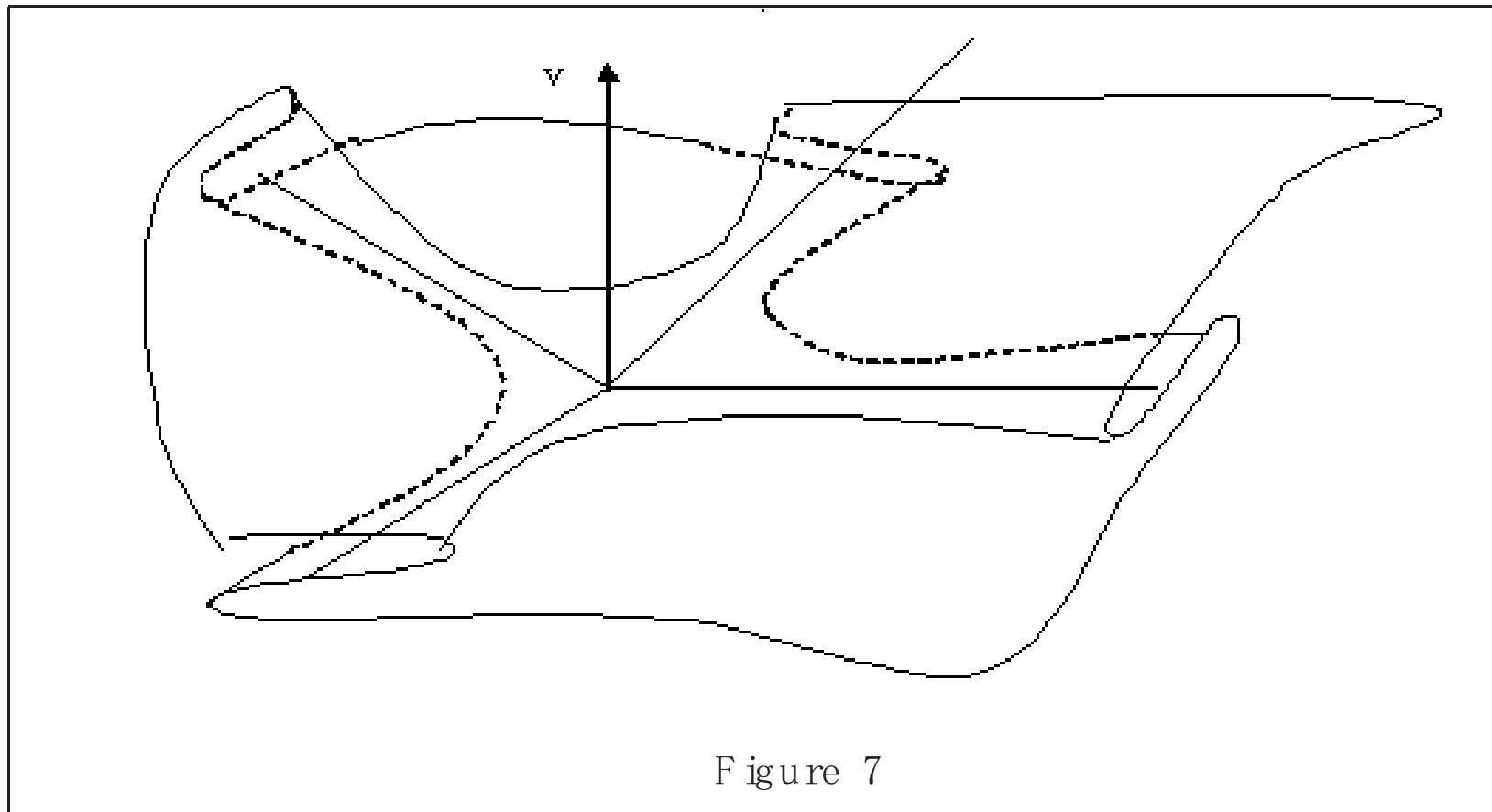
$$\Phi = -\frac{p^2}{K_0} + \frac{3K_0 + 4\mu}{8\mu} v^2 + F(a', \alpha', b', \beta', \gamma')$$

