Научный доклад об основых результатах подготовленной научно-квалификационной работы (диссертации) по теме: Специальная Кэлерова геометрия и теории Ландау Гинзбурга

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Overview

- Special Kähler geometry is geometry of coupling constants of low-energy supersymmetric effective theories in superstring compactifications.
- Compactified superstring backgrounds have a form $\mathbb{R}^{1,3} \times \mathcal{X}^6$ and coupling constants of low-energy theory are expressed through the geometry of $\mathcal{X}^6$.
- The main result is a method of computation of the special geometry in superstring compactifications using supersymmetric N=(2,2) Landau-Ginzburg orbifolds.
- We also connect our computations with the localization computations in Gauge Linear Sigma Models (GLSM) via mirror symmetry and the correspondence between non-linear sigma models and GLSM.

Classical worldsheet approach to type II superstring theory is based on 2d CFT with N=(2,2) extended supersymmetry. The flat superstring background is described by a linear sigma model with a target space $\mathbb{R}^{1,9}$.

The spectrum of the theory consists of various excitation modes of different propagating strings. The most interesting states are massless and form a supergravity multiplet in 10 dimensions. N=(1,1) and N=(2,0) theories are called superstrings of type IIA and IIB.

$$
\begin{aligned}
IIA &: (G_{MN}, B_{MN}, \Phi, C^1_M, C^3_{MNP}),\\
IIB &: (G_{MN}, B_{MN}, \Phi, C^0, C^2_{MN}, C^{4+}_{MNPQ}).
\end{aligned}
$$

The low-energy dynamics of the massless particles is descibed by 10-dimensional supergravity.

Superstring compactification is curved background target space $\mathbb{R}^{1,3} \times \mathcal{X}$ which is invariant under 4-dimensional $N = 2$ super-Poincaré algebra.

Metric on the background is G_{MN}, therefore a curved background is considered as a coherent state in the superstring theory. The background can be more complicated and include other coherent states such as branes and fluxes. We consider the simplest backgrounds since they are required in the more realistic cases.

Harmonic tensors on $\mathcal{X}$ produce 4-dimensional particles via Kaluza-Klein mechanism.

$$S[\phi] = \int_{\mathbb{R}^{1,3}\times\mathcal{X}} \mathrm{d}^{10}w\, \partial_M\phi\partial^M\phi = -\int_{\mathbb{R}^{1,3}\times\mathcal{X}} \mathrm{d}^4x\mathrm{d}^6z\, \phi\Delta\phi.$$

10-dimensional fields decompose in eigenfunctions of compact kinetic Laplace operator

$$\phi(w) = \phi(x,y) = \sum_n \phi^n(x) f_n(z), \quad \Delta_{\mathcal{X}} f_n(z) = \lambda_n f_n(z).$$

$$S[\phi] = \sum_n \int_{\mathbb{R}^{1,3}} \mathrm{d}^4x\, \partial_M\phi^n\partial^M\phi^n + \lambda_n^2(\phi^n)^2.$$

Zero modes are massless and appear in the low-energy 4d theory.

N=2 d=4 superalgebra has 8 superharges $Q^A_\alpha, \bar{Q}^B_{\dot\alpha}$ and $su(2)_R$ symmetry rotates them. N=2 super Yang-Mills is a partcular case of N=1 Yang-Mills.
N=1 Yang-Mills have vector multiplets and (anti)chiral multiplets. A N=2 vector multiplet consists of one N=1 vector and one chiral multiplet:

$$(A_\mu, \lambda_\alpha, \tilde\lambda_\alpha, \phi) = (A_\mu, \lambda_\alpha) + (\tilde\lambda_\alpha, \phi).$$

N=1 chiral multplet has the following kinetic term

$$\frac{1}{2} g_{i\bar{j}}(\phi)\, \partial^\mu \phi^i \partial_\mu \overline{\phi^j} + g_{i\bar{j}}(\phi)\, \bar{\tilde\lambda}^j \not{D} \tilde\lambda^i,$$

where $g_{i\bar{j}} = \partial_i \overline{\partial_j} K(\phi, \bar\phi)$.

N=1 vector multiplet kinetic term is

$$\frac{1}{8\pi}\left(Im(\tau_{ij})\, F^i_{\mu\nu} F^{j,\mu\nu} - Re(\tau_{ij})\, F^i_{\mu\nu} (\star F)^{j,\mu\nu} \right) - \frac{1}{2\pi} Im(\tau_{ij})\, \bar\lambda^j \not{D} \lambda^i,$$

where $\tau_{ij}(\phi)$ is a holomorphic function of ϕ.

