Small-Scale Turbulent Dynamo

M. Chertkov,[1] G. Falkovich,[2] I. Kolokolov,[2,3] and M. Vergassola[4]
[1]*Department of Physics, Princeton University, Princeton, New Jersey 08544*
[2]*Physics of Complex Systems, Weizmann Institute, Rehovot 76100, Israel*
[3]*Budker Institute of Nuclear Physics, Novosibirsk 630090, Russia*
[4]*CNRS, Observatoire de la Côte d'Azur, B.P. 4229, 06304 Nice Cedex 4, France*
(Received 18 June 1999)

Kinematic dynamo theory is presented here for turbulent conductive fluids. We describe how inhomogeneous magnetic fluctuations are generated below the viscous scale of turbulence where the spatial smoothness of the velocity permits a systematic analysis of the Lagrangian path dynamics. We find analytically the moments and multipoint correlation functions of the magnetic field at small yet finite magnetic diffusivity. We show that the field is concentrated in long narrow strips and describe anomalous scalings and angular singularities of the multipoint correlation functions which are manifestations of the field's intermittency. The growth rate of the magnetic field in a typical realization is found to be half the difference of two Lyapunov exponents of the same sign.

PACS numbers: 47.65.+a, 47.10.+g

It is believed that the magnetic fields of planets, stars, and galaxies have their origin in dynamo action driven by motions of conducting fluids [1–3]. Inhomogeneous flow stretches magnetic lines amplifying the field while the field produces electric currents that dissipate energy and diffuse the field due to finite resistivity. The outcome of the competition between amplification and diffusion depends on the type of flow. We consider the long-standing problem of how turbulence excites inhomogeneous fluctuations of magnetic field [4–6]. Since the growth rate is proportional to velocity gradients, the fastest growth is for the fluctuations shorter than the viscous scale of turbulence; they are the first to reach saturation and strongly influence the subsequent evolution of the system [7]. It is then important to have a systematic description of the field that has emerged from the linear dynamo phase. A consistent description of the long-time evolution of the small-scale field with the account of diffusion remained elusive for a long time [3]; only the second moment has been found [5,6]. When diffusivity κ is small, the field is almost frozen into the fluid and is expected to grow exponentially like an infinitesimal material line element. To what extent this is offset by a transversal contraction that eventually brings diffusion into play depends on the statistics of stretching and contraction. We find below the growth rate of the field in a typical realization $\gamma = \langle \log B \rangle / t$ for arbitrary velocity statistics and derive analytically the whole function $E_n = \log \langle B^{2n} \rangle / t$ for short-correlated velocity. Both E_n and γ are finite at $\kappa \to +0$ (known as fast dynamo [3,8]). It is interesting that $\gamma = \lambda_1$ in a perfect conductor while $\gamma \le \lambda_1/2$ at whatever small κ, with λ_1 being the growth rate of a material line element. We also find many point correlation functions necessary to describe a dynamo-generated field which is intermittent in space.

Consider the kinematic stage of a dynamo when the only equation to solve is that for the magnetic field,

$$\partial_t \mathbf{B} + (\mathbf{v} \cdot \partial)\mathbf{B} = (\mathbf{B} \cdot \partial)\mathbf{v} + \kappa \triangle \mathbf{B}, \qquad (1)$$

while the velocity statistics is presumed to be known. In many astrophysical applications the viscosity-to-diffusivity ratio is large and there is a wide interval of scales between viscous and diffusive cutoffs, where velocity is spatially smooth while the magnetic field has the nontrivial spatial structure described below. Note that there is no folding of magnetic lines (only stretching and contraction) in this interval. For smooth velocity, we substitute $\mathbf{v} = \hat{\sigma}\mathbf{r}$ introducing the local strain $\sigma_{\alpha\beta} = \partial v_\alpha / \partial r_\beta$. Given the initial condition, the solution of (1) is then conveniently written in Fourier space [9,10],

$$\mathbf{B}(\mathbf{k}, t) = \hat{W}(t)\mathbf{B}[\mathbf{k}(0), 0] \exp\left(-\kappa \int_0^t k^2(t')dt'\right),$$