$$\frac{\partial \Phi}{\partial v} = 0$$

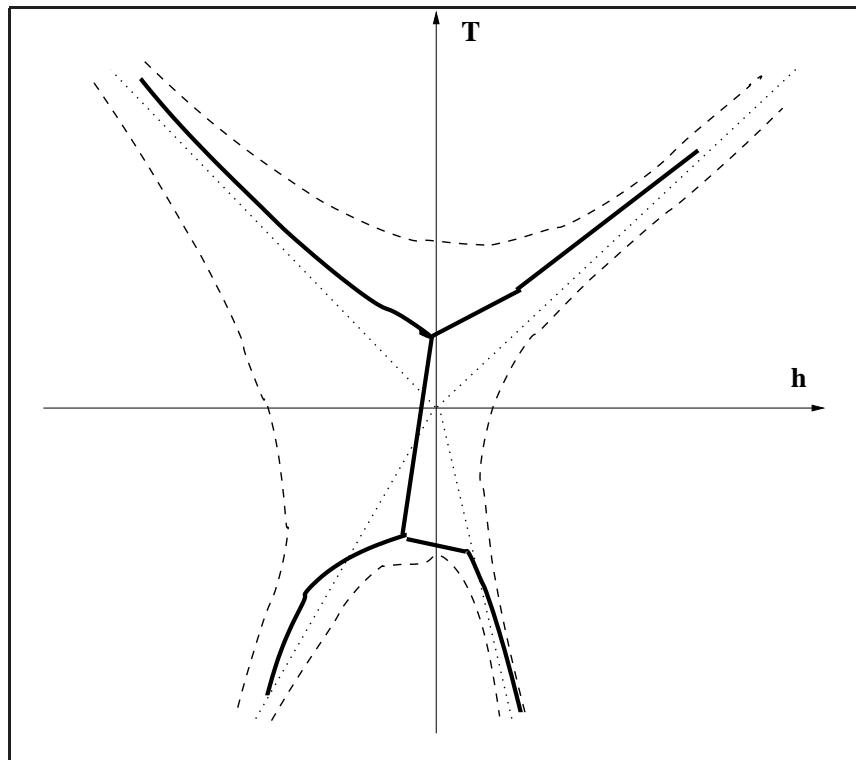
where

$$a' = a + q \frac{v - p}{K_0}, \quad \alpha' = \alpha + J \frac{v - p}{K_0}, \quad b' = b - \frac{3q^2}{3K_0 + 4\mu}, \dots$$