Fermionic kinetic terms are equal via the N=2 $su(2)_R$ symmetry which implies

$$\frac{\partial}{\partial\phi^i}\tau_{jk} = \frac{\partial^3 K(\phi,\bar{\phi})}{\partial\phi^i\partial\phi^j\partial\overline{\phi^k}} \quad \Longrightarrow \quad \tau_{ij} = \partial_i\partial_j F(\phi).$$

The Kähler potential (= the kinetic term) is given by

$$K(\phi,\bar{\phi}) = i(\phi^i\overline{\partial_i F(\phi)} - \overline{\phi^i}\partial_i F(\phi)) = \Pi_i(\phi)\Sigma^{ij}\overline{\Pi_j(\phi)},$$

where $\Pi = (\phi, \partial F(\phi))$ and Σ^{ij} is a symplectic unit.

This geometry describes coupling constants of N=2 d=4 vector multiplets in terms of a holomorphic prepotential $F(\phi)$.

N=2 supergravity multiplet is

$$(E_\mu^M, \psi_{\mu,\alpha}, \tilde{\psi}_{\mu,\dot{\alpha}}, A_\mu).$$

The graviphoton A_μ is mixed with "photons" from vector multiplets.

Introduce n+1 vector multiplets $(A_\mu^I, \lambda_\alpha^I, \bar{\lambda}_{\dot{\alpha}}^I, \Phi^I)$ and one gauge symmetry $\Phi^I \to e^{f(\Phi)}\Phi^I$ which kills a redundant scalar and fermions. The remaining gauge field is identified with the graviphoton. On the space with coordinates Φ^I there is a global special geometry with the metric

$$K^{tot}(\Phi, \bar{\Phi}) = i(\Phi^i \overline{\partial_i F(\Phi)} - \overline{\Phi^i} \partial_i F(\Phi)),$$

On the physical factor space the induced metric is

$$e^{-K(\Phi,\bar{\Phi})} = i(\Phi^i \overline{\partial_i F(\Phi)} - \overline{\Phi^i} \partial_i F(\Phi)) = \Pi_i(\phi) \Sigma^{ij} \overline{\Pi_j(\phi)},$$

Under gauge transformations this metric does not change since $K \to K + f + \bar{f}$ if $\Phi \to e^f \Phi$. The vector $\Pi(\phi)$ consists of $2h + 2$ elements.

Superstring compactification background should be invariant with respect to d=4 N=2 super-Poincaré algebra. In particular, variations of 2 gravitini should vanish

$$\langle \delta_\epsilon \psi_{\mu,\alpha} \rangle = \langle \nabla_\mu \epsilon_\alpha \rangle = 0$$

which implies that there is a covariantly spinor ϵ on $\mathcal{X}$. This forces the holonomy on $\mathcal{X}$ to be $su(3)$, that is $\mathcal{X}$ is a complex three-dimensional Calabi-Yau manifold.

Harmonic forms on Xc generate Kaluza-Klein massless particles and coupling constants of these particles are proportional to integrals of the corresponding harmonic forms.

$$\text{Harmonic forms } \mathrm{Ker}\Delta_{\mathcal{X}} \quad \Longleftrightarrow \quad \text{Cohomology elements } H^*(\mathcal{X})$$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{2,2} & & 0 & \\ 1 & & h^{2,1} & & h^{1,2} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

2-forms or Kähler moduli

$$\omega_{i\bar{j}}\,\mathrm{d}z^i\overline{\mathrm{d}z^j}.$$

3-forms holomorphic volume form

$$\Omega = \Omega_{123}(z)\,\mathrm{d}z^1\mathrm{d}z^2\mathrm{d}z^3$$

and complex moduli

$$\chi^a_{ij\bar{k}}\,\mathrm{d}z^i\mathrm{d}z^j\overline{\mathrm{d}z^k} = \frac{\partial}{\partial\phi^a}\Omega - \kappa_a\Omega.$$

Calabi-Yau cohomology and Special geometry

The coupling constants of kinetic terms are equal to

$$\frac{1}{\mathrm{Vol}}\int_{\mathcal{X}} \chi^a \wedge \overline{\chi^b} = \partial_i \overline{\partial_j} \log \int_{\mathcal{X}} \Omega \wedge \overline{\Omega}$$

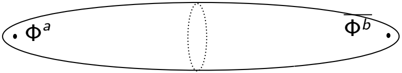
and do not have instanton corrections in type IIB superstring theory.

$$e^{-K} = \int_{\mathcal{X}} \Omega \wedge \overline{\Omega} = \Pi_i(\phi)\Sigma^{ij}\overline{\Pi_j(\phi)} = \omega_i(\phi)C^{ij}\overline{\omega_j(\phi)},$$

where the period integrals or brane amplitudes are

$$\omega_i(\phi) = \int_{q^i} \Omega,$$

q^i form a basis of 3-dimensional cycles in $H_3(\mathcal{X})$ and $(C^{-1})^{ij} = q^i \cap q^j$.



