Majorana state on the surface of a disordered 3D topological insulator

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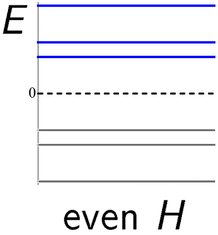
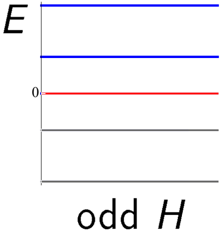
13 September 2012
The Science of Nanostructures:
New Frontiers in the Physics of Quantum Dots
Chernogolovka

Odd and even classes of H and the Majorana fermion



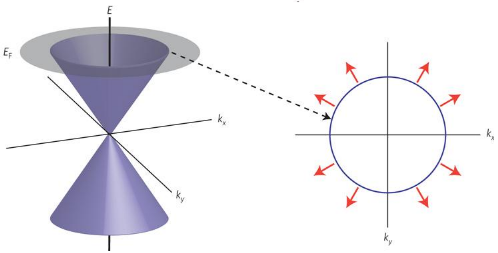


Majorana level protected by BdG symmetry

$$\gamma^\dagger = \gamma$$

- Bogolyubov-de Gennes hamiltonian has built-in $\mathcal{C}$-symmetry $\mathcal{C}H\mathcal{C} = -H$, breaking levels into conjugate $\pm E$ pairs and, possibly, a self-conjugate Majorana fermion
- symmetries like time-reversal $\mathcal{T}$ or spin rotation symmetry guarantee H to be even.
- H with only $\mathcal{C}$-symmetry belongs to the D-class of symmetry (provided $\mathcal{C}^2 = 1$).

Surface states of a 3D TI with $\mathcal{T}$-symmetry



- Single Dirac cone ($\mathrm{Bi_2Se_3}, \mathrm{Bi_2Te_3}$ etc)

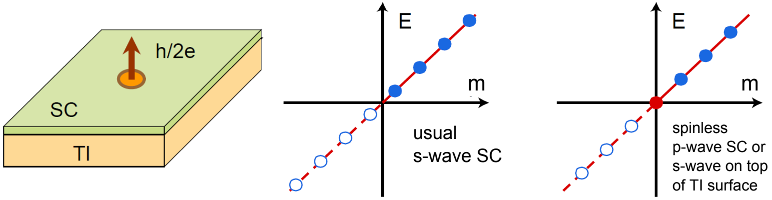
$$H_0 = v_f(\mathbf{s}\cdot\mathbf{p}),$$

- Spin-polarized electron states

$$\Psi = \begin{pmatrix} 1 \\ \pm\frac{p_x+ip_y}{|p|} \end{pmatrix} e^{i\mathbf{p}\cdot\mathbf{r}} \qquad E = \pm v_f|p|$$

- $\mathcal{T}$-symmetry connects states with opposite $\mathbf{p}$ and $\mathbf{s}$.

Majorana fermion in a vortex core



- s-wave superconductivity is induced by proximity effect
- vortex breaks $\mathcal{T}$-symmetry and produces an odd hamiltonian:

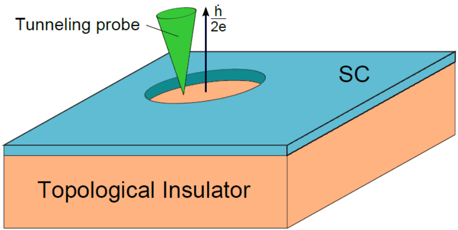
$$H = (v_f \mathbf{s} \cdot \mathbf{p} - \mu)\tau_z + \Delta(r)(\tau_x \cos\theta + \tau_y \sin\theta)$$

- non-degenerate de Gennes spectrum

$$E_m = \omega_0 m, \qquad \omega_0 \sim \Delta^2 / E_f,$$

- $m = 0$ is a Majorana state (Fu, Kane, 2008)

Setup and Goal



Strong disorder changes subgap spectrum significantly

Majorana level is protected by symmetry and stays at E=0

we find

- average local density of states (DoS) $\rho(r, E)$,
- $I(V, T)$ characteristics of a tunneling probe applied to the TI surface for any particular disorder realization and the average.
- special behaviour at zero-bias: $2e^2/h$ peak for B-class (ensemble with Majorana fermion) and dip to zero for D-class (no Majorana fermion)

