Hamiltonian Theory of Fractionally Filled Chern Bands

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Acknowledgements

G. M. and R. Shankar arXiv:1108.5501, 1207.2133

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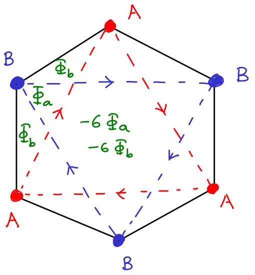
Outline

- QHE without external flux: Chern Bands
- Evidence for FQH-like States in Flat CBs
- Previous work
- The Composite Fermion mapping
- Hall conductivity and Hall Crystals
- Conclusions and open questions

QHE without Flux

Volovik, Phys. Lett. A, 128, 277 (1988): Showed that because ^3He in its A phase breaks time-reversal symmetry and is a $p + ip$ superconductor (like $\nu = \frac{5}{2}$), there should be an analogue of the QHE in a thin slab geometry.

Haldane, PRL 61, 2015 (1988): Constructed a lattice model with time-reversal breaking due to a periodic flux, but no net flux. When a band is full it exhibits the QHE with a chiral edge mode. The QHE arises because of a nontrivial Berry curvature in the Brillouin Zone, making the band a "Chern Band".



$$\varphi = 2\pi\,\frac{\Phi_a + 2\Phi_b}{\Phi_0}$$

$$H = -t\sum_{\langle ij\rangle}\left(c_i^{\dagger} c_j + h.c\right) - t_2 \sum_{\triangle}\left(c^{\dagger}_{Ai} c_{Aj} e^{i\varphi} + h.c\right)$$

$$-t_2 \sum_{\triangledown}\left(c^{\dagger}_{Bi} c_{Bj} e^{i\varphi} + h.c\right) - M \sum\left(c^{\dagger}_{Ai} c_{Ai} - c^{\dagger}_{Bi} c_{Bi}\right)$$

The Chern Number

First define the wave functions $\Psi_{\vec{k}}(\vec{x})$ labelled by crystal momentum $\vec{k}$. Now the Bloch functions are $u_{\vec{k}}(\vec{x}) = e^{-i\vec{k}\cdot\vec{x}}\Psi_{\vec{k}}(\vec{x})$. The the Berry connection, or Berry gauge field is defined by $\vec{A}(\vec{k}) = i\langle u_{\vec{k}}|\nabla_{\vec{k}}|u_{\vec{k}}\rangle$ and the Berry flux or Chern flux density is $b(\vec{k}) = \nabla_{\vec{k}} \times \vec{A}(\vec{k})$. The Chern number is $C = \frac{1}{2\pi}\int d^2k\ b(\vec{k})$. The dimensionless Hall conductance of the filled band is C. Thouless, Kohmoto, Nightingale, and den Nijs, PRL 49, 405 (1982).

Like a filled LL, so what about a fractionally filled Chern band?

Previous work

Band engineering to make the Chern band flat: E. Tang, J.-W. Mei, and X.-G. Wen, PRL 106, 236802 (2011); K. Sun, Z. Gu H. Katsura, and S. Das Sarma, PRL 106, 236803 (2011): Take a multi-band model and play with parameters until the band of interest becomes nearly flat.

T. Neupert, L. Santos, C. Chamon, and C. Mudry, PRL 106, 236804 (2011): Add long-range hoppings to make it flat. These authors also carried out the first numerics to show that an incompressible FQH-like state exists here for suitable repulsive interactions.

X.-L. Qi, PRL 107, 126803 (2011): Mapped single-particle states from the Chern band to Landau gauge basis for LL. See also Y.-L. Wu, N. Regnault, and B. A. Bernevig, arXiv:1206.5773. J. Maciejko, X.-L. Qi, A. Karch, and S.-C. Zhang, PRL 105, 246809 (2010): B. Swingle, M. Barkeshli, J. McGreevy and T. Senthil, PRB 83, 195139 (2011); Y.-M. Lu and Y. Ran, PRB 85, 165134 (2012): Parton constructions.

Evidence for FQH-like states

Several groups have found numerical evidence for the analogues of $\nu = \frac{1}{3},\ \frac{1}{5}$ states in Chern Bands. D. N. Sheng, Z.-C. Gu, K. Sun, and L. Sheng, arXiv:1102.2658

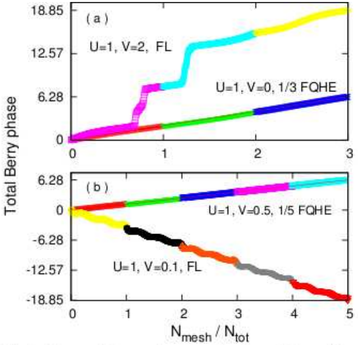


FIG. 3: The total Berry phase as a function of N_{mesh}/N_{tot}, which measures the ratio of the area in the boundary phase space for (a) 1/3 filling at $N_s = 30$; (b) 1/5 filling at $N_s = 40$. In fractional-quantum-Hall phase, a linear curve is observed whose slope is determined by filling factor, as expected. In the Fermi liquid phase, we observed large fluctuation and nonuniversal behaviors, indicating the absence of the topological quantization.