$\langle \Phi^a | e^{-TH} | \Phi^b \rangle$

Consider the following weighted projective space

$$\mathbb{P}^4_{(k_1:\ldots:k_5)} := \mathbb{C}^5 \backslash \{0\} = /\mathbb{C}^* = \{(x_1 : \ldots : x_5) \mid x_i \sim \lambda^{k_i} x_i,\ \bar{x} \neq 0\}.$$

When $k_i = 1$ we have an ordinary projective space. Each variable has integral degree (U(1) charge) k_i.

$W(x)$ is weighted homogeneous $\iff$ $W(\lambda^{k_i} x_i) = \lambda^d W(x_i) \implies$ its zero locus $\mathcal{X} = \{W = 0\} \subset \mathbb{P}^4_k$ is well-defined.

$W(x)$ is non-degenerate if $\mathrm{d}W(x) = 0 \iff x = 0 \iff \mathcal{X}$ is not too singular.

$W(x)$ defines a Calabi-Yau manifold $\iff \sum_{i=1}^5 k_i = d$. We consider Calabi-Yau deformations

$$W(x, \phi) = W_0(x) + \sum_{s=1}^{h} \phi_s e_s(x).$$

such that manifolds with different ϕ have different complex structure

The holomorphic volume form is explicitly

$$\Omega = \frac{x_5 \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3}{\partial W(x,\phi)/\partial x_4} = \oint_{x_5=0} \oint_{W=0} \frac{\mathrm{d}^5 x}{W(x,\phi)}.$$

The periods of such a form are

$$\omega_i(\phi) = \int_{q_i} \Omega = \int_{Q_i} \frac{\mathrm{d}^5 x}{W(x,\phi)}.$$

A good example of such a Calabi-Yau is s quintic threefold in the ordinary projective space $\mathbb{P}^4$:

$$X = \{(x_1 : \cdots : x_5) \in \mathbb{P}^4 \mid W(x,\phi) = 0\},$$

$$W(x,\phi) = W_0(x) + \sum_{t=0}^{100} \phi_t e_t(x), \; W_0(x) = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5$$

and $e_t(x)$ are the degree 5 monomials such that each variable has the power that is a non-negative integer less then four.

N=(2,2) supersymmetric Landau-Ginzburg theory has the superspace Lagrangian

$$L = \int \mathrm{d}^4\theta\, K(X, \bar{X}) + \int \mathrm{d}^2\theta\, W(X, \Phi) + h.c.,$$

where Φ^a are deformation parameters and the chiral superfields X_i are complex coordinates in $\mathbb{C}^5$. Theory is conformal if $W(X, \Phi)$ is weighted homogeneous

$$W(\lambda^{k_i} X) = \lambda^d W(X).$$

Consider the discrete gauge symmetry $Q : X_i \to e^{2\pi i k_i/d} X_i$ and corresponding Landau-Ginzburg orbifold on $\mathbb{C}^5/Q$. Its chiral ring is

$$\mathcal{R}^Q = \frac{\mathbb{C}[x_1, \ldots, x_5]^Q}{(\partial_1 W, \ldots, \partial_5 W)}.$$

which decomposes as

$$\mathcal{R}^Q = \langle 1 \rangle \oplus (\mathcal{R}^Q)^1 \oplus (\mathcal{R}^Q)^2 \oplus \langle \mathrm{Hess}\, W \rangle.$$

We choose a basis $e_a(x)$ of the chiral ring.

The disk one-point functions (brane amplitudes) in Landau-Ginzburg theory are given by oscillatory integrals:

$$\int_{Q^i_+} e_a(x) e^{-W(x,\phi)} \mathrm{d}^5 x,$$

where the cycles Q_i^+ are the steepest descent contours or Lefschetz thimbles $Q_i^+ \in H_5(\mathbb{C}^5, \mathrm{Re}(W) \gg 0)$.