where the wave vectors evolve as $\mathbf{k}(t') = \hat{W}^T(t, t')\mathbf{k}(t)$ and the final condition is $\mathbf{k}(t) = \mathbf{k}$. The evolution matrix $\hat{W}$ satisfies $d\hat{W}(t,t')/dt = \hat{\sigma}(t)\hat{W}(t,t')$, with $\hat{W}(t',t') = \mathbf{1}$ and $\hat{W}(t) = \hat{W}(t,0)$. We adopt here the methods of Lagrangian path analysis [11,12] developed recently for the related problem of passive scalar. The moments of $\mathbf{B}$ are to be calculated by two independent averagings: first, (trivial) average over initial statistics and, second, average over velocity statistics. Without any loss of generality, we assume the initial statistics to be homogeneous, isotropic, and Gaussian, with zero mean and the variance $\langle B_\alpha(\mathbf{k},0) B_\beta(\mathbf{k}',0)\rangle = P_{\alpha\beta}(\mathbf{k}) k^2 f(k^2) \delta(\mathbf{k} + \mathbf{k}')$. The solenoidal projector $P_{\alpha\beta} = \delta_{\alpha\beta} - k_\alpha k_\beta / k^2$ ensures that $\mathbf{B}$ is divergence free. The initial magnetic noise is concentrated at the scale L; we use $f(k^2) = L^5 \exp(-k^2 L^2)$ whenever an explicit calculation is performed. Inhomogeneous advection produces smaller and smaller scales and balances with diffusion at the scale $r_d = \sqrt{\kappa/\lambda_1}$; the magnetic Reynolds number L/r_d is assumed to be large. Inhomogeneous advection also produces larger scales; the theory below is valid until $L\exp(\lambda_1 t)$ is less than the viscous scale.

The wave vectors are of order $1/L$ initially and for some period they remain much larger than $1/r_d$. This is the stage where dynamics is insensitive to diffusion so that the field is frozen into the fluid as in a perfect conductor; see [3,9] and (6) below. At some time $t_d \propto \ln(L/r_d)$, the wave vectors reach $1/r_d$, transversal contractions bring diffusion into play, and the new regime starts, which is the main subject of this paper. It is supposed that t_d is much larger than the velocity correlation time τ and we can then carry over from random matrix theory the well-known Iwasawa decomposition (see, e.g., [13]): $\hat{W}(t) = \hat{R}\hat{D}\hat{S}$. For any fixed time, $\hat{R}$ is an SO(3) rotation matrix, $\hat{D}$ is diagonal with $D_{ii}(t) = \exp[\rho_i(t)]$, and the shearing matrix $\hat{S}$ is upper triangular with unit elements on the diagonal. The sum of ρ_i's vanishes by incompressibility and the ratios ρ_i/t tend at $t \to \infty$ to the three Lyapunov exponents λ_i (arranged in decreasing order); see, e.g., Ref. [14].

Since $\mathbf{B}$ is expressed via $\hat{W}$, our aim now is to reformulate the average over $\hat{\sigma}$ into that over $\hat{W}$. The average over $\hat{R}$ is equivalent to the integration over the directions of the vectors involved. The matrix $\hat{S}$ tends for each realization to a time-independent form. Indeed, it will be shown below that the magnetic field moments grow exponentially in time and the dominant contributions come from realizations with $\rho_1 \gg \rho_2$ at $t \to \infty$. For such realizations the matrix $\hat{S}$ is frozen at large times; the eigendirections of $\hat{W}^T\hat{W}$ do not fluctuate in time and fix an orthogonal basis, $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$, with respective stretching (contraction) rates $2\lambda_1, 2\lambda_2, 2\lambda_3$ [13]. We shall also see below that the time-independent random matrix elements of $\hat{S}$ influence only constant factors, not of interest for the space-time dependencies studied here. The problem is reduced then, on one hand, to the integration over the angles of the vectors involved and, on the other hand, to the average over the statistics of ρ_1 and ρ_2.