N. Regnault and B. A. Bernevig, arXiv:1105.4867

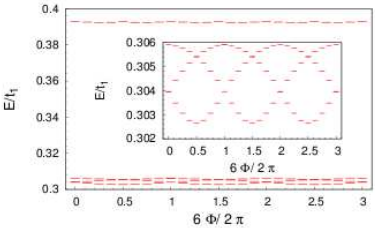


FIG. 2: Evolution of the 3-fold degenerate groundstate upon flux insertion along the y direction at $N = 10, N_x = 5$ and $N_y = 6$. The 3-fold degenerate groundstates spectral flow into each other (inset) separated at each point in the flux insertion from the first excited state (the only one of the excited states shown here) which does not exhibit spectral flow with any of the other states.

Comparison to the LLL

One is used to understanding the FQHE by flux attachment to make Composite Bosons or Composite Fermions. The density projected to the LLL satisfies the Magnetic Translation Algebra (S. M. Girvin, A. H. MacDonald, and P. M. Platzman, PRB 33, 2481 (1986))

$$[\rho_{GMP}(\vec{q}), \rho_{GMP}(\vec{q}\,')] = 2i \sin\left(\frac{\vec{q} \times \vec{q}\,' l^2}{2}\right) \rho_{GMP}(\vec{q}+\vec{q}\,')$$

So what is the problem?

Problem 1: In a Chern band there is no external flux. So the usual picture of the attached flux cancelling the external flux in an average sense does not make sense.

Problem 2: It is difficult to attach flux on a lattice. Flux naturally lives on the plaquettes while charges live on the sites. Attaching fractions of a flux makes sense (Fradkin, PRB 42, 570 (1990), Lopez, Rojo, and Fradkin, PRB 49, 15139 (1994)), but an integer number of flux quanta are equivalent to zero!

Back to Algebra

However, there seems to be a sense in which the Chern band is like a Landau level. The density operator algebra in the Chern band is "close" to that of the LLL. As $\vec{q},\ \vec{q}' \to 0$ it satisfies

$$[\rho_{Ch}(\vec{q}), \rho_{Ch}(\vec{q}')] = i\vec{q} \times \vec{q}'\ \rho_{Ch}(\vec{q}+\vec{q}')\ +\ \textit{other stuff}$$

Unfortunately, the algebra does not close. S. A. Parameswaran, R. Roy, and S. L. Sondhi, arXiv:1106.4025. See also, M. O. Goerbig, arXiv: 1107.1986

The Hamiltonian approach

Here is the way we introduce Composite Fermions (Murthy and Shankar, RMP 75, 1101 (2003)). Start with electronic guiding center coordinates, R_{ex}, R_{ey}, which satisfy

$$[R_{ex}, R_{ey}] = -il^2$$

where $l = \frac{1}{\sqrt{eB}}$ is the magnetic length. The Hilbert space is "too small". At filling ν introduce auxiliary pseudovortex guiding center coordinates R_{vx}, R_{vy} defined by the CCR

$$[R_{vx}, R_{vy}] = il^2/c^2 = il^2/(2\nu)$$

The CF Substitution in ρ_{GMP}

The expanded Hilbert space has the right size for a 2D fermion, the Composite Fermion, which sees a field $B^* = B(1-2\nu) = B(1-c^2)$, and has cyclotron $(\eta_x,\ \eta_y)$ and guiding center $(R_x,\ R_y)$ coordinates satisfying

$$[\eta_x, \eta_y] = \frac{il^2}{1-c^2} = i(l^*)^2 \qquad [R_x, R_y] = -i(l^*)^2$$

$$\vec{R}_e = \vec{R} + c\vec{\eta} \qquad \vec{R}_v = \vec{R} + \vec{\eta}/c$$

Express ρ_e in terms of CF operators.

CF Hartree-Fock and beyond

Since the CFs see a reduced field B^* at the right fractions they fill up an integer number of CF-LLs. This is found as a natural HF solution in our Hamiltonian theory, and allows us to compute gaps, temperature-dependent polarizations, and the effects of disorder (Murthy PRL 103, 206802 (2009)). The problem is that we have too many states in the Hilbert space, and we need to project to the physical space by constraining the auxiliary coordinates $\vec{R}_v$. This can be done in a conserving approximation (time-dependent HF = RPA + Ladders).