The intersection pairing is $(C^{-1})^{ij} = Q^i_+ \cap Q^j_-$ and the Kähler potential of the tt^* metric is

$$e^{-K} = C^{ij} \int_{Q^i_+} e^{-W(x,\phi)} \mathrm{d}^5 x \overline{\int_{Q^i_-} e^{W(x,\phi)} \mathrm{d}^5 x}.$$

This Special geometry coincides with the one on a Calabi-Yau hypersurface $\mathcal{X} = \{W = 0\} \subset \mathbb{P}^4_k$ as follows from the formula

$$\int_{Q_i} \frac{\mathrm{d}^5 x}{W(x,\phi)} = \int_{Q_i^+} e^{-W(x,\phi)} \mathrm{d}^5 x$$

and intersection matrices C^{ij} coincide.

$$\int_\gamma \Omega = \int_{T(\gamma)} \frac{\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4\mathrm{d}x_5}{W_0(x)},$$

Consider a nearby Milnor fiber $\{W(x,\phi) = w\} \subset \mathbb{C}^5$.

$$\int_{T(\gamma_w)} \frac{\mathrm{d}^5 x}{W(x,\phi_1) - w} = \int_\gamma \frac{\mathrm{d}^5 x}{W(x,\phi_1)} \sum_{n=0}^{\infty} \left(\frac{w}{W(x,\phi_1)} \right) = \int_\gamma \frac{\mathrm{d}^5 x}{W(x,\phi_1)}.$$

due to weighted homogeneity. Using this and inserting the 1 we have

$$\int_{T(\gamma)} \frac{\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4\mathrm{d}x_5}{W(x,\phi_1)} = \int_{T(\gamma_w)} \frac{\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4\mathrm{d}x_5}{W(x,\phi_1) - w} =$$

$$= z \int_{w>0} e^{-w/z} \left(\int_{T(\gamma_w)} \frac{\mathrm{d}^5 x}{W(x,\phi_1) - w} \right) \mathrm{d}w, \quad (1)$$

Now we take a residue at $W = w$ in the inner integral in (1)

$$z \int_{w>0} e^{-w/z} \left(\int_{T(\gamma_w)} \frac{\mathrm{d}^5 x}{W(x,\phi_1) - w} \right) \mathrm{d}w = z \int_{\Gamma_z := \cup_w \gamma_w} e^{-w/z} \frac{\mathrm{d}^4 x \mathrm{d}w}{\partial W(x,\phi_1)/\partial x_5}.$$

For any $x \in \Gamma_z$ we have $W(x,\phi) = w$. The last step is a coordinate change

$$z \int_{\Gamma_z} e^{-w/z} \frac{\mathrm{d}^4 x \mathrm{d}w}{\partial W(x,\phi_1)/\partial x_5} = z \int_{\Gamma_z} e^{-W(x,\phi_1)/z} \mathrm{d}^5 x.$$

Oscillatory integral cohomology

We focus on computation of special geometry for a LG orbifold. Stokes formula for oscillatory integrals implies

$$\int e^{-W} D_{-}\alpha = \int e^{-W}(\mathrm{d}\alpha - \mathrm{d}W \wedge \alpha) = 0,$$

so oscillatory integrands $e_a(x)\,\mathrm{d}^5x$ form a cohomology group $H^5_{D_-}(\mathbb{C}^5)^Q$ which is dual to steepest descent contours $H_5(\mathbb{C}^5, \mathrm{Re}(W) \gg 0)^Q$.

Define a basis of cycles by duality formula

$$\langle \Gamma^a_+, e_b(x)\,\mathrm{d}^5x\rangle = \int_{\Gamma^a_+} e^{-W_0} e_b(x)\,\mathrm{d}^5x = \delta^a_b.$$

The cycles Γ^a_+ are not actual cycles but complex linear combinations of cycles.

Using the duality it is very easy to find an intersection matrix of cycles $\Gamma^i_+ \cap \Gamma^j_- = (\eta^{-1})^{ij}$, where η^{ij} is a topological residue pairing of Landau-Ginzburg theory in the appropriate gauge

$$\eta^{ij} = Res\,\frac{e_i(x)\,e_j(x)\,\mathrm{d}^5x}{\partial_1 W_0 \cdots \partial_5 W_0}.$$

Main working formula

We use the formula for the Kähler potential for a Landau-Ginzburg orbifolds in the basis of cycles Γ^i_+:

$$e^{-K} = \eta^{ij} \int_{\Gamma^i_+} e^{-W(x,\phi)} \mathrm{d}^5 x \int_{\overline{\Gamma^i_-}} \overline{e^{W(x,\phi)} \mathrm{d}^5 x},$$

where the last conjugation is due to the fact that $\Gamma^i_\pm$ are linear combinations of cycles with complex coefficients.

We denote

$$\sigma_i(\phi) := \int_{\Gamma^i_+} e^{-W(x,\phi)} \mathrm{d}^5 x, \quad \overline{\Gamma^j_-} = \mathbf{M}^k_j \Gamma^k_-$$

for a matrix $\mathbf{M}^k_j$ which is called a real structure matrix. $\mathbf{M}\bar{\mathbf{M}} = 1$.