The moments of the magnetic field can be obtained by considering $\mathbf{B}^2(t)$ averaged over initial statistics,

$$\mathbf{B}^2(t) = \int d^3q f(q) e^{-2\kappa \mathbf{q}\hat{\Lambda}\mathbf{q}} q^2 \,\mathrm{Tr}[\hat{W}\hat{P}(\mathbf{q})\hat{W}^T],$$

then taking powers and averaging over velocity. We have changed the variables $\mathbf{q} = \hat{W}^T\mathbf{k}$ to reexpress the average in terms of the wave vectors at $t = 0$. When $t \gg |\lambda_3|^{-1}$ the main contribution to $\hat{\Lambda}(t) = \int_0^t dt'\, \hat{W}^{-1}(t')\hat{W}^{-1,T}(t')$ is given by $t' \gg |\lambda_3|^{-1}$. By changing the variables $\mathbf{q} = \hat{S}^T\mathbf{Q}$ we eliminate the constant $\hat{S}$ matrix from the diffusive exponent. The dependence on $\hat{S}$ remains only in the quadratic in the $\mathbf{Q}$ prefactor and in f so that in averaging over velocity we may replace $\hat{S}$ by the unit matrix. It follows that $\mathbf{q}\hat{\Lambda}\mathbf{q} = \int_0^t dt \sum Q_i^2 e^{-2\rho_i} \equiv U(Q,\rho)$ and in the $\mathbf{q}^2\,\mathrm{Tr}$ term the main contribution is $e^{2\rho_1}(Q_2^2 + Q_3^2)$. In any given realization the growth of the field is thus described by a simple formula

$$\mathbf{B}^2(t) \simeq \int d^3Q f(Q) \exp(-2\kappa U) e^{2\rho_1}(Q_2^2 + Q_3^2).$$

It shows that initially the diffusion is unimportant ($\kappa U \ll 1$) and B^2 grows as $e^{2\rho_1}$, i.e., as a square of a material line element. At $t \simeq t_d = |\lambda_3|^{-1}\log L/r_d$, the diffusive exponent starts to decrease substantially and the growth rate is reduced. Asymptotically for $t \gg t_d$, it is clear that the realizations and the $\mathbf{q}$'s dominating the growth are such that the quadratic form κU at the diffusive exponent remain $O(1)$. Note that for growing functions one has with exponential accuracy, $\int^t dt' \exp[-\rho_3(t')] \propto \exp[-\rho_3(t)]$. The integration over Q_3 is thus restricted within an exponentially small interval: the diffusion exponent remains $O(1)$ only for initial wave vectors with such a small projection on the contraction direction $\mathbf{n}_3$ that the respective component does not reach $1/r_d$ during the time t [9]. Neglecting Q_3^2 comparatively to Q_2^2 and omitting numerical factors, we get, after the integration over $\mathbf{Q}$,

$$\mathbf{B}^2(t) \simeq \exp[2\rho_1(t)]\{1 + (r_d/L)^2 \exp[-2\rho_3(t)]\}^{-1/2} \times \left\{1 + (r_d/L)^2 \int^t dt' \exp[-2\rho_2(t')]\right\}^{-3/2}. \quad (2)$$

The first figure bracket reduces for large times to $(L/r_d)\exp\rho_3$, and the geometrical factor, due to the orthogonality condition, to $\mathbf{n}_3$. In the second line the exponential term can be either comparable or larger than unity depending on the sign of ρ_2, which corresponds to the geometrical pictures (cone vs pancake in k space) illustrated in [9].

Moments of (2) should be averaged over the probability distribution $\mathcal{P}_t(\rho_1, \rho_2)$. When $t \gg \lambda_1^{-1}, \tau$, the theory of large deviations ensures that $\mathcal{P}_t(\rho_1,\rho_2) \propto \exp[-tH(\rho_1/t, \rho_2/t)]$, where the entropy $H(x,y)$ has a sharp minimum $H = 0$ at $x = \lambda_1$, $y = \lambda_2$ whose width decreases as $t^{-1/2}$ [15]; for the vector case see, e.g., Ref. [16]. The mean growth rate $\gamma(t) = \langle \ln B^2(t)\rangle/2t$ is then simply obtained by taking the logarithm of (2) and substituting with $\rho_i = \lambda_i t$ (strictly speaking, one must average the logarithm over the initial measure as well, yet this differs by a correction decreasing as t^{-1}, and the growth rate does not fluctuate at large time). At $t \ll t_d$ the growth rate $\gamma = \lambda_1$, as it has to be for a perfect conductor. During an intermediate stage $t \sim t_d$, γ decreases and, eventually at $t \gg t_d$, it comes to an asymptotic value γ_∞, independent of κ (so-called fast dynamo [3,8]):

$$\gamma_\infty = \min\{(\lambda_1 - \lambda_2)/2, (\lambda_2 - \lambda_3)/2\}. \quad (3)$$