Hamiltonian

- The disordered superconducting TI surface is described by

$$H = (v_f \mathbf{s} \cdot \mathbf{p} - \mu + V(\mathbf{r}))\tau_z + \Delta(r)(\tau_x \cos\theta + \tau_y \sin\theta)$$

with white-noise disorder potential $V(r)$:

$$\langle V(\mathbf{r}) V(\mathbf{r}') \rangle = \frac{\delta(\mathbf{r} - \mathbf{r}')}{\pi \nu \tau}$$

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- We consider the regime $E_{Th} = D/R^2 \ll \Delta \ll E_f$, implying

$$\Delta(r) = \begin{cases} 0, & r < R, \\ \Delta, & r \geq R. \end{cases}$$

Supersymmetric sigma-model action

$$S[Q] = \frac{\pi\nu}{8} \int d^2 r \, \mathrm{str} \left[D(\nabla Q)^2 + 4(i\epsilon\Lambda - \hat{\Delta})Q \right]$$

- Q is a 8×8 supermatrix in Nambu-Gor'kov (τ) and Particle-Hole (σ) space, obeying $Q^2 = 1$ and

$$Q = CQ^T C^T \qquad \text{with} \qquad C = \tau_x \begin{pmatrix} \sigma_x & 0 \\ 0 & i\sigma_y \end{pmatrix}_{FB},$$

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- $\epsilon = E + iG_t\delta(\mathbf{r} - \mathbf{r}_0)/4\pi\nu$ with E being the energy and the second term describing tunneling to the probe; $\Lambda = \sigma_z\tau_z$.

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Conditions $Q^2 = 1$ and $Q = \overline{Q}$ lead to the structure:

$$Q = V^{-1}\begin{pmatrix} Q_\mathrm{F} & 0 \\ 0 & Q_\mathrm{B} \end{pmatrix} V$$

where V contains 8 grassman variables, and $Q_{F,B}$ are parameterized by 4 angles each. $Q_{F,B} = \overline{Q}_{F,B}$.

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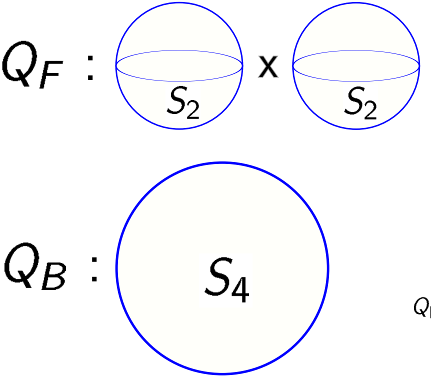

$$\begin{aligned} Q_{\mathrm{F}} &= [\tau_z \cos\theta_f + \sigma_z \sin\theta_f(\tau_x \cos\phi_f + \tau_y \sin\phi_f)] \\ &\quad \times [\sigma_z \cos k_f + \tau_z \sin k_f(\sigma_x \cos\chi_f + \sigma_y \sin\chi_f)] \end{aligned}$$

$$\begin{aligned} Q_{\mathrm{B}} &= \tau_z \cos\theta_b[\sigma_z \cos k_b + \sin k_b(\sigma_x \cos\chi_b + \sigma_y \sin\chi_b)] \\ &\quad + \sin\theta_b(\tau_x \cos\phi_b + \tau_y \sin\phi_b). \end{aligned}$$

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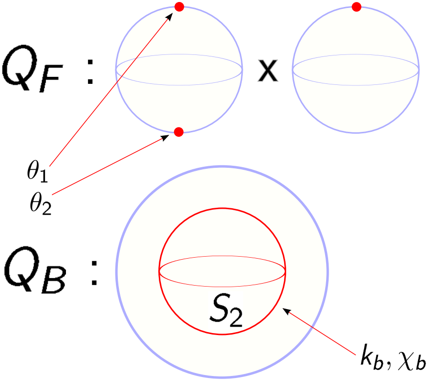