What about ρ_{Ch}?

This is great for the FQHE, but in the Chern band the density is not proportional to ρ_{GMP}. Here is where our central idea comes in. In any single band defined in a square BZ, let

$$\bar{\rho}(\vec{q}) = \sum_{\vec{p}\in BZ} c^{\dagger}(\vec{p}')c(\vec{p})e^{i\Phi(\vec{q},\vec{p})} \quad \Phi(\vec{q},\vec{p}) = \frac{q_x q_y}{4\pi} - p'_x N_y(q_y, p_y) + \frac{q_x p_y}{2\pi}$$

$$\vec{p} + \vec{q} = \vec{p}' + 2\pi\left(N_x(q_x, p_x)\hat{e}_x + N_y(q_y, p_y)\hat{e}_y\right)$$

These operators (i) Obey the GMP algebra, and (ii) For $\vec{q} = \vec{Q} + \vec{G}$ with $\vec{Q} \in BZ$, they form a complete set of operators.

Expansion of ρ_{Ch} and H in terms of ρ_{GMP}

This leads to the crucial identity

$$\rho_{Ch}(\vec{Q}) = \sum_{\vec{G}} C(\vec{Q}, \vec{G}) \rho_{GMP}(\vec{Q} + \vec{G})$$

The coefficients $C(\vec{Q}, \vec{G})$ are easily found by Fourier transformation.

How about the one-body energy?

$$H_{1b} = \sum_{\vec{p} \in BZ} \epsilon(\vec{p}) c^{\dagger}(\vec{p}) c(\vec{p}) = \sum_{\vec{G}} V(\vec{G}) \rho_{GMP}(\vec{G})$$

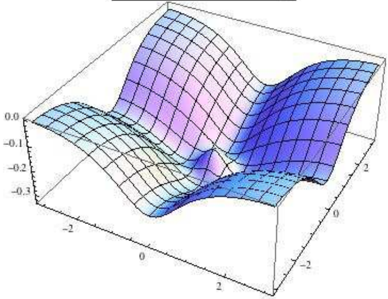
Now we can carry out the CF-substitution for any Chern band.

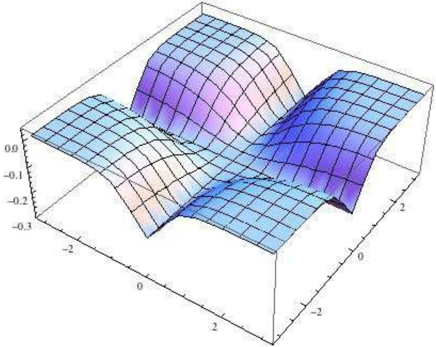
Example 1: $\nu = \frac{1}{3}$

The key difference between fractionally filled Chern bands and a LL is twofold: (i) The Chern density is varying, sometimes by an order of magnitude, and (ii) The kinetic energy competes with the interactions in determining the ground state. We will solve a simple model with both those properties, originating from two LLs with a periodic potential inducing both the above features. Here is a comparison of the Chern density of such a model and the Lattice Dirac model

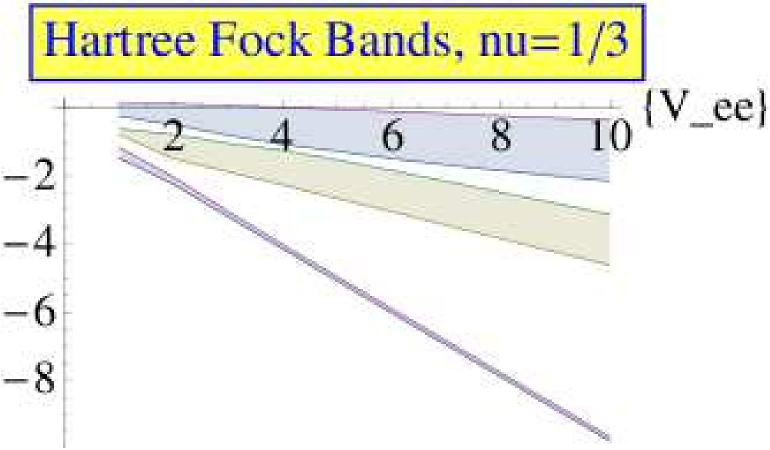
$$H_{LDM} = \sin(p_x)\sigma_x + \sin(p_y)\sigma_y + (1 - \cos(p_x) - \cos(p_y))\sigma_z$$