Our main formula becomes

$$e^{-K} = \sigma_i(\phi)\, \eta^{ik} \mathbf{M}^j_k\, \overline{\sigma_j(\phi)}.$$

We consider deformations of symmetric superpotentials/Calabi-Yau manifolds

$$W_0(x) + \sum_{s=1}^{h} \phi_s e_s(x),$$

where $W_0(x)$ has additional discrete symmetry group Π_{W_0} of the form $x_i \to \alpha_i x_i$ such that $W_0(\alpha \cdot x) = W_0(x)$. An example is $W_0(x) = \sum_{i=1}^{5} x_i^5$ and $\Pi_{W_0} = \mathbb{Z}_5^5$. We consider the case where the chiral ring decomposes into different one-dimensional representations.

Such a symmetry gives strong constraints on the formulas.

- We can pick a monomial basis of $\mathcal{R}^Q$ such that $\eta^{ij} = \mathrm{antidiag}\{1, 1, \ldots, 1\}$.
- Real structure matrix $\mathbf{M}_k^j = \mathrm{antidiag}\{A_1, A_2, \ldots, A_{2h+2}\}$.
- The Kähler potential is

$$e^{-K} = \sum_{a=1}^{2h+2} A_s |\sigma_a(\phi)|^2.$$

Real structure A_s is computed through integration over simple actual cycles which decompose into products of one-dimensional integrals.

Quintic CY manifold X be given as a solution of the equation

$$W(x,\phi)=\sum_{i=1}^{5}x_i^5+\sum_{\mathbf{s}=1}^{101}\phi_{\mathbf{s}}\prod_i x_i^{s_i}=0$$

$\mathbf{s}$=(s_1,s_2,s_3,s_4,s_5), $0\leq s_i\leq 3$, $\deg(\mathbf{s}):=\sum_{i=1}^{5}s_i=5$.
The complex structures Kähler potential in this case is

$$e^{-K(\phi)}=\sum_{\mu=0}^{203}(-1)^{\deg(\mu)/5}\prod\gamma\left(\frac{\mu_i+1}{5}\right)|\sigma_\mu(\phi)|^2,$$

$$\sigma_\mu(\phi)=\sum_{n_1,\dots,n_5\geq 0}\prod_{i=1}^{5}\frac{\Gamma(\frac{\mu_i+1}{5}+n_i)}{\Gamma(\frac{\mu_i+1}{5})}\sum_{m\in\Sigma_n}\prod_s\frac{\phi_s^{m_{\mathbf{s}}}}{m_{\mathbf{s}}!},$$

μ=$(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5)$, $0\leq\mu_i\leq 3$, $\quad\sum_{i=1}^{5}\mu_i=0,5,10,15.$

$$\gamma(x)=\frac{\Gamma(x)}{\Gamma(1-x)},\qquad \Sigma_n=\{m_{\mathbf{s}}\mid\sum_s m_{\mathbf{s}}s_i=5n_i+\mu_i\}$$

The Fermat hypersurfaces (around 100 threefolds) are given by

$$W(x,\phi)=\sum_{i=1}^{5}x_i^{d/k_i}+\sum_{\mathbf{s}=1}^{h}\phi_{\mathbf{s}}\prod_i x_i^{s_i}=0$$

$\mathbf{s}=(s_1,s_2,s_3,s_4,s_5)$, $0\le s_i\le d/k_i-1$, $\deg(\mathbf{s}):=\sum_{i=1}^{5}k_is_i=d$.
The complex structures Kähler potential in this case is

$$e^{-K(\phi)}=\sum_{\mu=0}^{2h+1}(-1)^{\deg(\mu)/d}\prod\gamma\left(\frac{k_i(\mu_i+1)}{d}\right)|\sigma_\mu(\phi)|^2,$$

$$\sigma_\mu(\phi)=\sum_{n_1,\dots,n_5\ge 0}\prod_{i=1}^{5}\frac{\Gamma(\frac{k_i(\mu_i+1)}{d}+n_i)}{\Gamma(\frac{k_i(\mu_i+1)}{d})}\sum_{m\in\Sigma_n}\prod_s\frac{\phi_s^{m_s}}{m_s!},$$

$\mu=(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5)$, $0\le\mu_i\le d/k_i-1$, $\quad\sum_{i=1}^{5}\mu_i=0,d,2d,3d$.

$$\Sigma_n=\{m_{\mathbf{s}}\mid\sum_s m_{\mathbf{s}}k_is_i=dn_i+k_i\mu_i\}$$

Consider a so-called invertible singularty

$$W_0(x) = \sum_{i=1}^{n} \prod_{j=1}^{n} x_j^{M_{ij}},$$

where M_{ij} is an invertible matrix.