variation at zero energy:
Usadel equation

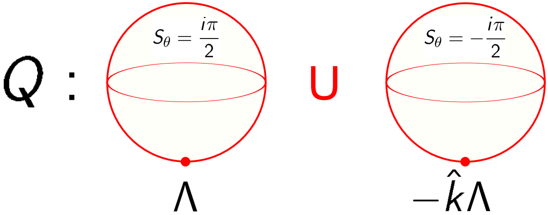
$$D\left[\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} - \boxed{\frac{\sin 2\theta}{2r^2}}\right] + 2\Delta(r)\cos\theta = 0$$

vortex term $\Longrightarrow \theta = 0, \pi$ at origin

$$\theta_1 = 2\arctan(r/R)$$

$$\theta_2 = \pi - 2\arctan(r/R)$$

$$S[Q] = \frac{\pi\nu}{8}\int d^2 r \,\mathrm{str}\,[D(\nabla Q)^2 + 4(i\epsilon\Lambda - \hat{\Delta})Q] + S_\theta[Q]$$



manifold consists of two disjoint parts

structure of the symmetry class D,

Change in $S_\theta[Q]$ means D-*odd*

Density of states at low energies

$$\rho(r,E) = \frac{\nu}{8}\,\mathrm{Re}\int DQ\,\mathrm{Str}[\hat{k}\Lambda Q(r)]\; e^{-S[Q]}.$$

Two manifold parts contribute to this integral with opposite signs, which distinguishes the D-*odd* (B) class from D-*even*. We find for $E \ll E_{Th}$

$$\rho(r,E) = \nu\, n(r) f(E/\omega_0),$$

$$n(r) = \cos\theta_1(r) = \frac{R^2 - r^2}{R^2 + r^2},$$

$$f(x) = \frac{\gamma}{\pi(x^2+\gamma^2)} + 1 - \frac{\sin(2\pi x)}{2\pi x}.$$

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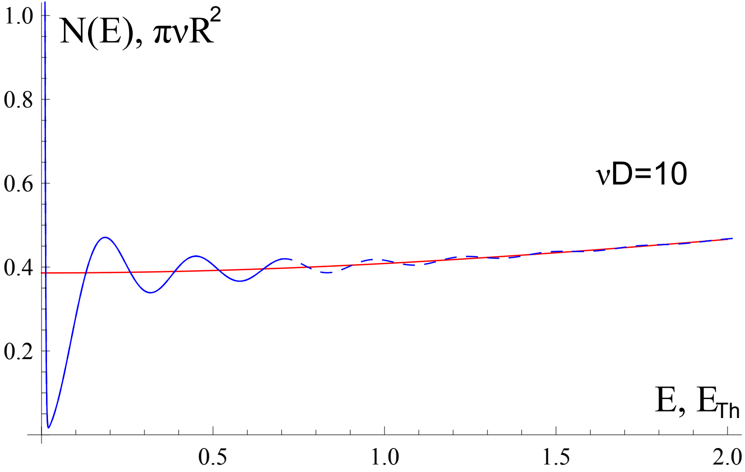
where $\gamma = G_t n(r_0)/2\pi \ll 1$ with r_0 being the position of the probe, and the level-spacing

$$\omega_0^{-1}=2\nu\int d^2r\,\cos\theta_1(r)=2\pi(\log 4-1)\nu R^2$$



Integrating $\rho(r, E)$ over $d^2 r$ we obtain

$$N(E) = \frac{\gamma}{2\pi(E^2/\omega_0 + \gamma^2\omega_0)} + \frac{1}{2\omega_0}\left[1 - \frac{\sin(2\pi E/\omega_0)}{2\pi E/\omega_0}\right]$$

When $G_t = 0$ (no probe) the first term becomes $\delta(E)/2$, characteristic for the B-class (D. A. Ivanov, 2002).