MLL Chern Density

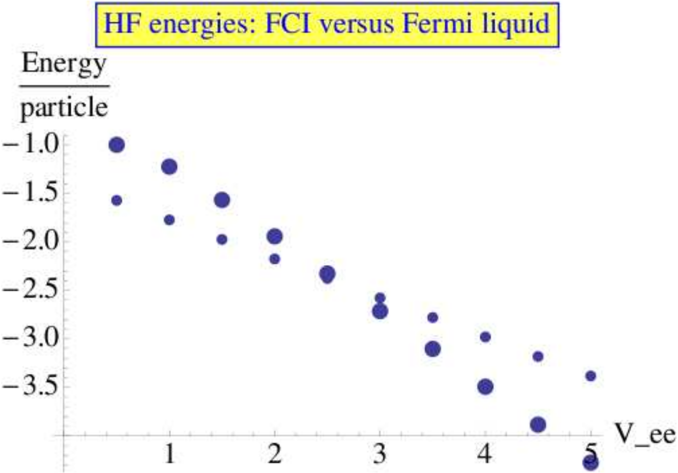




HF Bands



Ground state energy: FCI vs Fermi Liquid



σ_{xy}, e^*, and ground state degeneracy

Kol and Read, PRB 48, 8890 (1993): Analyzed FQHE in a periodic potential by various methods, including flux attachment and Chern-Simons theory. To understand their conclusions, let us first define a mean-field Composite Fermion Hall conductivity σ_{xy}^{CF}, which is an integer in units of $\frac{e^2}{h}$. This is the Chern number of all the filled bands of the CF's. In terms of this, the ground state degeneracy d, the electronic Hall conductance σ_{xy}, and the quasihole charge $e^*$are (for 2 flux attached)

$$d = 1 + 2\sigma_{xy}^{CF} \qquad \sigma_{xy} = \frac{\sigma_{xy}^{CF}}{1 + 2\sigma_{xy}^{CF}} \qquad e^* = e/(1 + 2\sigma_{xy}^{CF})$$

So, for $\nu = \frac{1}{3}$ all the quantum numbers are the same as in the Laughlin liquid. These are the states seen in numerics (Sheng et al, arXiv:1102.2568, Regnault and Bernevig, arXiv:1105.4867.

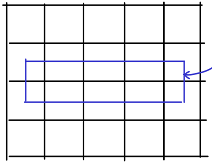
QHE with p/q flux per unit cell

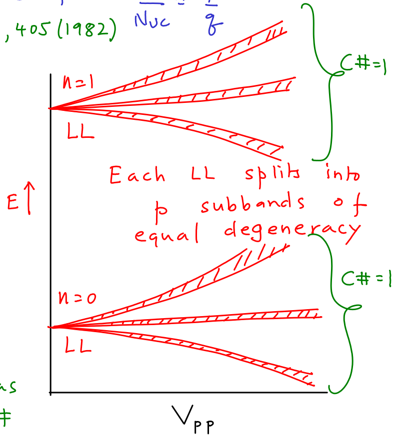
Generic features for $\frac{N_\phi}{N_{UC}} = \frac{p}{q}$

TKNN, PRL $\underline{49}$, 405 (1982)

The original UC has to be enlarged q times

Say $p = 3$, $q = 4$





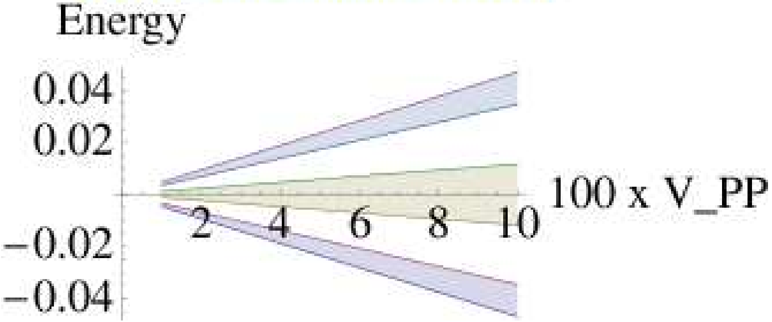
Each subband has integer Chern #

The total Chern # of all the subbands of a LL is 1

A Novel State

Now let us consider $\nu = \frac{1}{5} \Rightarrow N_e = \frac{1}{5} N_\phi$, while still attaching only two units of flux and still maintaining one quantum of flux per unit cell $N_\phi = N_{UC}$. The effective flux seen by the CFs is $N_\phi^{CF} = N_\phi - 2N_e = \frac{3}{5} N_\phi$, so the CF filling is $\nu^{CF} = \frac{N_e}{N_\phi^{CF}} = \frac{1}{3}$. Without a potential this state would be gapless. However, here the CFs see $\frac{3}{5}$ quanta of effective flux per unit cell, so each CFLL splits up into 3 subbands. Filling the lowest subband will give us a gapped state.