Note that $\gamma_\infty \geq 0$ and $\gamma_\infty \to 0$ as $\lambda_2 \to \lambda_1$, or $\lambda_2 \to \lambda_3$, corresponding to the zero growth rate for axially symmetric cases. Both for time-reversible flow statistics and for 2D flow, $\lambda_2 = 0$ and $\gamma_\infty = \lambda_1/2$. Note that three-dimensional fluctuations of the field grow in 2D flow until our approximation of linear velocity is valid [9]. For isotropic Navier-Stokes turbulence, numerical data suggest $\lambda_2 \simeq \lambda_1/4$ [17], so that our prediction for the long-time growth rate in a typical realization is $\gamma_\infty \simeq 3\lambda_1/8$.

The moments with $n > 0$ all grow in a random incompressible flow with a nonzero Lyapunov exponent since $E_n = \log\langle B^{2n}\rangle/2t$ is a convex function of n (due to Hölder inequality) with $E_0 = 0$ and $dE_n/dn(0) = \gamma \geq 0$. Even when $\gamma = 0$, E_n are positive for $n > 0$ if H has a finite width, that is, if the flow is random (for $n = 1$ this was stated in [18]). The growth of the $2n$th moment at $t \ll t_d$ is determined by the average of $\exp(2n\rho_1)$. For $t \gg t_d$, the expression to average is either $\exp(n\rho_1 - n\rho_2)$ (with $\rho_2 > \log r_d/L$) or $\exp(n\rho_2 - n\rho_3)$, depending on whether the entropy function favors positive or negative ρ_2 (cf. [9]). The formula $B^2 \propto \exp[(\lambda_1 - \lambda_2)t]$ was previously derived for a permanent strain [19]. Note in passing that at $t \gg t_d$ the magnetic flux (conserved in an ideal conductor) decreases with the rate $\gamma_\infty + \lambda_2 + \lambda_3 < 0$, independent of diffusivity. The function E_n is nonuniversal since it is determined by the saddle point of $\mathcal{D}\rho_{1,2}$ integration which depends on the particular form of the entropy function. The saddle point falls within the (universal) parabolic region of H around the minimum only for $n \ll (\lambda_1\tau)^{-1}$. Therefore, we calculate below, for a short-correlated strain, the temporal growth of the moments.

Here, we continue with the general case to establish what is universal in the different point correlation functions $F_{2n} = \langle\prod_{k=1}^n[\mathbf{B}(\mathbf{x}_{2k-1},t)\mathbf{B}(\mathbf{x}_{2k},t)]\rangle$. Its calculation is reduced to averaging $(2n-1)!!$ terms arising from the Wick decomposition in the Gaussian integration over the random initial condition. Each term is a product of n integrals generalizing that for B^2 with the inclusion of respective $\exp(i\mathbf{r}_j\hat{W}^{T,-1}\mathbf{q})$ in the integrand. The n vectors $\mathbf{r}_j$ are the differences between couples of $\mathbf{x}_k$'s. The new feature, with respect to the moments, is the presence of the rotation matrix $\hat{R}$ in the exponential factor. The $\mathbf{q}$ integrations proceed along the same lines as previously: in every one of the n integrals we change variables $\mathbf{q} = \hat{S}^T\mathbf{Q}$, and the dependence on $\hat{S}$ is entirely moved into the prefactors. Substituting $\hat{S}$ by the unit matrix and performing the $\mathbf{Q}$ integrations, we obtain the long-time asymptotics for any of the $(2n-1)!!$ contributions to F_{2n}:

$$\left\langle \frac{(L/r_d)^n \exp[n(\rho_1 - \rho_2)]}{[1 + e^{-2\rho_2} r_d^2/L^2]^{5n/2}} \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\varphi \int_0^{2\pi} d\phi \prod_{j=1}^{n} \left[2 - \frac{R_{2j}^2 e^{-2\rho_2}}{L^2}\right] \exp\left[-\frac{R_{2j}^2}{4[L^2e^{2\rho_2} + 2r_d^2]} - \frac{R_{3j}^2}{8r_d^2}\right]\right\rangle, \tag{4}$$

where $\mathbf{R}_j = \hat{R}_3[\varphi]\hat{R}_2[\theta]\hat{R}_3[\phi]\mathbf{r}_j$, and $\hat{R}_{2,3}$ stand for rotations around the Y and Z axes, respectively. We consider time-reversible statistics, where the scaling laws turn out to be universal. Let us explain the physical meaning of (4) and derive the correlation functions starting from $n = 1$. The realizations contributing have the advective exponent $\exp(i\mathbf{r}\hat{W}^{T,-1}\mathbf{q})$ of order unity. This requires $\rho_2 > \ln(r/L)$ and the direction of contraction $\mathbf{n}_3$ to be almost perpendicular to $\mathbf{r}$, which gives the geometrical factor $(L/r)\exp\rho_3$. At $\lambda_1 t > \ln(L/r)$ we then obtain

$$F_2(r,t) \simeq \frac{L}{r}\int_{-\infty}^{\infty} d\rho_1 \int_{\ln(r/L)}^{\infty} d\rho_2 e^{\rho_1-\rho_2-tH} \propto r^{-2-h}e^{E_2 t},$$

where $h = \partial H/\partial y$ is taken at $y = 0$ and at x given by the saddle point $\partial H/\partial x = 1$. Time reversibility means that $H(x,y) = H(x+y,-y)$ so that $h = 1/2$ and $F_2 \propto r^{-5/2}\exp(E_2 t)$. At $r \ll L$ and $\lambda_1 t \ll \ln(L/r)$, F_2 is r independent. This generalizes the consideration of [7] for arbitrary time-reversible statistics. To understand the simple geometrical picture behind this derivation, note that the integral over ρ_2 comes from $\rho_2 \simeq \ln(r/L)$. That means that the field configurations in the form of strips with width r dominate $F_2(r)$. The angular integral in (4) comes from $\varphi \simeq 1$, that is, the strips with the stretching direction almost parallel to $\mathbf{r}$ do not contribute (because of cancellations due to solenoidality).

For $n \geq 2$, the geometry of the vectors $\mathbf{r}_j$ becomes important. Let us consider the case where all of the vectors are in the same plane (their length being r). They can be either collinear or not. Almost orthogonality of $\mathbf{r}_j$ to $\mathbf{n}_3$ involves therefore either one angle or two, giving the angular factor $(L/r)\exp\rho_3$ or its square, respectively. The other difference concerns the behavior along $\mathbf{n}_2$. For noncollinear geometry, all vectors cannot be orthogonal to $\mathbf{n}_2$, and ρ_2 should then be constrained as $\rho_2 > \log r/L$. This is technically signaled by the fact that the integration over φ is not saddle point. Conversely, for collinear geometry all of the vectors can be orthogonal both to $\mathbf{n}_2$ and $\mathbf{n}_3$, giving an additional angular factor $(L/r)\exp\rho_2$: the saddle-point integrations over θ and φ pick $\theta = \pi/2$, $\varphi \simeq \exp[\rho_2]L/r$. In the rest of the integrals (either $n-1$ or $n-2$, respectively) the above angular constraints ensure that the advective exponents are $O(1)$ so the diffusive exponents $\exp(-2\kappa\mathbf{q}\Lambda\mathbf{q})$ become important. The calculation of these integrals is essentially the same as for the moments and this is where diffusion comes into play. The wave vectors should be quasiorthogonal to $\mathbf{n}_3$, giving either $n-1$ or $n-2$ factors $(L/r_d)\exp\rho_3$. The growth along $\mathbf{n}_2$ for a generic planar geometry is automatically controlled by the previous constraint $\rho_2 > \log r/L$; for collinear geometry it provides the bound $\rho_2 > \log r_d/L$. Simply speaking, the strips with the width r_d stretched along $\mathbf{r}$ contribute in the collinear case, while the width is r in the generic case. The resulting integrations over ρ_1 are saddle point and those over ρ_2 are dominated by the lower bounds. Finally,

$$F_{2n} \simeq e^{E_n t}\left(\frac{L}{r_d}\right)^{5n/2}\left(\frac{r_d}{r}\right)^2 \times \begin{cases} 1 & \text{collinear} \\ (r_d/r)^{3n/2} & \text{planar} \end{cases}. \tag{5}$$