Tunneling current

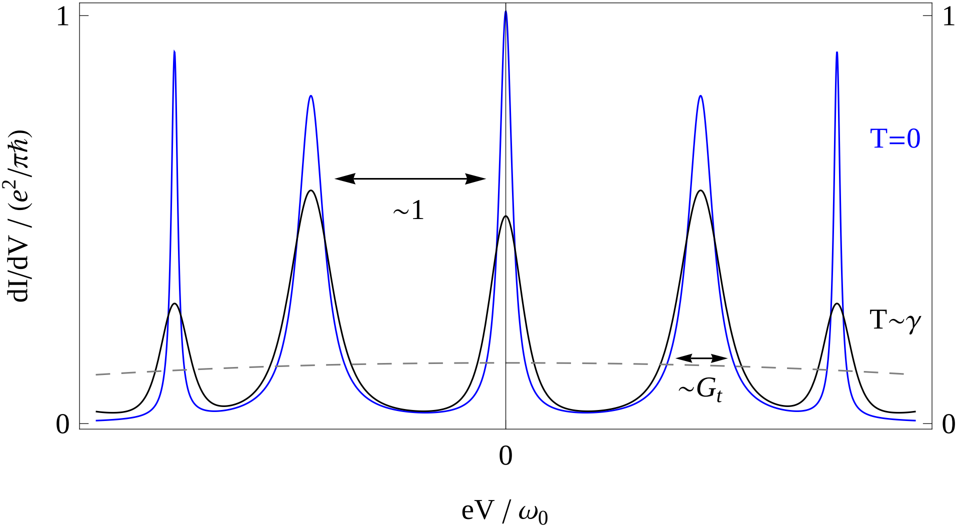
The current in a tunneling experiment is

$$I = \frac{eG_t}{h\nu} \int \rho(E, r_0)\big[f(E - eV) - f(E)\big]\, dE$$

with the Fermi distribution function $f(E)$. At $eV \ll \omega_0$ we get

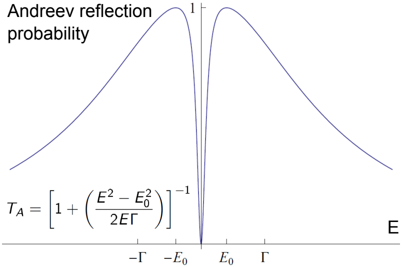
$$\frac{dI}{dV} = \begin{cases} \dfrac{2e^2\gamma^2}{h\big[\gamma^2 + (eV/\omega_0)^2\big]}, & T \ll \gamma\omega_0, \\ \dfrac{\pi e^2 \gamma\omega_0}{2hT\cosh^2(eV/2T)}, & \gamma\omega_0 \ll T \ll \omega_0. \end{cases}$$

At small T there is a Lorentz peak at $E = 0$ with a width $\sim G_t\omega_0$ and a universal height $e^2/\pi\hbar$ meaning perfect Andreev reflection (Law, Lee, Ng, 2009).

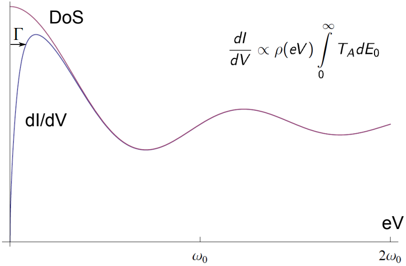


- All levels produce Lorentz-shaped resonances;
- Heights equal $\frac{2e^2}{h}\frac{\psi^\dagger \mathcal{C}\psi}{\psi^\dagger\psi}$, widths equal $G_t\frac{\psi^\dagger\psi}{\nu}$;
- Fermionic levels produce two peaks at $\pm E$, Majorana level creates one peak at $E = 0$. For D-*even* $I(0) \equiv 0$.

zero-bias conductance dip in class D



Symmetric levels interfere destructively, at E=0 Andreev reflection is absent



At small voltages Andreev current is suppressed, dI/dV is not proportional to DoS any more

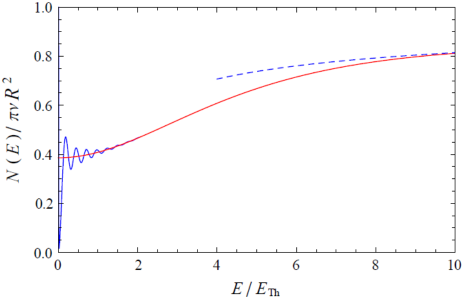
- Exactly at $E = 0$ conductance is either 0 or $2e^2/\hbar$. This follows from BdG-symmetry and unitarity for a single-channel S-matrix.
- in D-class (no Majorana fermion) $dI/dV = 0$. Processes involving symmetric Andreev levels $\pm E_j$ cancel each other out
- in B-class (one Majorana fermion) $dI/dV = 2e^2/\hbar$ which is the Majorana fermion contribution