Hartree Fock Bands, nu=1/5



The σ_{xy} surprise

Let us consider what the Hall conductance could be. We need to add up the Chern numbers of the occupied CF-subands to obtain σ_{xy}^{CF}. That depends on the way the total Chern index of 1 for the $n = 0$ CFLL splits up between the three subbands. Say the total Chern index of the filled two subbands is j, then

$$\sigma_{xy} = \frac{j}{1+2j} = \frac{1}{3},\ \frac{2}{5},\ \frac{3}{7}\cdots$$

But the filling factor $\frac{1}{5}$ is not on the list!! This is a state for which ν and σ_{xy} are different. Since the ground state is unique at the mean-field level, it does not break any lattice symmetries. So this state is not adiabatically connected to any liquid state. There is some evidence for such states for Bosons on a lattice in an external flux (G. Moller and N. R. Cooper, PRL 103, 105303 (2009))

Hall Crystals

MacDonald, PRB 28, 6713 (1983): Dana, Avron, Zak, J. Phys. C 18, L679 (1985): Kunz, PRL 57, 1095 (1986): Tesanovic, Axel, and Halperin, PRB 39, 8525 (1989)

There is a general gap-labelling theorem, which holds for a perfect crystal in a magnetic field with $\frac{p}{q}$ quanta of flux per unit cell. Each subband α separated from other subbands by a gap can be characterized by two integers (for noninteracting electrons) satisfying a Diophantine equation

$$p\sigma_{xy,\alpha} + qm_\alpha = 1$$

$\sigma_{xy,\alpha}$ is the Hall conductance of that subband in units of e^2/h, and m_α is an integer whose meaning will become clear shortly.

Adding over all subbands $\alpha = 1, \cdots, N$ we get

$$\frac{p}{q}\sigma_{xy} + M = \frac{N}{q} = \bar{n}$$

where we have defined the number of electrons per unit cell $\bar{n}$ and the integer $M = \sum_\alpha m_\alpha$. The physical meaning of M is the following: Drag the lattice adiabatically by one lattice unit. The amount of charge transported per unit cell by this process is M in units of e.

A simple example of a Hall crystal is $\nu = 1 - \frac{1}{13}$, where the holes at $\frac{1}{13}$ filling make a Wigner Crystal. Impose an external potential commensurate with the WC, and move it by one unit. The charge that moves is 1, and the Hall conductance is also 1.

One can think of the states which have no continuum liquid analogue as Hall Crystal states pinned by the lattice potential.

T-invariant Topological Insulators

So far we have looked at a time-reversal-broken Chern band. A time-reversal invariant topological insulator will have a pair of such Chern bands with "spin", which exchange under time-reversal. There have been several papers classifying the possible incompressible states of such systems.

M. Levin and A. Stern, PRL 103, 196803 (2009): Assumed that S_z was conserved

L. Santos, T. Neupert, S. Ryu, C. Chamon, and C. Mudry, arXiv:1108.2440: Generic, based on the $K-$Matrix approach of Wen and Zee.

Our approach can be easily extended to include two "spins" even if there are interactions between them. Thus, all the phenomena we are used to in the FQHE will apply to T-invariant TIs.

Conclusions

- A single Chern band with arbitrary Chern density in the BZ can be mapped into the LLL with a periodic potential. Flux attachment can then be applied to a fractionally filled Chern band.
- We find states which have been seen in numerics on fractionally filled Chern bands, but we also find states that do not have liquid analogues, for which the filling ν is not the Hall conductance. Such states may have been seen in numerics for Bosons in an external flux on a lattice.
- Our approach easily generalizes to fractionally filled time-reversal invariant Topological Insulators.

Open Questions

- Composite Fermions have applications beyond their utility in constructing incompressible states in the LLL. The most important class is the even denominator fractions in the Fermi liquid regime. Can such a CF-Fermi liquid be seen in a Chern band?
- So far we have ignored the band dispersions with width W, assuming that $V_{ee} \gg W$. However, our Hamiltonian theory allows us to keep both and study phase transitions as W/V_{ee} varies.
- The effects of disorder on the states that have no liquid analogue may be nontrivial, since they depend for their very existence on a perfect lattice.
- Is it possible to have a QHE or FQHE in a fractionally filled band with zero Chern number?