Here, $E_n = x_n - H(x_n,0)$ with $\partial H/\partial x(x_n,0) = n$. That the integrals over ρ_2 are all dominated by the lower bounds indicates the geometric nature of the scaling universality found: the field configurations that contribute are narrow strips (not ropes and layers as suggested in [9]) with one direction of stretching, one of contraction, and a neutral one. The factor $(r_d/r)^2$ is the probability for two points at distance r to lie within the same strip of width r_d. The peculiar nature of strips has another dramatic consequence for $n \geq 3$: the correlation functions are strongly suppressed in a generic situation when at least three vector $\mathbf{r}_j$'s do not lie in parallel planes. Indeed, they cannot then be on parallel strips, and nonzero correlation appears only because the strips have exponential diffusive tails. As a result, the factor $(r_d/r)^2$ in the planar formula is replaced by $\exp(-ar^2\sin^2\Theta/r_d^2)$, where $a \simeq 1$ and Θ is the minimal angle between a vector and the plane formed by another two vectors. This can be derived from (4), where all angular integrations are not saddle point now. For general irreversible velocity statistics, the r dependences are different yet the qualitative conclusions (that the correlation functions are not exponentially suppressed only for planar geometry and are anomalously large for collinear geometry) are generally valid. Angular anomalies are peculiar to the viscous interval, where advection by a smooth velocity preserves collinearity. Similar collinear anomalies have been described before for a passive scalar advected by a smooth velocity [20–22]. Note that the cliff-and-ramp structures observed in passive scalar experiments (see [23] for review) are probably also related to the strips.

It is left to find E_n for the standard Kazantsev-Kraichnan model of an isotropic short-correlated Gaussian strain with $\langle\sigma_{\alpha\beta}(t)\sigma_{\alpha\beta}(0)\rangle = 10\lambda_1\delta(t)$. Straightforward derivation gives Gaussian $\mathcal{P}_t(\rho_1,\rho_2)$ with $H = (\rho_1 + \rho_2/2 - \lambda_1 t)^2/\lambda_1 t + 3\rho_2^2/4\lambda_1 t$. Note that $\lambda_2 = 0$. Now we integrate the moments of (2) with such $\mathcal{P}_t$. We integrate $\exp(2n\rho_1)$ and get the answer for the perfect conductor [3,9],

$$\langle \mathbf{B}^{2n}\rangle \simeq \exp[2\lambda_1 n(2n+3)t/3]. \tag{6}$$

The main contribution to the nth moment comes from $\rho_1 = \lambda_1 t(4n+3)/3, \rho_2 = -2n\lambda_1 t/3$ so that (6) is valid until $L\exp\rho_3 > r_d$, that is, for $t < 3t_d/(2n+3)$ with $t_d \equiv \lambda_1^{-1}\ln(L/r_d)$. There is then a logarithmically wide crossover interval when the growth is nonexponential. The asymptotic regime starts at $t > 3t_d/(n+2)$ when unity in the first parenthesis of (2) may be neglected. The integral over ρ_1 now comes from $\rho_1 + \rho_2/2 = (n+2)\lambda_1 t/2$ while that over ρ_2 is dominated by the lower bound $\rho_2 \simeq \ln(r_d/L)$:

$$\langle \mathbf{B}^{2n}\rangle \simeq (L/r_d)^{5n/2}\exp(E_n t), \qquad E_n = \lambda_1 n(n+4)t/4. \tag{7}$$