We compute the period integrals

$$\int_{\Gamma_+} e^{-W_0(x)+\sum_s \phi_s e_s} \mathrm{d}^5 x = \sum_{m_1,\dots,m_h \geq 0} \frac{\phi_1^{m_1} \cdots \phi_h^{m_h}}{m_1! \cdots m_h!} \int_{\Gamma_-^a} e^{-W_0(x)} \prod_{i \leq 5} x_i^{\sum_{s=1}^h m_s s_i} \mathrm{d}^5 x.$$

All the monomials of W_0 belong to the Jacobi ideal themselves

$$\prod_j x_j^{M_{aj}} = \sum_k M_{ka}^{-1} x_k \partial_k W_0(x).$$

Which allows to shift exponent vectors of the integrands $x_i^{\sum_{s=1}^h m_s s_i} \mathrm{d}^5 x = \prod_{i \leq 5} x_i^{M_{ki}+a_i} \mathrm{d}^5 x$:

$$\prod_{i\le 5} x_i^{M_{ki}+a_i}\,\mathrm{d}^5x - D_-\left(\sum_b M^{-1}_{bk}\prod_{i\le 5} x_i^{a_i+\delta_{ib}}\,\mathrm{d}^5x/\mathrm{d}x_b\right) =$$
$$= \sum_b (a_b+1)M^{-1}_{bk}\prod_{i\le 5} x_i^{a_i}\,\mathrm{d}^5x.$$

Which implies a formula for the periods

$$\sigma_a(\phi) = \sum_{v_1,\dots,v_5\ge 0}\prod_{i\le 5}\left((a_j+1)M^{-1}_{ji}\right)_{v_i}\sum_{\sum_{s=1}^h m_s s_i = M_{ij}v_j + a_i}\frac{\phi_1^{m_1}\cdots\phi_h^{m_h}}{m_1!\cdots m_h!},$$
$$a = (a_1,\dots,a_5)\in\mathcal{R}_0,$$
$$\sum_{i\le 5} M_{ij}a_j = 0,\ d,\ 2d,\ 3d.$$

where the Pochhammer symbol is

$$(a)_m := \frac{\Gamma(a+m)}{\Gamma(a)}.$$

Real structure computation

If a cycle L_+ is an actual cycle, then

$$\operatorname{Im}\left[\int_{L_+} e^{-W}\left(e_a(x)\,\mathrm{d}^5x + \mathbf{M}_a^b e_b(x)\,\mathrm{d}^5x\right)\right] = 0.$$

We find real cycles using a following singular coordinate change

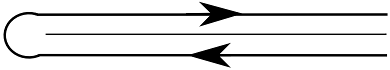
$$y_i := x^{M_i} = \prod_j x_j^{M_{ij}}.$$

The period integral becomes

$$\int_{L_+} x^k e^{-\sum_i x^{M_i}}\,\mathrm{d}^n x = \det M^{-1}\int_{L_+} y^{(k+1)M^{-1}-1} e^{-\sum_i y_i}\,\mathrm{d}^n y,$$

We can pick a contour to be a product of 5 Pochhammer contours in coordinates y:

$$z(s) = \rho(s)e^{i\theta(s)}$$



The integral above decomposes into a product of gamma functions with complex coefficients which allows to find the real structure $\mathbf{M}_a^b$.

2d N=(2,2) supersymmetric GLSM have superspace Lagrangians of the type

$$L = \int \mathrm{d}^4\theta \left(\sum_{i=1}^{N} \overline{\Phi}_i e^{Q_{ia} V_a} \Phi_i - \sum_a \frac{1}{2e_a^2} \overline{\Sigma}_a \Sigma_a \right) + \\ + \frac{1}{2} \left(- \int \mathrm{d}^2\tilde{\theta} \sum_{a=1}^{k} t_a \Sigma_a + \int \mathrm{d}^2\theta W(\Phi) + \text{h.c.} \right),$$

where Φ_i are 2d chiral multiplets which are charged with respect to the 2d vector multiplets V_a of U(1) with the charge matrix Q_{ia} and $W(\Phi)$ is gauge invariant.

The parameters $t_a = r_a + i\theta_a$ are complexified Fayet-Iliopoulos terms. The theory has the scalars potential

$$U = \sum_{a=1}^{k} \left(Q_{ia} |\phi_i|^2 - r_a \right)^2 + \sum_{i=1}^{k} \left| \frac{\partial W}{\partial \phi_i} \right|^2 .$$

Depending on r_a the vacuum manifold can be either a nontrivial manifold or a point $\phi = 0$. In the first case the theory flows to a nonlinear sigma model in the infrared. In the second case it flows to a Landau-Ginzburg model.