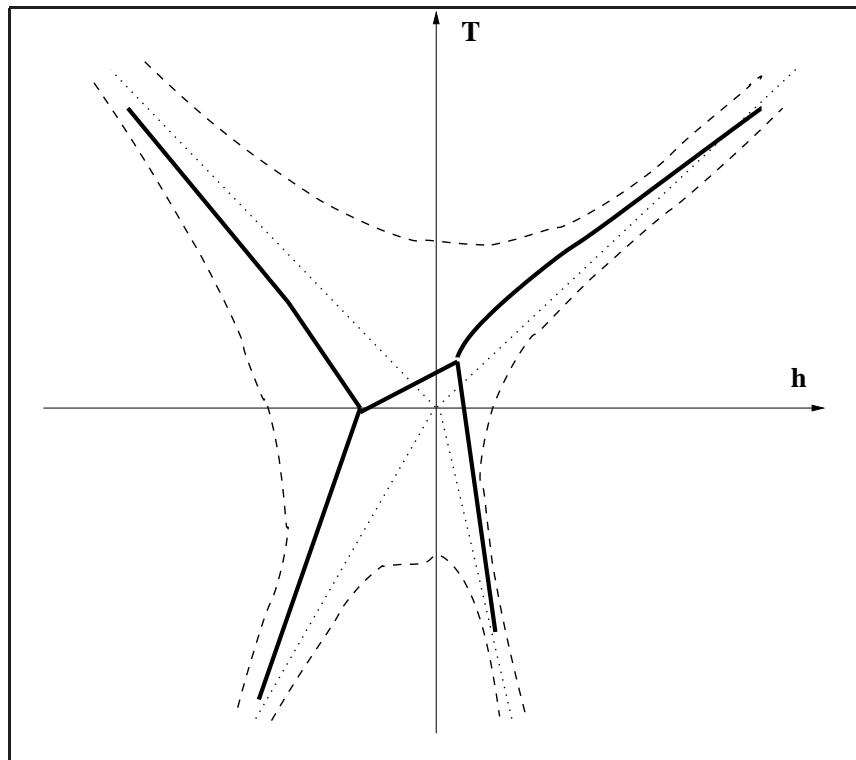
# Projection



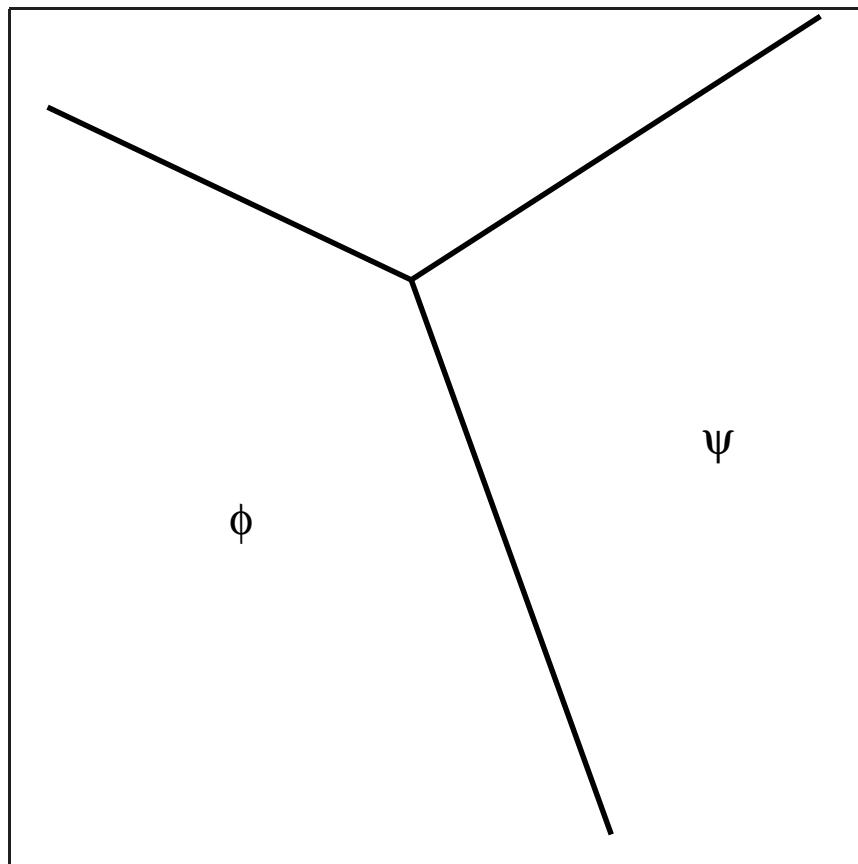
# Phase Diagram



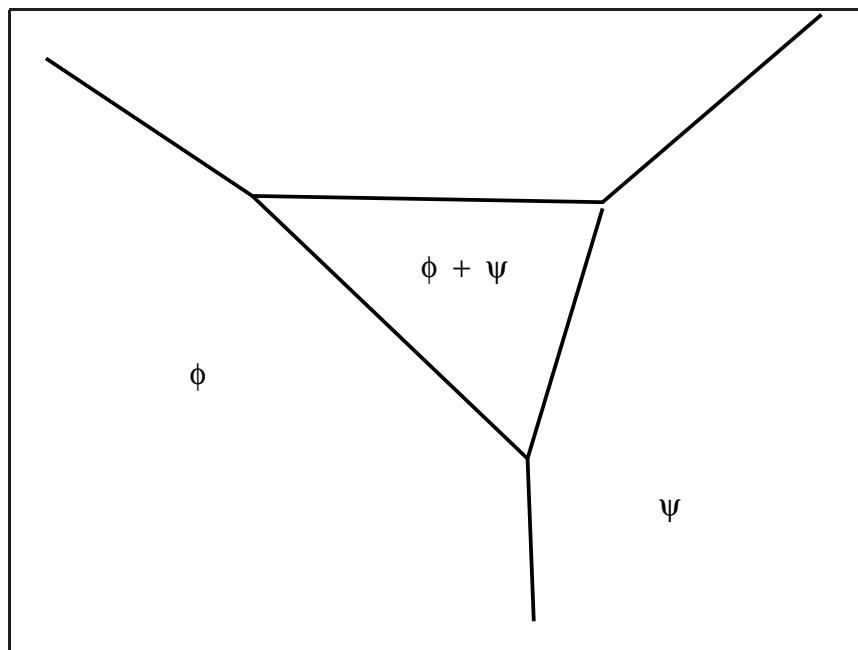
# Phase Diagram



# Phase Diagram



## Phase Diagram



# Summary