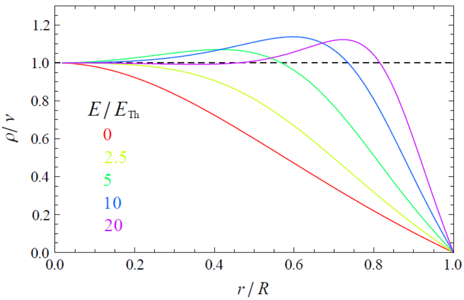
Conclusions

Conclusions [PRB, **86**, 035441 (2012)]

- Tunneling current in a vortex core on a TI surface has been studied in the strong disorder limit
- The average DoS has been calculated in the presence of a tunneling probe
- The current in a tunneling experiment has been studied for arbitrary disorder realizations
- The approach is universal and applies e.g. to Majorana fermions in nanowires which also fall into the $B-D$ class ([Bagrets, Altland 2012])
- DoS does not directly translate into conductance – zero-bias conductance is quantized, leading to a dip in dI/dV at $V=0$ in class D

Density of states at high energies





At $E \gg E_{Th}$ we use a mean-field formula

$$\rho(r, E) = \nu \operatorname{Re} \cos\theta(r, E)$$

It yields, in particular

$$N = \pi\nu R^2 \left(1 - (2 - \sqrt{2})\sqrt{\frac{E_{Th}}{E}}\right),$$

plotted in dashed blue on the left figure. The red curve is numerical.

Topological term

For our parameterization, it explicitly reads

$$S_\theta[Q] = \frac{i}{4}\int d^2r\left[\sin\theta_f\left(\nabla\theta_f \times \nabla\phi_f\right) + \sin k_f\left(\nabla k_f \times \nabla\chi_f\right)\right]$$

It can be written invariantly as a Wess-Zumino-Wittem term

$$S_\theta[Q] = \frac{i\epsilon_{abc}}{24\pi}\int_0^1 dt\int d^2r\,\mathrm{str}\left[\mathcal{Q}^{-1}(\nabla_a\mathcal{Q})\mathcal{Q}^{-1}(\nabla_b\mathcal{Q})\mathcal{Q}^{-1}(\nabla_c\mathcal{Q})\right]$$

where Q is extended onto a third dimension t. The variation $\delta S_\theta[Q]$ only depends on $Q = \mathcal{Q}|_{t=1}$. If $Q^2 = 1$, then $\delta S_\theta[Q] \equiv 0$ equals exactly zero, so that $S_\theta[Q] = const$ over any connected part of the manifold, thus playing the role of a topological term.

$$T_A = \left[1 + \left(\frac{E^2 - E_0^2}{2E\Gamma}\right)\right]^{-1}$$

For $\Gamma \ll \omega_0$ we get at zero temperature

$$\frac{dI}{dV} \propto \rho(eV) \int_0^\infty T_A dE_0 \tag{1}$$

this is easily generalized to finite temperatures

Majorana fermion

- self-conjugate (hermitian) particle

$$\gamma^\dagger = \gamma,$$

- energy equal to exactly zero, since
  $[H, \gamma^\dagger] = E\gamma^\dagger$ and $[H, \gamma] = -E\gamma$
- half a conventional fermion

$$c^\dagger = \gamma_1 + i\gamma_2$$
$$c = \gamma_1 - i\gamma_2$$

- if γ is made out of electronic operators $\psi, \psi^\dagger$ then

$$\gamma = \lambda^* \psi^\dagger + \lambda \psi$$

Majorana fermion and superconductivity

- electron-hole mixing is necessary for γ, and is provided by superconductivity

$$H = \begin{pmatrix} H_0 & \Delta \\ \Delta^* & -\Theta^{-1} H_0 \Theta \end{pmatrix}$$

  with time-reversal $\Theta = s_y K$ (s_y acts on spin, K is c.c.),
- H has a built-in $\mathcal{C}$-symmetry

$$\mathcal{C} H \mathcal{C} = -H$$

  $\mathcal{C} = \tau_y \Theta$ is charge conjugation (τ_y acts in Nambu space).
  $\mathcal{C}^2 = 1$, energy levels split into $-E, E$ pairs

$$\Psi_{-E} = \mathcal{C} \Psi_E \qquad\qquad \gamma = \mathcal{C}\gamma$$

- BdG double-counting means a single γ-eigenstate counts as half a conventional fermion