In the nonlinear sigma model case the vacuum manifold is a Hamliltonian reduction

$$Y_r = \left\{ (\phi_1, \ldots, \phi_N) \in \mathbb{C}^N \;\middle|\; \sum_{a=1}^{N} Q_{al} |\phi_a|^2 = r_l,\ l = 1, \ldots, k,\ \frac{\partial W}{\partial \phi_a} = 0 \right\} / U(1)^k.$$

This manifold is isomorphic to a hypersurface $\mathrm{d}W = 0$ in a toric variety

$$\mathbb{C}^N // (\mathbb{C}^*)^k,$$

where the action of $(\mathbb{C}^*)^k$ is defined by the $k \times N$ charge matrix Q_{al}.

The classical way to describe a toric variety is a fan $\{v_{lj}\}_{l \le N, j \le 5}$. Integral vectors v_l satisfy $\sum_{l=1}^{N} Q_{al} v_l = 0$.

Vectors v_i of a fan and spans of several of them (cones) are in one-to-one with $(\mathbb{C}^*)^k$ invariant cycles in the toric variety Y_r.

In the recent years the partition function of GLSM was computed in a supersymmetric background on S^2 using the supersymmetric localization:

$$Z_{S^2} = \sum_m \int \left(\prod_{j \le k} \frac{\mathrm{d}\sigma_j}{2\pi} \right) Z_{class}(\sigma, m) \prod_{i \le N} Z_{\Phi_i}(\sigma, m),$$

where the classical action is

$$Z_{class} = e^{-4\pi i r_l \sigma_l - i\theta_l m_l}$$

and the one-loop determinant of a chiral field Φ_i is

$$Z_{\Phi_i} = \frac{\Gamma(q_i/2 - i\sum_l (Q_{il}\sigma_l - m_l/2))}{\Gamma(1 - q_i/2 - i\sum_l (Q_{il}\sigma_l + m_l/2))}.$$

Shortly after localization computation there was proposed a conjecture that Z_{S^2} computes e^{-K} on the Kähler moduli space of the vacuum manifold Y_r.

The mirror symmetry relates special geometry on the moduli spaces of Kähler and complex structure defomations of two different families of Calabi-Yau manifolds Y_r and $\mathcal{X}_\phi$ through a mirror map $r = r(\phi)$.

We proved the mirror version of the Jockers et al conjecture by direct computations in the cases where we are able to compute special geometry using our method.

The mirror version should state that

$$\int_{\mathcal{X}} \Omega \wedge \overline{\Omega} = Z_{S^2}(Y_r).$$

Under a suitable mirror map.

We use a version of Batyrev mirror symmetry for hypersurfaces in toric varieties. Consider a family of Calabi-Yau varieties defined by the equation (for example the quintic)

$$W(x,\phi) = \sum_{i=1}^{5} x_i^5 + \sum_{l=1}^{101} \phi_l e_l(x) = \sum_{i=1}^{106} C_a(\phi) \prod_{j=1}^{5} x_j^{v_{ij}},$$

where we introduced the exponent matrix v_{ij}. Vectors v_i define integral points of a polytope in $\mathbb{R}^5$.

The Batyrev mirror symmetry implies that to get a mirror manifold we need to consider a fan with vectors v_i, construct a toric variety with this fan and a hypersurface Y_r inside this toric variety is a mirror quintic.

Mirror quintic as a GLSM

For the quintic the vectors of the fan are

$$v_{ij} = \begin{cases} 5\delta_{i,j}, & 1 \le i \le 5, \\ s_{i-5,j}, & 6 \le i \le 106. \end{cases}$$

We build a GLSM whose vacuum manifold is a mirror quintic. We easily reconstruct the charge matrix Q_{al}

$$Q_{ai} = \begin{cases} s_{ai}, & 1 \le i \le 5, \\ -5\delta_{i-5,a}, & 6 \le i \le 106. \end{cases}$$

such that

$$\sum_{i \le 106} Q_{ai} v_i = 0.$$

Elements Q_{ai} form a basis in linear relations among v_i and force $\sum_a m_a Q_{ai} \in \mathbb{Z}$ due to the charge quantization condition.

To write the superpotential of the GLSM it is convenient to separate the chiral fields as

$$\Phi_i = \begin{cases} S_i, & 1 \leq i \leq 5, \\ P_{i-5}, & 6 \leq i \leq 106. \end{cases}$$

The superpotential is

$$W_Y := P_1 \, G(S_1, \ldots, S_5; P_2, \ldots, P_{101}).$$

And the scalar potential whose zeroes define a mirror quintic is

$$U(\phi) = \sum_{l=1}^{101} \frac{e_l^2}{2} \left(\sum_{i=1}^{5} s_{li} |S_a|^2 - 5|P_l|^2 - r_l \right)^2 + \frac{1}{4} |G(S_1, \ldots, S_5; P_2, \ldots, P_{101})|^2 +$$
$$+ \frac{1}{4} |P_1|^2 \sum_{i=1}^{5} \left| \frac{\partial G}{\partial S_i} \right|^2 + \frac{1}{4} |P_1|^2 \sum_{l=2}^{101} \left| \frac{\partial G}{\partial P_l} \right|^2 .$$