For $n = 1$, this was obtained by Kazantsev [6]. The difference between (6) and (7) formally means that the two limits $t \to \infty$ and $\kappa \to 0$ do not commute (called dissipative anomaly). The physical reason is quite clear: realizations with continuing contraction along two directions contribute most in a perfect conductor, while with diffusion present, one direction is neutral. A magnetic field initially concentrated in the ball with the radius L will have the fastest growth rate γ if the ball turns into a strip with the dimensions $L(L/r_d)^{1/2}\exp(3Dt)$, r_d, and $L(r_d/L)^{1/2}\exp(-3Dt)$. The nth moment is given by strips with the dimensions $L(L/r_d)^{1/2}\exp[3(n+2)Dt/2]$, r_d, and $L(r_d/L)^{1/2}\exp[-3(n+2)Dt/2]$. In conclusion, we have related the growth rate of the small-scale dynamo to the Lypunov exponents of the flow and described analytically the strip structure of the magnetic field.

Very fruitful discussions with E. Balkovsky, A. Fouxon, U. Frisch, A. Gruzinov, V. Lebedev, A. Pouquet, B. Shraiman, and A. Vulpiani are gratefully acknowledged. This work was supported by JRO (M. C.), by the Einstein Center (I. K.), by the grants from the Minerva Foundation and Mitchell Research Fund at the Weizmann Institute (G. F.) and by the Russian Foundation for Basic Research (I.K.).

[1] H. K. Moffatt, *Magnetic Field Generation in Electrically Conducting Fluids* (Cambridge University Press, Cambridge, England, 1978).
[2] E. N. Parker, *Cosmic Magnetic Fields, Their Origin and Activity* (Clarendon Press, Oxford, 1979).
[3] S. Childress and A. Gilbert, *Stretch, Twist, Fold: The Fast Dynamo* (Springer-Verlag, Berlin, 1995).
[4] G. K. Batchelor, Proc. R. Soc. London A **261**, 405 (1950).
[5] R. Kraichnan and S. Nagarajan, Phys. Fluids **10**, 859 (1967).
[6] A. P. Kazantsev, Sov. Phys. JETP **26**, 1031 (1968).
[7] R. Kulsrud and S. Anderson, Astrophys. J. **396**, 606 (1992).
[8] S. Vainshtein and Ya. Zeldovich, Sov. Phys. Usp. **15**, 159 (1972).
[9] Ya. Zeldovich, A. Ruzmaikin, S. Molchanov, and V. Sokolov, J. Fluid Mech. **144**, 1–11 (1984).
[10] A. M. Soward, in *Lectures on Solar and Planetary Dynamos*, edited by M. Proctor and A. Gilbert (Cambridge University Press, Cambridge, England, 1994).
[11] B. Shraiman and E. Siggia, Phys. Rev. E **49**, 2912 (1994).
[12] M. Chertkov, G. Falkovich, I. Kolokolov, and V. Lebedev, Phys. Rev. E **51**, 5609 (1995).
[13] I. Goldhirsch, P. Sulem, and S. Orszag, Physica (Amsterdam) **27D**, 311 (1987).
[14] A. Crisanti, G. Paladin, and A. Vulpiani, *Products of Random Matrices* (Springer-Verlag, Berlin, 1993).
[15] R. Ellis, *Entropy, Large Deviations and Statistical Mechanics* (Springer-Verlag, Berlin, 1985).
[16] E. Balkovsky and A. Fouxon, chao-dyn/9905020.
[17] S. Girimaji and S. Pope, J. Fluid Mech. **220**, 427 (1990).
[18] A. Gruzinov, S. Cowley, and R. Sudan, Phys. Rev. Lett. **77**, 4342–4345 (1996).
[19] J. Pearson, J. Fluid Mech. **5**, 274–288 (1959).
[20] E. Balkovsky, M. Chertkov, I. Kolokolov, and V. Lebedev, JETP Lett. **61**, 1049 (1995).
[21] B. Shraiman and E. Siggia, Comp. Rend. Ac. Sc. (CRAS) **321**, 279 (1995).
[22] E. Balkovsky, G. Falkovich, V. Lebedev, and M. Lysiansky, Phys. Fluids **11**, 2269 (1999).
[23] K. Sreenivasan, Proc. R. Soc. London A **434**, 165 (1991).