The partition function of the GLSM above is given by a 101-fold contour integral

$$Z_{S^2} = \sum_{m_l \in V} \int_{\mathcal{C}_1} \cdots \int_{\mathcal{C}_{101}} \prod_{l=1}^{101} \frac{d\tau_l}{(2\pi i)} \left(z_l^{-\tau_l + \frac{m_l}{2}} \bar{z}_l^{-\tau_l - \frac{m_l}{2}} \right) \times$$

$$\times \frac{\Gamma\left(1 - 5(\tau_1 - \frac{m_1}{2})\right)}{\Gamma\left(5(\tau_1 + \frac{m_1}{2})\right)} \prod_{a=1}^{5} \frac{\Gamma\left(\sum_l s_{la}(\tau_l - \frac{m_l}{2})\right)}{\Gamma\left(1 - \sum_l s_{la}(\tau_l + \frac{m_l}{2})\right)} \prod_{l=2}^{101} \frac{\Gamma\left(-5(\tau_l - \frac{m_l}{2})\right)}{\Gamma\left(1 + 5(\tau_l + \frac{m_l}{2})\right)},$$

where

$$z_l := e^{-(2\pi r_l + i\theta_l)},$$

and summation is over m_l such that $\sum_a m_a Q_{ai} \in \mathbb{Z}$ for all i.
To connect with our previous computations we compute the integral at $r_a << 0, \ |z_a| >> 0$. The contours can be deformed to the right picking up the residues at

$$5\left(\tau_l - \frac{m_l}{2}\right) - 1 = p_1, \ 5\left(\tau_l - \frac{m_l}{2}\right) = p_l;$$

$$p_1 = 1, 2, \ldots, \ p_l = 0, 1, \ldots \quad \text{so that} \quad p_l + 5m_l > 0.$$

After computing the residues the partition function reduces to

$$Z_{S^2} = \pi^{-5} \sum_{p_1>0, p_l \geq 0} \sum_{\bar{p}_l \in \Sigma_p} \prod_l \frac{(-1)^{p_l}}{p_l! \bar{p}_l!} z_l^{-\frac{p_l}{5}} \bar{z}_l^{-\frac{\bar{p}_l}{5}}$$
$$\prod_{i=1}^{5} \Gamma\left(\frac{1}{5}\sum_{l=1}^{h} s_{li} p_l\right) \Gamma\left(\frac{1}{5}\sum_{l=1}^{h} s_{li} \bar{p}_l\right) \sin\left(\frac{\pi}{5}\sum_{l=1}^{h} s_{li} \bar{p}_l\right),$$

where the set Σ_p - is a set of all $\{\bar{p}_l\}$ such that
$\sum_a (\bar{p}_a - p_a) Q_{ai}/5 = \sum_a m_a Q_{ai} \in \mathbb{Z}$.
After a rearrangement this formula becomes

$$Z_{S^2} = \sum_{\boldsymbol{a}} (-1)^{|\boldsymbol{a}|} \prod_{i=1}^{5} \frac{\Gamma\left(\frac{a_i}{5}\right)}{\Gamma\left(1 - \frac{a_i}{5}\right)} |\sigma_{\boldsymbol{a}}(\boldsymbol{z})|^2,$$

where

$$\sigma_{\boldsymbol{a}}(\boldsymbol{z}) = \sum_{n_i \geq 0} \prod_{i=1}^{5} \frac{\Gamma\left(\frac{a_i}{5} + n_i\right)}{\Gamma\left(\frac{a_i}{5}\right)} \sum_{\boldsymbol{p} \in S_{\boldsymbol{a},\boldsymbol{n}}} \prod_{l=1}^{101} \frac{(-1)^{p_l} z_l^{-\frac{p_l}{5}}}{p_l!}.$$

Conclusion

The formula for partition function on S^2 coincides with the special geometry on the moduli space of the quintic itself after a simple mirror map

$$z_a = -\phi_I^{-5}.$$

We constructed an explicit correspondence between a family of Calabi-Yau manifolds $\mathcal{X}_\phi$ and the Gauge Linear Sigma Model whose vacuum manifold Y_r is a mirror of $\mathcal{X}_\phi$ and checked that special geometries coincide after a very simple mirror map.

The partition function gives an analytic continutation of the special geometry and may be used to compute various correlation functions in superstring theory.

Thank you for your attention!